Large N from Localization
and
Large N from Seiberg-Witten

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Based on:

J.R. and K. Zarembo, JHEP 1311 (2013) 130 , arXiv:1309.1004
J.R., JHEP 1412, 169 (2014), arxiv:1411.2602
J.R., Phys.Lett. B748 (2015) 19-23 , arXiv:1504.02958

PHYSICS ON THE RIVIERA 2015
Sestri Levante, 16-18 September 2015

Our questions today

- study exact results in supersymmetric gauge theories both at large *N* and finite *N*
  "Exact" *= all order in coupling, including both perturbative and non-pert contributions*

- How to extract the large *N* limit from the Seiberg-Witten curve?

- $\mathcal{N} = 4$ SYM has a smooth behavior from weak to strong coupling, but what about deformations of this?
  – Add $\mathcal{N} = 2$ mass deformation. This does not have a smooth behavior. There are phase transitions at specific values, $\lambda = 35.4$, etc.
  – $\mathcal{N} = 2$ massive fundamental matter. The resulting theory has two quantum phases.

Localization

Consider SU(N) $\mathcal{N}$ = 2 supersymmetric YM theories on **S⁴** , radius R

Vector multiplet $$(A_\mu, \psi^1_\alpha, \psi^1_\alpha, \Phi + i\Phi')$$

Matter hypermultiplet mass M $$(\phi, \chi_\alpha, \tilde{\chi}_\alpha, \tilde{\phi})$$ adjoint or fundamental

Exact partition function for $\mathcal{N}$ = 2 supersymmetric YM theories on $\mathbf{S}^4$, with arbitrary matter content . [Pestun, 0712.2824]

Partition function localizes to a finite dimensional integral over Coulomb moduli

$$\langle \Phi \rangle = diag(a_1, \ldots, a_N)$$ VEV of scalar of vector multiplet

$$Z(g) = \int d^{N-1}a \prod_{i<j} (a_i - a_j)^2 \, \mathrm{e}^{-S_{cl}(a)} \, z_{1-loop}(a) \quad \left| z_{inst}(a; g^2) \right|^2$$

$Z = Z(g)$

Exact *g* dependence

$$S_{cl} = \frac{1}{4g^2} \int_{S^4} d^4x \sqrt{g} \ \mathrm{R} \ \mathrm{tr}\,\Phi^2 = R^2 \frac{8\pi^2}{g^2} \sum_i a_i^2$$

$z_{1\text{-}loop}$ is expressed in terms of a single function $H(x) \equiv \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n \mathrm{e}^{-\frac{x^2}{n}}$

$$Z_{inst} = \sum_{k=0}^{\infty} q^k \, z_k(M, a, \varepsilon_1, \varepsilon_2) \quad , \qquad q = e^{2\pi i \tau}, \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

$$\varepsilon_1 = \varepsilon_2 = \frac{1}{R}$$

Z = Z(g) is given in terms of a complicated integral which must still be computed to be able to understand how the partition function depends on the coupling.

The one-loop factor

$$H(x) \equiv \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n \mathrm{e}^{-\frac{x^2}{n}}$$

Related to Barnes G-function:

$$H(x) \equiv e^{-(1+\gamma)x^2} G(1+ix) G(1-ix)$$

The different multiplets contribute as follows:

Vector multiplet $$\prod_{i<j} H^2(a_i - a_j)$$

Adjoint hypermultiplet $$\prod_{i<j} \frac{1}{H(a_i - a_j - M) H(a_i - a_j + M)}$$

Fundamental hypermultiplet $$\prod_{i} \frac{1}{H(a_i + M)}$$

How can we find *Z(g)*?
The integrals are extremely complicated.
As usual, when something is complicated, we consider limits

I) Large N, R arbitrary ($\lambda = g^2 N$ fixed)
This implies two big simplifications that will allow us to determine Z exactly.

a) At $N \rightarrow$ Infinity the integral is exactly determined by a saddle-point.
b) Instantons do not contribute. $z_{inst} \rightarrow 1$, since

$$|q| = e^{-\frac{8\pi^2}{g^2}} = e^{-\frac{8\pi^2 N}{\lambda}} \xrightarrow[\lambda\ fixed]{N\rightarrow\infty} 0$$

II) Finite *N* (e.g. SU(2)) **but *R* → Infinity**

a) The integral is also exactly determined by a saddle-point, as long as a saddle-point exists.
b) Instanton contribution will be incorporated exactly using Seiberg-Witten curve.

