Hodge dual on supermanifolds and supersymmetric actions

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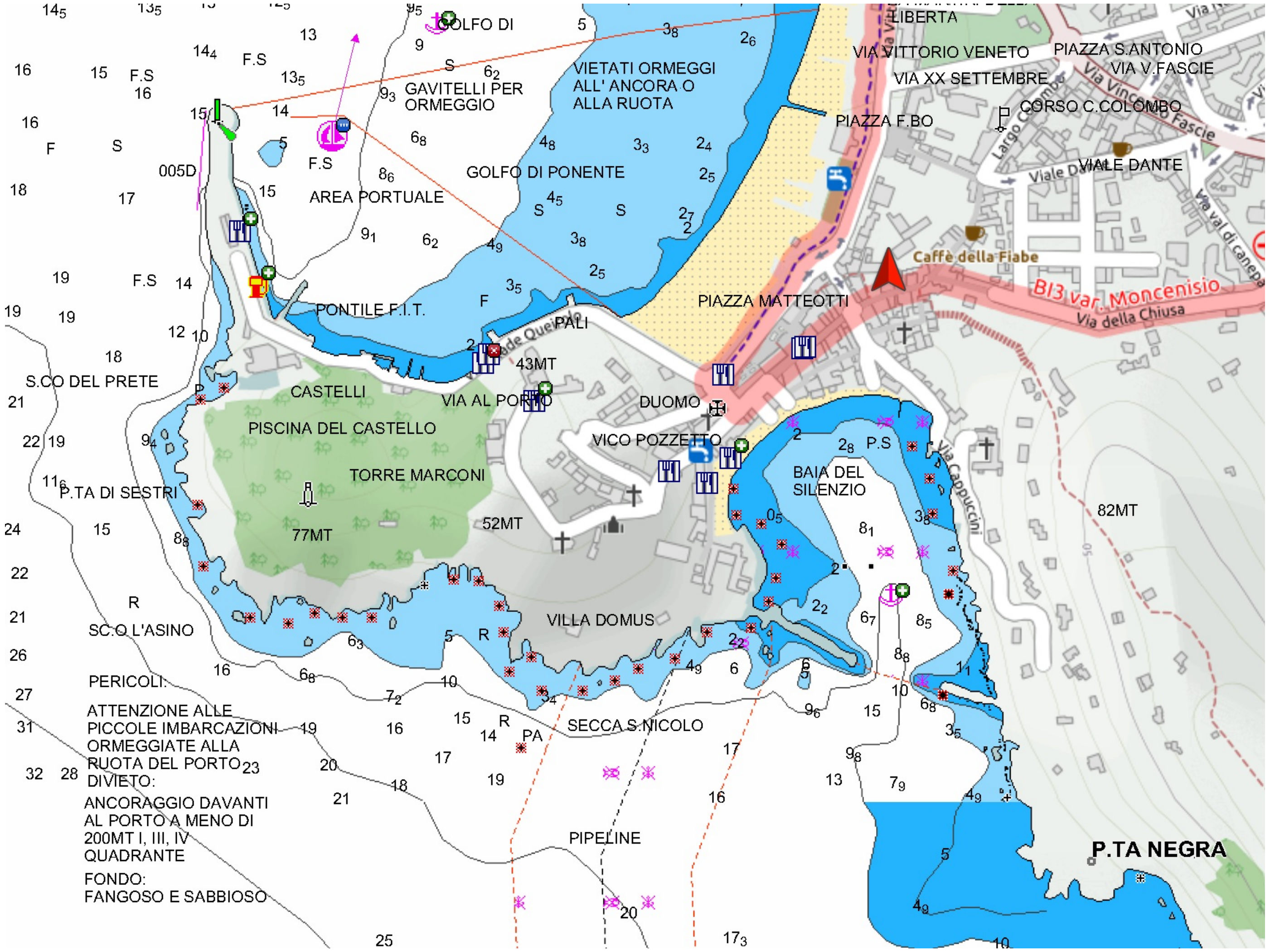

