Aspects of entanglement of/between disjoint regions in CFT & Holography

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SISSA

2D CFT	**P. Calabrese, J. Cardy and E.T.**	**[1206.3092]**	**PRL**
		[1210.5359]	**JSTAT**
		[1408.3043]	**JPA**
	C. De Nobili, A. Coser and E.T.	**[1501.04311]**	**JSTAT**
	A. Coser, E.T. and P. Calabrese	**[1503.09114]**	**JSTAT**
		[1508.00811]	
$\mathbf{AdS_4/CFT_3}$	**P. Fonda, L. Giomi, A. Salvio, E.T.**	**[1411.3608]**	**JHEP**
	P. Fonda, D. Seminara, E.T.	**[to appear]**	

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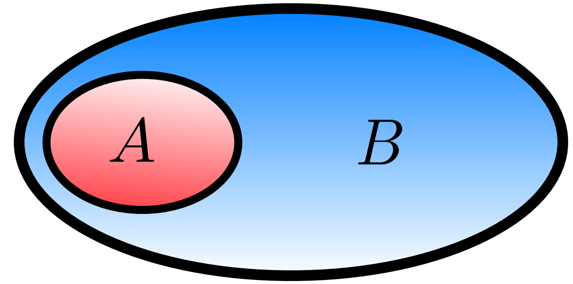
Outline

- Introduction & some motivations
- Holographic entanglement entropy (HEE) in AdS_4/CFT_3
  - Arbitrary shapes domains with smooth boundaries
- Entanglement in 2D CFT:
  - Entanglement negativity for adjacent and disjoint intervals
  - Partial transpose in the XY spin chain
  - Partial transpose of the free fermion
  - Entanglement negativity at finite temperature
- Conclusions & open issues

Mutual Information & Entanglement Negativity

- Ground state $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



Reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$

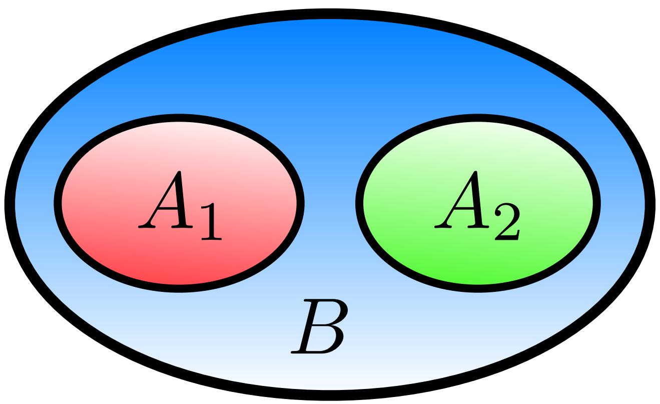
Rényi entropies

Entanglement entropy $S_A \equiv -\mathrm{Tr}(\rho_A \log \rho_A) = \lim_{n\to 1} \frac{\log(\mathrm{Tr}\rho_A^n)}{1-n} = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\,\rho_A^n$

- $S_A = S_B$ for pure states

- *Tripartite* system $\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B$

$\rho_{A_1 \cup A_2}$ is mixed



Entanglement between A_1 and A_2?

- $S_{A_1 \cup A_2}$: entanglement between $A_1 \cup A_2$ and B
  The mutual information $S_{A_1} + S_{A_2} - S_{A_1 \cup A_2}$ gives an upper bound
- A *computable* measure of the entanglement is the *logarithmic negativity*

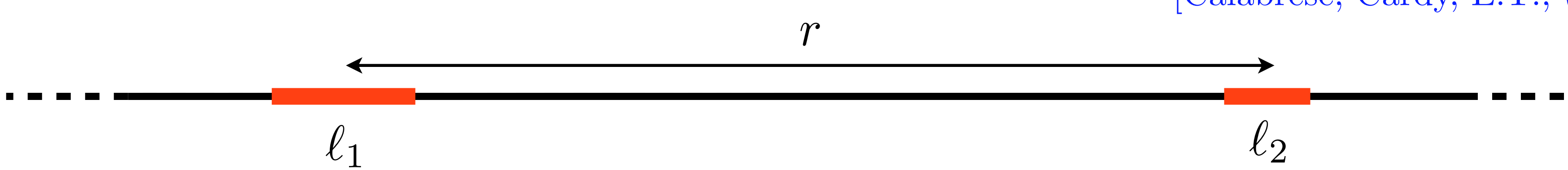
Why disjoint intervals?

