Holography and Cartan Geometry in Topological Superconductors

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Physics on the Riviera 2015
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September 15, 2015

What is a topological phase of matter?

- Characterization: topological quantum numbers (Z, Z_2, etc.)
- New paradigm: states of matter not described by Landau's spontaneous-symmetry-breaking theory
- Continuous systems: quantum Hall states, quantum liquids
- Lattice systems: topological insulators, topological superconductors, Weyl semimetals, etc.
- 2+1-dimensional non-Abelian phases support anyons: Ising anyons, Fibonacci anyons, Majorana fermions, Z_n parafermions, etc.
- In 3+1 dimensions, there appear string-like excitations: fractional statistics of loops and strings
- Most of the free-fermion topological phases are classified in terms of Dirac Hamiltonians
- In the low-energy regime, the topological behavior is described by suitable topological field theories
- Bulk-edge correspondence

Schwarz–type TQFTs: Chern-Simons and BF theories

- No local propagating degrees of freedom
- The coupling constants do not run along the RG flow
- Physical Observables: expectation values of Wilson loops and Wilson surfaces
- Quantum excitations: Anyons in (2+1)-D and loops/strings in (3+1)-D

Example: U(1) Chern-Simons Theory in the Abelian FQHE

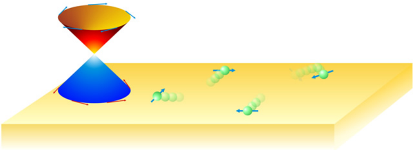
$$S = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

Bulk-edge correspondences: $\mathrm{TFT}_{2+1}/\mathrm{CFT}_{1+1}$, $\mathrm{TFT}_{3+1}/\mathrm{TFT}_{2+1}$

Topological Superconductors (and Insulators)

Symmetry				Spatial Dimension d								
Class	T	C	S	1	2	3	4	5	6	7	8	$\cdots$
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

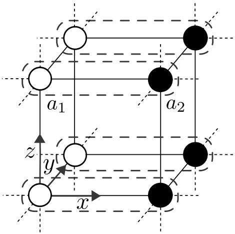
Protected gapless surface states - Majorana cones for TS: $\gamma^\dagger = \gamma$



3D Lattice Model

We consider two species of fermion a_k with $k = 1, 2$ on a periodic cubic lattice. The unit cell, positioned at $j = (j_x, j_y, j_z)$, consists of two sites lying along the x-axis and the interactions are taken to be at most between next-to-nearest neighbours:

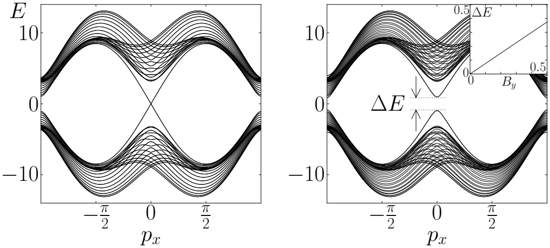
$$H = \sum_j \left(\sum_k \mu a^\dagger_{k,j} a_{k,j} + \sum_{k,k',s} t_{kk's} a^\dagger_{k,j} a_{k'j+s} + \sum_{k,k',s} \Delta_{kk's} a_{k,j} a_{k',j+s} \right) + h.c.$$



Effective Bogoliubov-de Gennes Hamiltonian

$$H = \sum_p \psi_p^\dagger h(p) \psi_p, \qquad h(p) = \begin{pmatrix} \epsilon(p)\mathbb{I} & \Theta(p) \\ \Theta(p)^\dagger & -\epsilon(p)\mathbb{I} \end{pmatrix},$$

with $\epsilon(p)$ and $\Theta(p) = i(\mathbf{d}(p)\cdot\boldsymbol{\sigma})\sigma_y$ tunneling and pairing functions respectively.



Close to ground state, H can be written as a Dirac Hamiltonian!

3D Winding Number and Symmetries

We can recast the Hamiltonian in a off-diagonal form

$$h(p) = \begin{pmatrix} 0 & q(p) \\ q^\dagger(p) & 0 \end{pmatrix},$$

such that the 3D winding number is defined as follows

$$\nu_{3D} = \int_{BZ} \frac{d^3 p}{24\pi^2} \epsilon^{\mu\nu\lambda} tr \left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\nu q) \right]$$

If we perturbe the system without breaking TR and PH symmetries then the winding number does not change:

topology comes from the momentum space!