Example - $\mathcal{N}=4$ Super Yang-Mills theory on S^4

- Instantons do not contribute. $Z_{inst} = 1$

- with our rules, 1-loop corrections cancel

Gaussian matrix model: $$Z = \int d^{N-1}a \prod_{i<j}(a_i - a_j)^2\, \mathrm{e}^{-\frac{8\pi^2}{g^2}\sum_i a_i^2}$$

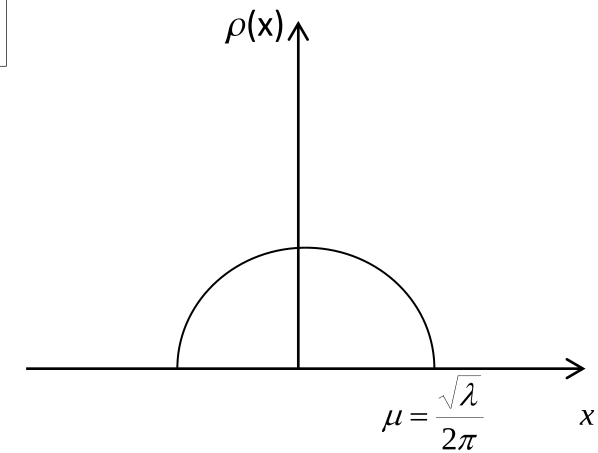
At large *N* the integral is dominated by a saddle-point.

$$S\,[a] = \frac{8\pi^2 N}{\lambda}\sum_{i=1}^{N} a_i^2 - \frac{1}{2}\sum_{i,j}\ln(a_i - a_j)^2 \quad \rightarrow \quad \sum_{j\neq i}\frac{1}{a_i - a_j} = \frac{8\pi^2 N}{\lambda}\, a_i$$

Introducing the eigenvalue density: $$\rho(x) = \frac{1}{N}\sum_{i=1}^{N}\delta(x - a_i)$$

the saddle-point equation becomes

$$-\!\!\!\!\!\!\int_{-\mu}^{\mu} dy\, \rho(y)\, \frac{1}{x-y} = \frac{8\pi^2}{\lambda}x \quad \Rightarrow \quad \rho(x) = \frac{8\pi}{\lambda}\sqrt{\frac{\lambda}{4\pi^2} - x^2}$$



Eigenvalues are distributed in a semicircle (Wigner's law)

$$W(C) = \int_{-\mu}^{\mu} dx\, \rho(x)\, \mathrm{e}^{2\pi x} = \frac{8\pi}{\lambda}\int_{-\mu}^{\mu} dx \sqrt{\frac{\lambda}{4\pi^2} - x^2}\, \mathrm{e}^{2\pi x} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

There is a smooth dependence with λ all the way from 0 to infinity.

Physical origin of phase transitions at R = infinity

They are generic for $\mathcal{N} = 2$ gauge theories.
Why? Look at spectrum

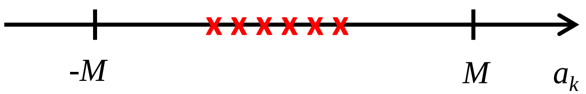
Mass spectrum in the background $\langle \Phi \rangle = diag(a_1, \ldots, a_N)$

$$\text{vector multiplet}: \quad m_{ij}^{\mathrm{v}} = |a_i - a_j|$$

$$\text{adjoint hyper}: \quad m_{ij}^{\mathrm{h}} = |a_i - a_j \pm M|$$

$$\text{(anti)fundamental hyper}: \quad m_i^{\mathrm{h}} = |a_i \pm M|$$

$\lambda << 1$



The a_i computed at the saddle-point are functions of the coupling.