- One interval on the infinite line at $T = 0$
  [Holzhey, Larsen, Wilczek, (1994)]
  [Calabrese, Cardy, (2004)]

$$S_A = \frac{c}{3} \log \frac{\ell}{a} + \text{const}$$

- Two intervals A_1 and A_2: $\text{Tr}\rho^n_{A_1 \cup A_2}$ for small intervals w.r.t. to other characteristc lengths of the system

[Headrick, (2010)]
[Calabrese, Cardy, E.T., (2011)]



$$\text{Tr}\rho^n_A = c_n^2 (\ell_1 \ell_2)^{-c/6(n-1/n)} \sum_{\{k_j\}} \left(\frac{\ell_1 \ell_2}{n^2 r^2} \right)^{\sum_j (\Delta_j + \bar{\Delta}_j)} \langle \prod_{j=1}^{n} \phi_{k_j} \left(e^{2\pi i j/n} \right) \rangle^2_{\mathbf{C}}$$

$\text{Tr}\rho^n_A$ for disjoint intervals contains *all* the data of the CFT
(conformal dimensions and OPE coefficients)

- Generalization to higher dimensions [Cardy, (2013)]

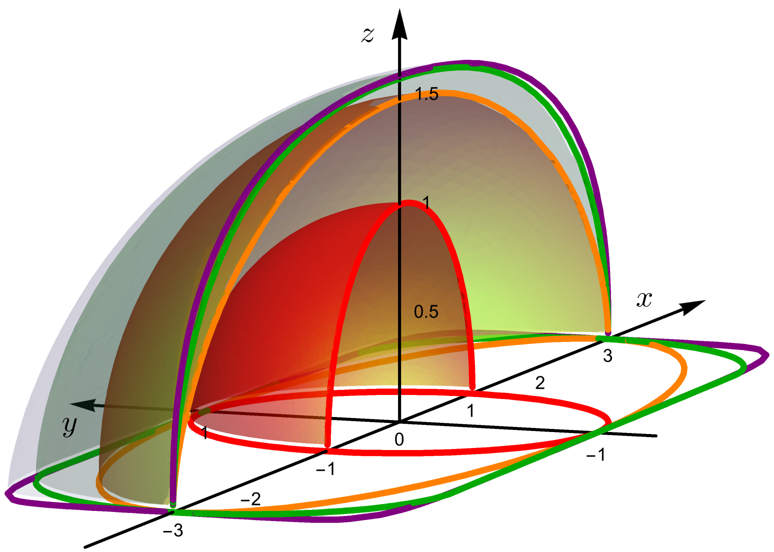
Holographic entanglement entropy in AdS(4)

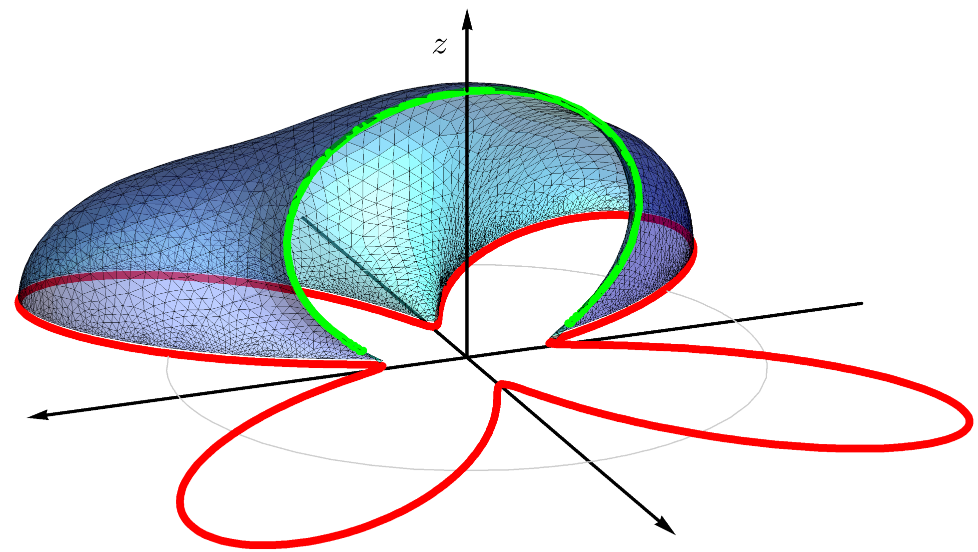
- Constant time slice in AdS_{d+1}
  Surfaces γ_A s.t. $\partial\gamma_A = \partial A$
  Find the *minimal area* surface $\hat{\gamma}_A$

$$S_A = \frac{\text{Area}(\hat{\gamma}_\varepsilon)}{4G_N^{(d+1)}}$$

[Ryu, Takayanagi, (2006)]