Importantly, the 3D winding number is not associated to any physical observable, i.e. there are no quantized transport properties in the 3D bulk.

A New Bulk-Boundary Correspondence

On the boundary, in the z-direction, there appear massless Majorana particles that become massive by introducing a Zeeman field: mass is an effective quantity in condensed matter!

The effective Hamiltonian that describes the low energy limit of these 2D superconducting states breaks TR symmetry, due to the presence of non-zero B_y, so it behaves as a TSC in the class D (p-wave superconductor).
We can define a 2D winding number ν_{2D} which characterizes the topological phase of the boundary as a whole, finding

$$\nu_{\mathrm{3D}} = \nu_{\mathrm{2D}}$$

P. Finch, J. de Lisle, G. P. and J. K. Pachos, arXiv:1408.1038, Phys. Rev. Lett. 114, 016801 (2015)

Thermal Effects and Geometry

1) The quantized thermal effects are the only physical observables related to the topological phases in topological superconductors. The thermal currents are related to the Majorana zero modes.
2) In the Luttinger theory (1969) the de-equilibrating effect of a small temperature gradient can be precisely compensated for by a fictitious gravitational potential ϕ provided that

$$\frac{1}{T}\frac{\partial T}{\partial x} = -\frac{1}{c^2}\frac{\partial \phi}{\partial x}$$

3) In 1+1-D, there exists a thermal analog of quantum Hall effect defined by a quantized thermal Hall conductance κ_{th}

$$\kappa_{\text{th}} = \frac{c}{6}\pi T, \qquad T \to 0$$

This effect is related to the gravitational anomaly of the CFT (Cappelli et al., 2002).

Effective Dirac Theory in Cartan Geometry

Dirac Hamiltonian for 3+1-D TSC in class DIII

$$H = \int d^3x \ \psi^\dagger (\alpha^j p_j + i\,\beta\gamma^5 m)\psi,$$

$$S^{3D}[\psi, \overline{\psi}] = \int d^4x \ \overline{\psi}\,(i\,\Gamma^\mu \partial_\mu - m)\psi,$$

$\overline{\psi} = \psi^\dagger \Gamma^0$, $\Gamma^0 = i\beta\gamma^5$ and $\Gamma^j = \Gamma^0 \alpha^j$, (Westman, 2013).

$\psi \to e^{\frac{i}{2}\theta_{AB} J^{AB}}\psi, \quad \psi^\dagger \to \psi^\dagger e^{-\frac{i}{2}\theta_{AB} J^{AB}}, \{A, B\} = \{0, 1, 2, 3, 4\}$

$$\left[J^{AB}, J^{CD}\right] = -i\left(\eta^{AC} J^{BD} - \eta^{AD} J^{BC} - \eta^{BC} J^{AD} + \eta^{BD} J^{AC}\right),$$

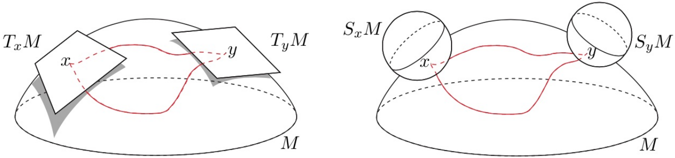
The mass term is invariant under SO(3,2) global transformations!

Effective Topological Field Theory

Dirac-Cartan action in an effective curved background with a AdS tangent space

$$S^{3D}[\psi, \overline{\psi}, A_\mu] = \int d^4x \, |e| \, \overline{\psi} \, (i \, \widehat{\Gamma}^\mu D_\mu - m)\psi,$$

with $D_\mu = \partial_\mu + A_\mu$, the Cartan connection A_μ takes values in the AdS algebra, while $\widehat{\Gamma}^\mu = e^\mu_A \Gamma^A$ with $\Gamma^4 = \gamma^5$.