Therefore every mass will be a function of the coupling.
At weak coupling $\lambda<<1$, eigenvalues are small. As λ increases, they grow and eventually some eigenvalue may hit the singularity where some of the hypermultiplet becomes massless.

This produces a discontinuity in the free energy F = - ln Z, which contains the term

$$S_{1-loop} = \sum_{i,j} m_{ij}^2 R^2 \ln m_{ij}^2 R^2 = \sum_k \quad |a_k \pm M|^2 \ln |a_k \pm M| + \ldots$$

Typically, the third derivative of the free energy is discontinuous.
The theory undergoes a phase transition.

$\mathcal{N} = 2$ SQCD with $2N_f$ massive hypermultiplets

J.R and K. Zarembo, arxiv:1309.1004

We assume $N_f < N$, in which case the theory is **asymptotically free**.
The partition function computed by localization is given by [Pestun, 0712.2824]

$$Z = \int d^{N-1}a \quad \frac{\prod_{i<j} (a_i - a_j)^2 H^2(a_i - a_j)}{\prod_i H(a_i + M)^{N_f} H(a_i - M)^{N_f}} \; \mathrm{e}^{2(N-N_f)\ln\Lambda \sum_i a_i^2} \quad \left| z_{inst}(a; g^2) \right|^2$$

Dynamically generated scale

$$\Lambda R = \mathrm{e}^{-\frac{4\pi^2}{\lambda(1-\zeta)}} \quad , \qquad \zeta \equiv \frac{N_f}{N}$$

Take large R (large sphere), $a_k R >> 1$

$$H(xR) \xrightarrow[R\to\infty]{} -\tfrac{1}{2}(xR)^2 \ln (xR)^2 + \ldots$$

The integral is determined by the saddle-point

$$Z = \int d^{N-1}a \;\; e^{-R^2 S(a_k)}$$

$$S = \sum_i \left(-2(N - N_f)(\ln\Lambda + \tfrac{3}{2}) a_i^2 - \tfrac{1}{2} N_f (a_i + M)^2 \ln(a_i + M)^2 - \tfrac{1}{2} N_f (a_i - M)^2 \ln(a_i - M)^2\right)$$
$$+ \tfrac{1}{2} \sum_{i,j} (a_i - a_j)^2 \ln(a_i - a_j)^2$$

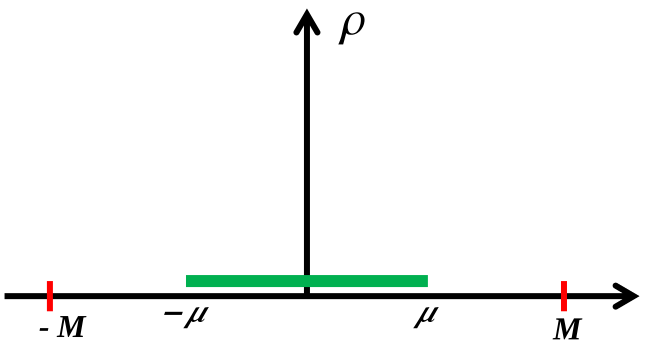
Introducing the eigenvalue density ρ as before. Differentiating the equation once, we find