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1. Why supermanifolds

2. Why integration on supermanifolds

3. Integral representation of the Hodge dual

LC, Catenacci, Grassi 1409.0192 , Nucl.Phys.B889 (2014) 419
1503.07886, Nucl.Phys.B899 (2015)112
1507.01421, Nucl.Phys.B899 (2015) 570

1. Why supermanifolds

To interpret (local) supersymmetry variations of the fields as the effect of a Grassmann coordinate transformation

This idea can be extended to gauge transformations as well —> supergroup manifolds

Thus diff.s, supersymmetry, gauge transformations are all diffeomorphisms in the supergroup manifold G

They are invariances of an action invariant under group manifold diff.s

$\boxed{\text{Dynamical fields: vielbeins on G}}$

Example: G = superPoincaré

generators	vielbein	G-coordinates
$P_a \longleftrightarrow$	V^a	x
$M_{ab} \longleftrightarrow$	ω^{ab}	y^{ab}
$Q_\alpha \longleftrightarrow$	ψ^α	θ^α

2. Why integration on supermanifolds

To obtain a diff-invariant action on the supergroup manifold: integral of a n-form on a n-dim supergroup manifold G

But $n > d$, where d is the dimension of spacetime.

How can we obtain a d-dim field theory where the fields depend only on d space-time coordinates (for ex. d=4) ?

Start with a Lagrangian d-form and integrate it on a d-dim submanifold S of G

Group geometric construction of supergravity theories, Torino group 80's
Review: LC, D'Auria, Fré

Action in *d* dim. ⟷ Action in *n* dim.

$$i : S \to G$$

$$\boxed{\int_S i^* L^{(d)} \quad = \quad \int_G L^{(d)} \wedge \eta_S}$$

where $L^{(d)}$ is a *d*-form Lagrangian (on *G*)
S is a (bosonic*) *d*-dim surface embedded in *G*,
η_S is the *Poincaré dual* of *S*

- If S described locally by the vanishing of *n-d* coordinates *t*

$$\eta_S = \delta(t^1) \cdots \delta(t^{n-d}) \; dt^1 \wedge \cdots \wedge dt^{n-d}$$

a singular closed localization (*n-d*) - form. Projects on the submanifold *S* (*t=0*) and orthogonally to $dt^1 \wedge \cdots \wedge dt^{n-d}$.

*diffeomorphic to d-dim Minkowski spacetime

This action, being the integral of a *n*-form on the *n*-dim supergroup manifold ***G***, is invariant under ***G***-diffeomorphisms

$$0 = \delta_\epsilon \int_G L^{(d)} \wedge \eta_S = \int_G \ell_\epsilon L^{(d)} \wedge \eta_S + \int_G L^{(d)} \wedge \ell_\epsilon \eta_S$$

→ a change of ***S***, generated by the Lie derivative along a tangent vector ϵ, can be compensated by a diffeomorphism applied to the fields in ***L***

Action principle

- The action

$$I[\phi, S] = \int_G L(\phi) \wedge \eta_S$$

depends on fields ϕ (contained in *L*) and on the submanifold *S*

- must vary both ϕ and *S*
- since variation of embedding of *S* is equivalent to variation of fields, just vary ϕ with *S* fixed and arbitrary (variational principle does not determine S)

$\rightarrow$ $\boxed{\text{Field equations: } \quad \frac{\delta L}{\delta \phi} = 0}$

d-1 form equations holding on *G*

Finally, dependence of the fields on the extra coordinates:

- disappears for "gauge coordinates"
- for "supersymmetry coordinates" θ , the fields at θ are related to the fields at $\theta = 0$

An output of the field equations:

- horizontality of the curvatures: no legs in gauge directions
- rheonomy of curvatures: legs in θ directions related to legs in x directions