(Wise, 2009)

Importantly

$$A_\mu = \omega_\mu + \frac{1}{l}\, e_\mu = \frac{i}{4}[\gamma_a, \gamma_b]\, \omega_\mu^{ab} + \frac{i}{2\, l}\, \gamma_a\, e_\mu^a,$$

$$F_{\mu\nu} = R_{\mu\nu} - \frac{1}{4\, l^2}(e_\mu^a e_\nu^b - e_\nu^a e_\mu^b)[\gamma_a, \gamma_b] + \frac{1}{l}\, T_{\mu\nu},$$

Integrating out the spinor field, the topological part of the corresponding effective action is given by

$$S_{\rm top}^{\rm 3D}[A_\mu] = \frac{\nu_{\rm 3D}}{192\pi} \int d^4x\, \epsilon^{\mu\nu\alpha\beta}\, {\rm tr}\, F_{\mu\nu} F_{\alpha\beta} =$$

$$= \frac{\nu_{\rm 3D}}{192\pi} \int d^4x\, \epsilon^{\mu\nu\alpha\beta}\, {\rm tr} \left[R_{\mu\nu} R_{\alpha\beta} + \frac{1}{l^2} \Big(T_{\mu\nu} T_{\alpha\beta} - R_{\mu\nu} e_\alpha e_\beta \Big) \right].$$

Pontrjagin and Nieh-Yan invariants respectively, here l is related to the AdS radius.

2+1-D Exotic Gravity on the Gapped Boundary

By using the Stokes theorem, we derive the following topological action on each gapped boundary

$$S^{\rm 2D}_{\rm top}[\omega_\mu, e_\mu] = \frac{\nu_{\rm 3D}}{192\pi}\int d^3x\, \epsilon^{\mu\nu\lambda}{\rm tr}\left(\omega_\mu\partial_\nu\omega_\lambda + \frac{2}{3}\,\omega_\mu\omega_\nu\omega_\lambda + \frac{1}{l^2}\,T_{\mu\nu}e_\lambda\right)$$
$$= \frac{\nu_{\rm 3D}}{384\pi}\left({\rm CS}[A^+_\mu] + {\rm CS}[A^-_\mu]\right),$$

where $A^\pm_\mu = \omega_\mu \pm \frac{1}{l}\, e_\mu$ $(\mu = 0, 1, 2)$ take values in $so(2,1)$ such that $so(2,1)\times so(2,1) \simeq so(2,2)$.
Exotic gravity (Witten, 1988) with a negative cosmological constant $\Lambda = -\frac{1}{l^2}$, breaks time-reversal symmetry and parity.

This is compatible with the existence of a p-wave topological superconductor on the gapped boundary of the 3+1-D TSC in class DIII. We can now study the defect lines on the boundary: they should trap Majorana zero modes!

AdS_{2+1}/CFT_{1+1}

There exist asymptotic conditions that are invariant under the action of $SO(2,2)$ and the asymptotic symmetries have well defined canonical operators. Thanks to these conditions, it is possible to identify two independent Virasoro algebras with central charge c_l and c_r, while the chiral central charge is $c = c_l - c_r$. Brown and Henneaux (1986) found that $c_l = c_r = \frac{3l}{2G}$ in Einstein theory, while in the exotic case (Klemm and Tagliabue, 2008)

$$c_l = -c_r = 24\pi \times \frac{\nu_{3D}}{192\pi}$$

no conformal anomaly!. The THE is related to the Lorentz anomaly. The corresponding total chiral central charge related to the defect lines on the boundary is given by

$$c_{tot} = \frac{1}{2}\,\nu_{3D} = \frac{1}{2}\,\nu_{2D}$$

Majorana fermions!

Conclusions

In several topological systems, relativistic quantum field theories emerge in the low-energy regime: Majorana and Dirac fermions, anyons and gauge bosons are the quasi-particles that appear close to the ground state.
In TSC, fermions can be coupled to an effective Cartan connection in order to take into account thermal effects.

Main results of this work:

- We have derived an effective topological theory describing the 2+1-D exotic gravity with negative cosmological constant on the boundary of a 3+1-D TSC. This theory supports also exotic BTZ black hole solutions.
- Thanks to the AdS_{2+1}/CFT_{1+1} we have derived a total chiral central charge which is agreement with the existence of Majorana fermions trapped by defect lines on the gapped boundary of TSC.

G. P. and J. K. Pachos, arXiv:1507.03236.