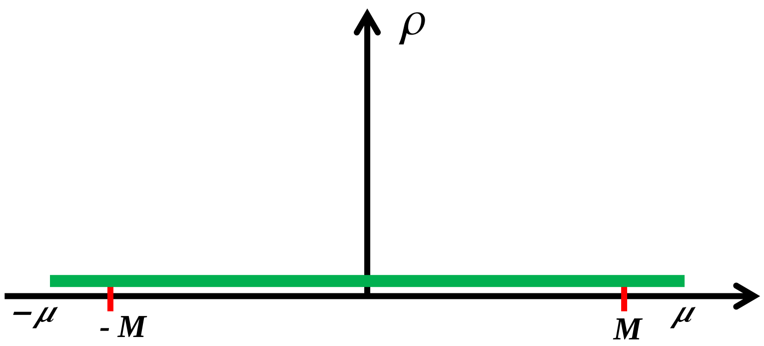
$$2\int_{-\mu}^{\mu} dy \;\; \rho(y) \ln\frac{(x-y)^2}{\Lambda^2} = \zeta \ln\frac{(x^2-M^2)^2}{\Lambda^4}$$

By differentiating again, we get

$$-\!\!\!\!\!\!\int_{-\mu}^{\mu} dy \;\; \frac{\rho(y)}{x-y} = \frac{\zeta}{x+M} + \frac{\zeta}{x-M}$$

Two cases:





The RHS has poles at x = M which may or may not lie within the eigenvalue distribution.
The solution to the integral equation is different in each case.

1. Weak coupling phase $\mu < M$. $(\Lambda < M/2)$

The poles sit outside the eigenvalue distribution.

$$\rho(x) = \frac{1-\zeta}{\pi\sqrt{\mu^2 - x^2}} + \zeta\, M\sqrt{M^2 - \mu^2}\, \frac{1}{\pi\sqrt{\mu^2 - x^2}}\, \frac{1}{M^2 - x^2}$$

The first saddle-point equation determines $\mu = \mu\,(\Lambda/M)$

The phase transition thus occurs when $\mu = M$, i.e. at $M_c = 2\,\Lambda$

2. Strong coupling phase $\mu > M$. $(\Lambda > M/2)$

The poles sit within the eigenvalue distribution.

$$\rho(x) = \frac{1-\zeta}{\pi\sqrt{\mu^2 - x^2}} + \frac{\zeta}{2}\delta(x+M) + \frac{\zeta}{2}\delta(x-M)$$