Invariances

- Diff.s on *G* are invariances of the action:

$$\delta_\epsilon \int_G L(\phi) \wedge \eta_S = \int_G \ell_\epsilon L(\phi) \wedge \eta_S + \int_G L(\phi) \wedge \ell_\epsilon \eta_S$$

- If second term vanishes, diff.s applied only to the fields in *L* are also invariances of the action

- This happens when

$$\boxed{i_\epsilon dL = 0}$$

(use $\ell_\epsilon = i_\epsilon d + d i_\epsilon$, $d\eta_S = 0$ and integration by parts)

Example: N=1 supergravity in d=4

Action

$$I_{SG} = \int_{M^4} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} + 4\ \bar{\psi} \wedge \gamma_5 \gamma_d \rho \wedge V^d$$

with $R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}$

$\rho = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi$

Invariances (diff.s on superPoincaré group manifold*)

- ordinary x-diff.s
- local Lorentz rotations
- local supersymmetry

$$\delta_\epsilon V^a = i\ \bar{\epsilon}\gamma^a\psi$$

$$\delta_\epsilon \psi = d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon$$

*soft group manifold

- Diff invariance relies on existence of a top form

$$\delta_\epsilon \int (top\ form) = \int (di_\epsilon + i_\epsilon d)(top\ form) = \int d(i_\epsilon\ top\ form)$$

since *d(top form)* = 0

- Are there top forms also on supermanifolds ?

- Can we integrate them ?

We know how to integrate functions on a supermanifold (Berezin integration).

Integration of functions on supermanifolds

- Example: real superspace $\mathbb{R}^{n|m}$

n bosonic coordinates x^i
m fermionic coordinates θ^α

- Integration of functions

$$f(x,\theta) = f_0(x) + \cdots + f_m(x)\ \theta^1 \cdots \theta^m$$

If the real function $f_m(x)$ is integrable in $\mathbb{R}^n$,

the Berezin integral of $f(x,\theta)$ is defined as

$$\boxed{\int_{\mathbb{R}^{n|m}} f(x,\theta)[d^n x d^m \theta] = \int_{\mathbb{R}^n} f_m(x) d^n x}$$

Can we define integration of forms on supermanifolds via Berezin integration of functions ?

After all, integration of usual (bosonic) forms

$$\omega = \omega_{[i_1 \cdots i_p]}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

is defined via Riemann-Lebesgue integration of functions

$$\int_{M^p} \omega = \int \omega_{[i_1 \cdots i_p]}(x) \epsilon^{i_1 \cdots i_p} d^p x$$

Berezin for bosonic forms

Usual integration theory of differential forms for bosonic manifolds can be rephrased in terms of Berezin integration.

The idea is to interpret the differentials *dx* as anticommuting variables $\xi = dx$, similar to the Grassmann coordinates θ

Then the *p*-form

$$\omega = \omega_{[i_1 \cdots i_p]}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

is reinterpreted as a function on a supermanifold $M^{p|p}$ with coordinates *x* and ξ

$$\omega(x,\xi) = \omega_{[i_1 \cdots i_p]}(x) \xi^{i_1} \cdots \xi^{i_p}$$

➡ The Berezin integral of this function is

$$\int_{M^{p|p}} \omega(x,\xi)[d^p x d^p \xi] = \int \omega_{[i_1 \cdots i_p]}(x) \epsilon^{i_1 \cdots i_p} d^p x$$

and reproduces $\int_{M^p} \omega$

Top forms for supermanifolds ?

- There seems to be a problem:
  forms on a supermanifold can be of arbitrarily high order,
  since the $d\theta$ commute !

$$\omega = \omega_{[i_1 \cdots i_r](\alpha_1 \cdots \alpha_s)}(x,\theta)\; dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge d\theta^{\alpha_1} \wedge \cdots \wedge d\theta^{\alpha_s}$$

$d\omega \neq 0$ —-> forms of this type **cannot be top forms**

- Then how can we define integration of forms on a supermanifold ?