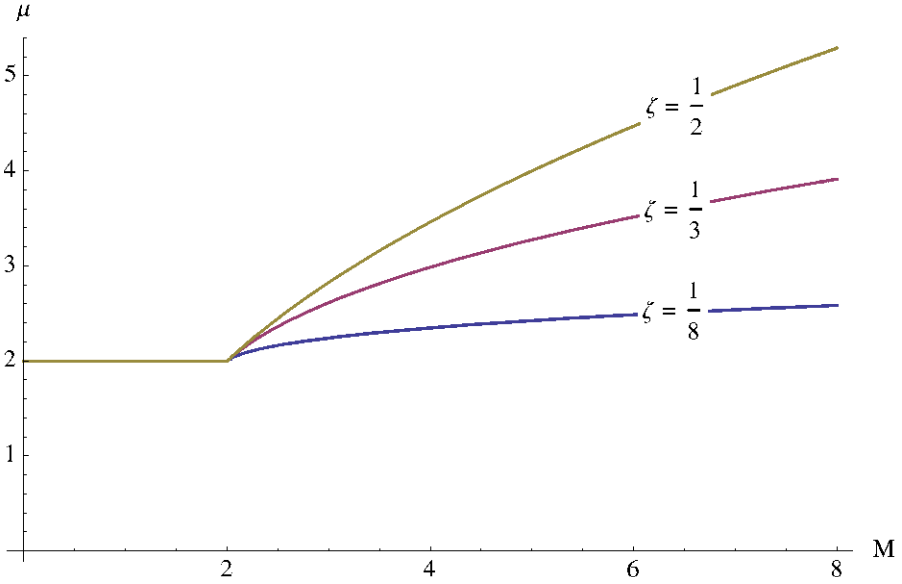
$$\mu = 2\Lambda$$

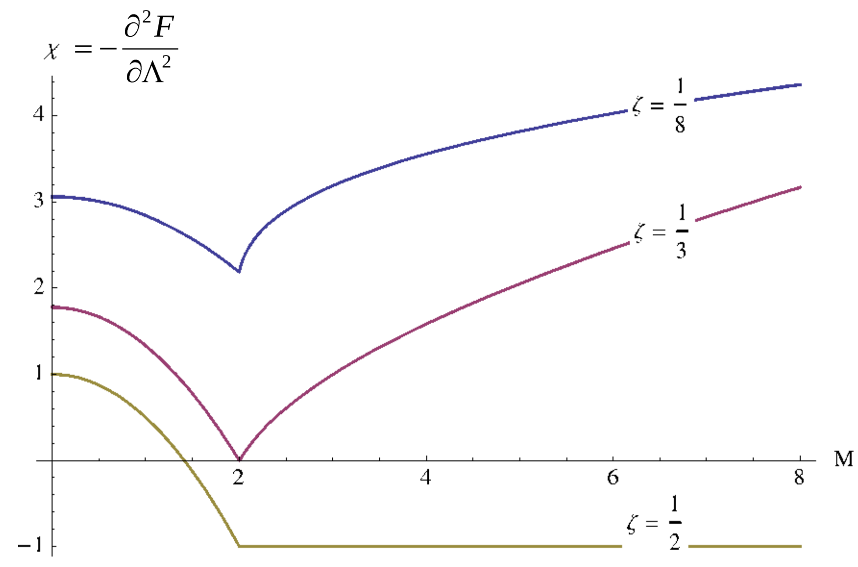
Free energy

Computing

$$\frac{\partial F}{\partial \log \Lambda} = -\left\langle x^2 \right\rangle = -\int_{-\mu}^{\mu} dx\, \rho(x)\, x^2$$

in each phase we find a discontinuity in the third derivative of *F*
Thus the transition is **third order**.





The width of the eigenvalue distribution μ and F’’ as functions of the quark mass M.

From Seiberg-Witten

Understanding how to reproduce these results from SW curve will allow us to study low rank groups including instanton effects

SW computes holomorphic prepotential $\mathcal{F}(a_k)$ in flat spacetime.

Write

$$Z = \int Da \quad |\mathcal{Z}|^2$$

$$Nekrasov: \quad 2\pi i \mathcal{F}(a) = \lim_{\varepsilon_{1,2}\to 0} \varepsilon_1 \varepsilon_2 \ln \mathcal{Z}$$

or

$$2\pi i \mathcal{F}(a) = \lim_{R\to\infty} \frac{1}{R^2} \ln \mathcal{Z}$$

This implies that

$$\boxed{\ln Z = -F = R^2 (2\pi i \mathcal{F}(a_k) - 2\pi i \bar{\mathcal{F}}(a_k))}$$

The identity includes coupling of the scalar to the curvature! –proportional to ***R²*** (provides classical contribution)

Large N saddle-point equations

$$\frac{\partial S}{\partial a_k} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{F}}{\partial a_k} = 0$$

But $\quad \dfrac{\partial \mathcal{F}}{\partial a_k} \equiv a_{Dk}$

Thus large N saddle occurs at a particular degenerating point of the SW curve where all the periods a_{Dk} vanish

What does the condition $a_{Dk} = 0$ mean? (more generally, massless dyon singularity)