- Answer: consider ω as a function of the differentials

$$\omega = \omega(x, \theta, dx, d\theta)$$

with *n+m* bosonic variables $x, d\theta$
and *m+n* fermionic variables θ, dx

- Use then Berezin integration on the function ω on the "double" supermanifold $M^{n+m|n+m}$

- The only functions of $x, \theta, dx, d\theta$ that can be integrated on $M^{n+m|n+m}$ are the "integral top forms" containing all the dx differentials, and all the $d\theta$ differentials inside delta functions:

$$\omega = \omega_{[i_1 \cdots i_n][\alpha_1 \cdots \alpha_m]}(x,\theta)\ dx^{i_1} \cdots dx^{i_n} \delta(d\theta^{\alpha_1}) \cdots \delta(d\theta^{\alpha_m})$$

NB ω has compact support as a function of the even variables $d\theta$: it is in fact a *distribution* with support at the origin, so that the integral over those variables makes sense.

- Note that $\delta(d\theta^\alpha)\,\delta(d\theta^\beta)$ = - $\delta(d\theta^\beta)\,\delta(d\theta^\alpha)$

to be consistent with $\int \delta(d\theta)\delta(d\theta')d(d\theta)d(d\theta') = 1$

- In analogy with the Berezin integral for bosonic forms:

$$\boxed{\begin{aligned} \int_{M^{n|m}} \omega &= \int_{M^{n+m|n+m}} \omega(x,\theta,dx,d\theta)[d^n x d^m \theta d^n(dx) d^m(d\theta)] \\ &\equiv \int_{M^{n|m}} \omega_{[i_1\cdots i_n][\alpha_1\cdots\alpha_m]}(x,\theta)\epsilon^{i_1\cdots i_n}\epsilon^{\alpha_1\cdots\alpha_m}[d^n x d^m \theta] \end{aligned}}$$

→ consistent theory of integration on supermanifolds

books: Berezin, Manin, DeWitt, Rogers
review articles: Kac, Leites, Voronov, Nelson, Deligne and Morgan, Witten
theory of integral forms initiated in Bernstein and Leites (1977)

including integration on a (bosonic) submanifold of a supergroup manifold,
necessary to give a sound mathematical basis to the group-geometric method outlined above.

3. Hodge dual for supermanifolds

LC,Catenacci,Grassi

Based on Fourier transform of superforms.

Again, superforms can be seen as functions of $x, \theta, dx, d\theta$

Then we just need to define Fourier transform of functions of $x, \theta, dx, d\theta$. Introducing the dual variables y, ψ, η, b :

$$\boxed{\mathcal{F}(\omega)(x, \theta, dx, d\theta) \equiv \int_{\mathbb{R}^{n+m|n+m}} \omega(y, \psi, \eta, b) e^{i(xy+\theta\psi+dx\eta+d\theta b)} [d^n y d^m \psi d^n \eta d^m b]}$$

defines the Fourier transform of a superform ω in $\mathbb{R}^{n|m}$

Integral representation of the Hodge dual

A **partial Fourier transform** only on the “differential variables”:

$$\boxed{(\star\omega)(x,\theta,dx,d\theta) \equiv \int_{\mathbb{R}^{m|n}} \omega(x,\theta,\eta,b) e^{i(dx\eta + d\theta b)} [d^n\eta d^m b]}$$

Examples:

$$\star 1 = dx^1 \wedge \cdots \wedge dx^n \wedge \delta(d\theta^1) \wedge \cdots \wedge \delta(d\theta^m)$$

$$\star\star = (-1)^{p(p-n)} \qquad \text{on } p\text{-superforms}$$

Isomorphism

$$\star : \Omega^{(p|0)} \longleftrightarrow \Omega^{(n-p|m)}$$

between finite dimensional spaces, generalizes Poincaré duality

Thank you !