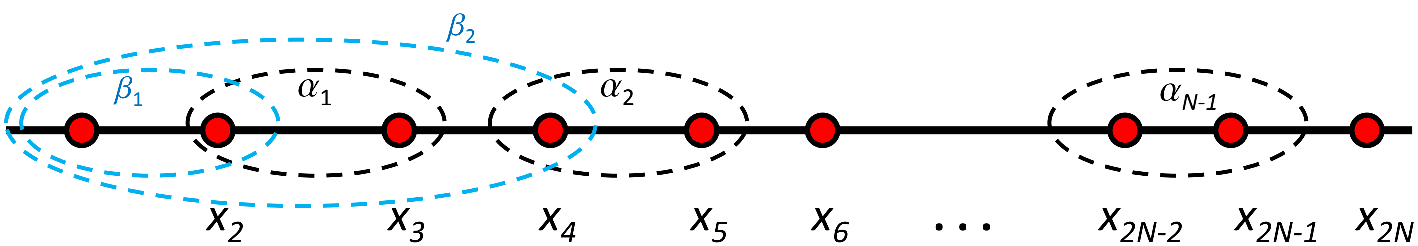
Consider a SW curve

$$y^2 = p(x) \quad , \qquad p(x) = x^{2N} + \ldots$$

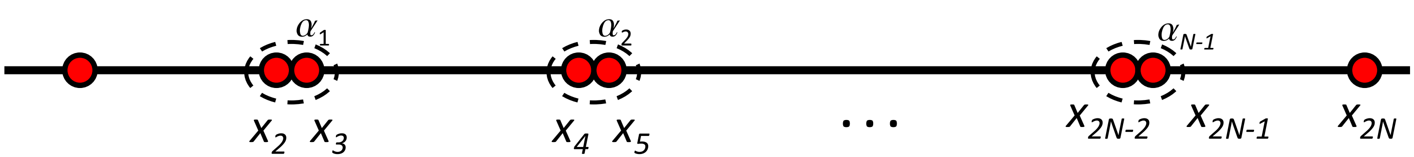
$2N$ branch points $x_1, \cdots, x_{2N}$

p(x) depends on
- masses
- coupling
- moduli parameters $\{u_k\}$, $k = 1, \ldots, N-1$

Define homology cycles α_n and β_n



$$a_k = \oint_{\beta_k} \lambda \quad , \qquad a_{Dk} = \oint_{\alpha_k} \lambda \qquad , \qquad \lambda = \text{SW meromorphic form}$$



N-1 conditions for N-1 unknowns $\{u_k\}$
Substituting the solution for $\{u_k\}$ into the prepotential, we find the free energy $F(M, \lambda)$ at large N

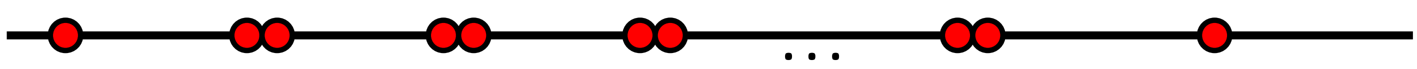
Example: SQCD

Consider the Seiberg-Witten curve that describes $\mathcal{N}$ = 2 SU(N) gauge theory coupled to two fundamental hypermultiplets of mass M

$$y^2 = C(x)^2 - G(x) \equiv p(x)$$

$$C(x) = \prod_{i=1}^{N}(x - u_i) \quad , \quad \sum_{i=1}^{N} u_i = 0 \quad , \quad G(x) = \Lambda^{2N-2N_f}(x+M)^{N_f}(x-M)^{N_f}$$

We are interested in the degenerating limit



We must demand that N-1 roots of $p(x)$ are double roots, i.e. we must find the u_i for which $p(x)$ takes the form

$$p(x) = (x-a)(x-b)\prod_{i=1}^{N-1}(x-c_i)^2$$

The general condition is that $p'(x)$ shares the same roots c_i as $p(x)$

$$p'(x) = 2\prod_{i=1}^{N}(x-u_i)^2 \sum_{i=1}^{N}\frac{1}{x-u_i} - N_f \Lambda^{2N-2N_f}(x^2-M^2)^{N_f}\left(\frac{1}{x+M} + \frac{1}{x-M}\right)$$

Using $p(x) = 0$

$$p'(x) = \Lambda^{2N-2N_f}(x^2-M^2)^{N_f}\left(\sum_{i=1}^{N}\frac{2}{x-u_i} - \frac{N_f}{x+M} - \frac{N_f}{x-M}\right)$$

In the continuum, large *N* limit, the equation is transformed into an integral equation

$$2\int_{-\mu}^{\mu} dy \ \frac{\rho(y)}{x-y} = \frac{\zeta}{x+M} + \frac{\zeta}{x-M} \qquad , \qquad \zeta \equiv \frac{N_f}{N}$$

reproducing exactly the same integral equation that we found from localization.

The parameter μ is determined by demanding that the roots also solve *p(x)* = 0

$$\prod_{i=1}^{N} (x-u_i)^2 = \Lambda^{2N-2N_f} (x^2 - M^2)^{N_f}$$

Taking the logarithm and going to the continuum, we find

$$2\int_{-\mu}^{\mu} dy \ \rho(y) \ln \frac{(x-y)^2}{\Lambda^2} = \zeta \ln \frac{(x^2 - M^2)^2}{\Lambda^4}$$

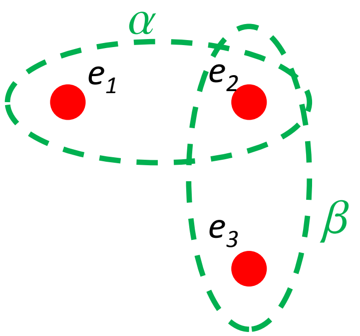
which reproduces the first derivative of the saddle-point equation in the localization partition function

Finite N

Example: $\mathcal{N}$= 2 SU(2) SYM with two flavors

$$y^2 = (x^2 - \tfrac{1}{64}\Lambda^4)(x-u) + \tfrac{1}{4}M^2\Lambda^2 x - \tfrac{1}{32}M^2\Lambda^4$$

$$\lambda = -\frac{\sqrt{2}}{4\pi}\frac{y\,dx}{x^2 - \frac{1}{64}\Lambda^4}$$



The branch points are at the three roots e_1, e_2, e_3 of the cubic polynomial.
•The cycle α defining a_D *surrounds* e_1 , e_2 .
•The cycle β defining a *surrounds* e_2 , e_3 .

Our aim is to compute the prepotential at one of the singularities of the curve where $a_D = 0$.
The singularities are located at

$$u_1 = -M\Lambda - \frac{\Lambda^2}{8} \quad , \quad u_2 = M\Lambda - \frac{\Lambda^2}{8} \quad , \quad u_3 = M^2 + \frac{\Lambda^2}{8}$$

At $u \to u_3$, and $M < \Lambda/2$, the cycle α defining a_D shrinks to zero size, $e_1 \to e_2$ and $a_D \to 0$.

If, instead, $M > \Lambda/2$, then $e_2 \to e_3$ the cycle α does not shrink. In this case a_D is different from 0 in the whole complex u-plane.

At $M = \Lambda/2$, we have all e_1 , e_2 , e_3 branch points collapse. At this point $a \to M$ and the hypermultiplet becomes massless.

It is an *Argyres-Douglas* point, first found in [Argyres, Plesser, Seiberg Witten].
Thus this point represents the critical point of our phase transitions.

Seiberg-Witten

Computes the exact holomorphic prepotential $\mathcal{F}(a)$ as a function of a_k labelling the Coulomb vacua:

$$a_k = \oint_{\beta_k} \lambda \quad , \qquad a_{Dk} = \oint_{\alpha_k} \lambda$$

$$\frac{\partial \mathcal{F}}{\partial a_k} = a_{Dk}(a) \quad , \qquad k = 1, \dots, N$$

where λ is a certain meromorphic one form in an auxiliary genus N-1 Riemann surface

$\mathcal{F}(a)$ determines the low energy effective action

$$L = \tfrac{1}{4\pi} \mathrm{Im}\left(\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A_k} A_k + \int d^4\theta \, \tfrac{1}{2} \frac{\partial^2 \mathcal{F}(A)}{\partial A_k \partial A_j} W_k^\alpha W_{j\alpha} \right)$$

Localization

Computes $Z(g)$ and VEV of ½ BPS Wilson loops <W> for SU(N) SYM compactified on a four-sphere

$$Z(g) = \int d^{N-1}a \prod_{i<j} (a_i - a_j)^2 \, e^{-\frac{8\pi^2}{g^2}\sum_i a_i^2} \, z_{1-loop}(a) \quad |z_{inst}(a)|^2$$

These are very different observables.

- In localization, we integrate over a_k.
- In SW, the final result (prepotential) depends on a_k.

Both quantities contain exact information on the coupling.

Is there any way to connect these results?

Idea: take decompactification limit $\boldsymbol{R}$ = infinity
An integral with a large parameter (if we are lucky enough) may be dominated by a saddle-point $\{a_1^* \dots a_k^*\}$

CLAIM: *In the decompactification limit, the free energy F = - log Z is exactly given by the prepotential at its minimum:*

$$F(g) = -R^2 \, \mathrm{Re}(4\pi i \mathcal{F}) \qquad , \qquad \frac{\partial \mathcal{F}}{\partial a_k} = a_{Dk} = 0$$

It represents a particular vacuum where the N-1 alpha cycles are collapsed to zero size.

Operator product expansion

Use localization to compute non-perturbative physics.
Example: all-order OPE.

Consider dynamically generated scale Λ_{eff} << M. Then observables admit an expansion

$$O = (\Lambda_{eff})^{\Delta} \sum_{n=0}^{\infty} C_n \left(\frac{\Lambda_{eff}}{M} \right)^{2n}$$

-The mass M in the denominator arises from expanding the effective action in local operators.
-Powers of Λ_{eff} in the numerator come from the VEV of the local operators generated by the OPE.

These VEV involve non-perturbative physics and are difficult to calculate.

Having the exact formula for the free energy of SQCD, we can now compute the OPE.
For M >> Λ , we can integrate out the hypermultiplet fields.
What remains is pure gauge N = 2 SYM with dynamical scale

$$\Lambda_{eff} = M^{\zeta} \Lambda^{1-\zeta}$$

We find

$$F = \Lambda_{eff}^2 \left(-2 + \zeta \frac{\Lambda_{eff}^2}{M^2} + \tfrac{2}{3} \zeta (1-2\zeta) \frac{\Lambda_{eff}^4}{M^4} + \tfrac{4}{3} \zeta (1-2\zeta)(5-8\zeta) \frac{\Lambda_{eff}^6}{M^6} + \dots \right)$$

Lessons

1. Massive $\mathcal{N} = 2$ supersymmetric gauge theories exhibit quantum phase transitions at critical couplings. Transitions occur when extra massless states contribute to the free energy.

2. The complete free energy (both at finite and large N) is exactly given in terms of the prepotential evaluated at a singularity of the Seiberg-Witten curve

$$F(g) = -R^2 \,\mathrm{Re}(4\pi i \mathcal{F}) \qquad , \quad \frac{\partial \mathcal{F}}{\partial a_k} = a_{Dk} = 0$$

where all α_i cycles shrink to zero size.

3. At the critical point of the phase transitions a pair of conjugate homology cycles shrink simultaneously. These are Argyres-Douglas points of the curve, where mutually non-local states become massless.

4. Recent important results:

- [Karch, Robinson, Uhlemann, arXiv:1509.00013] : Add $N_f << N$ massive fundamental matter to $\mathcal{N} = 4$ theory. The theory contains the expected phase transitions. The free energy can be computed either by:
a) exact localization formulas o
b) or in terms of the holographic dual, probe D7 branes in AdS_5xS^5, leading to a striking match.

- [Hollowood, Kumar. arxiv:1509.00716] Seiberg-Witten and saddle-points analysis for $\mathcal{N} = 2^*$ theory *at finite* N. They compute finite N partition function for λ<35.45