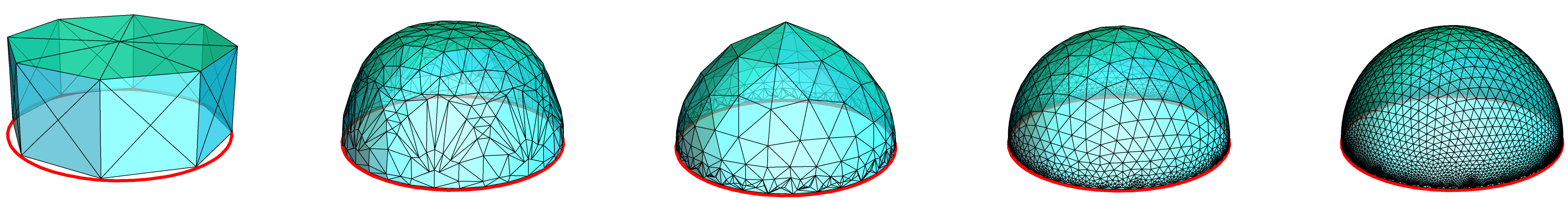
- Holographic dual of Wilson loops [Maldacena, (1998)] [Rey, Yee, (1998)]





- For arbitrary shapes of ∂A and AdS_4 we employ a numerical method based on *Surface Evolver* (by Ken Brakke)

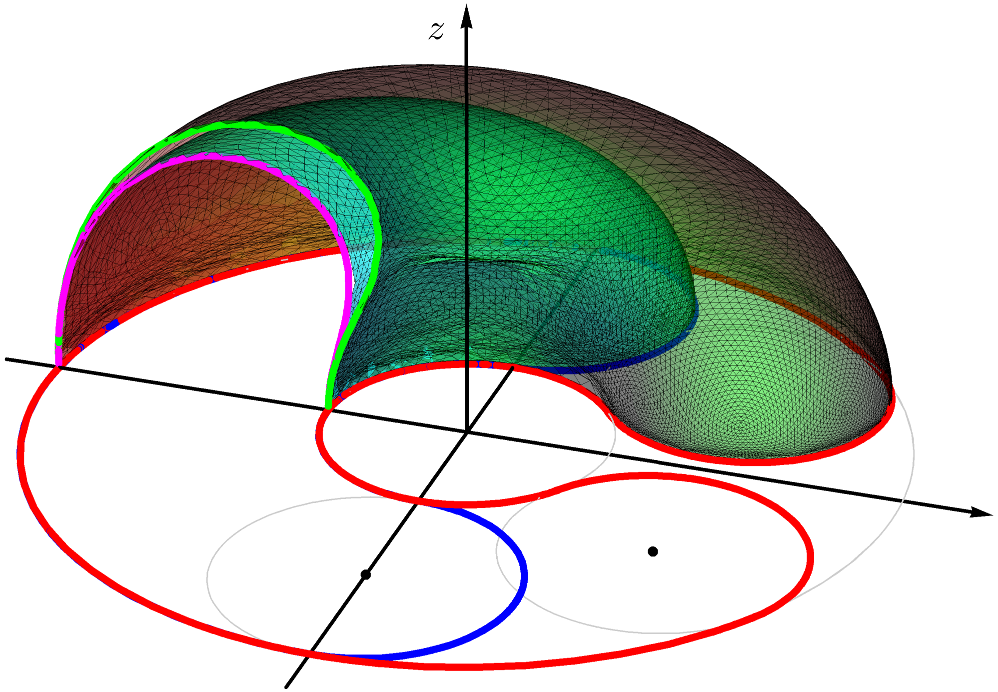
[Fonda, Giomi, Salvio, E.T., (2014)]

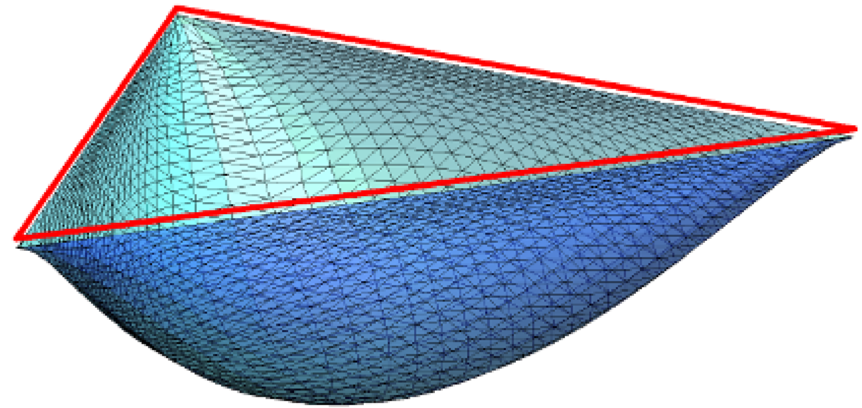


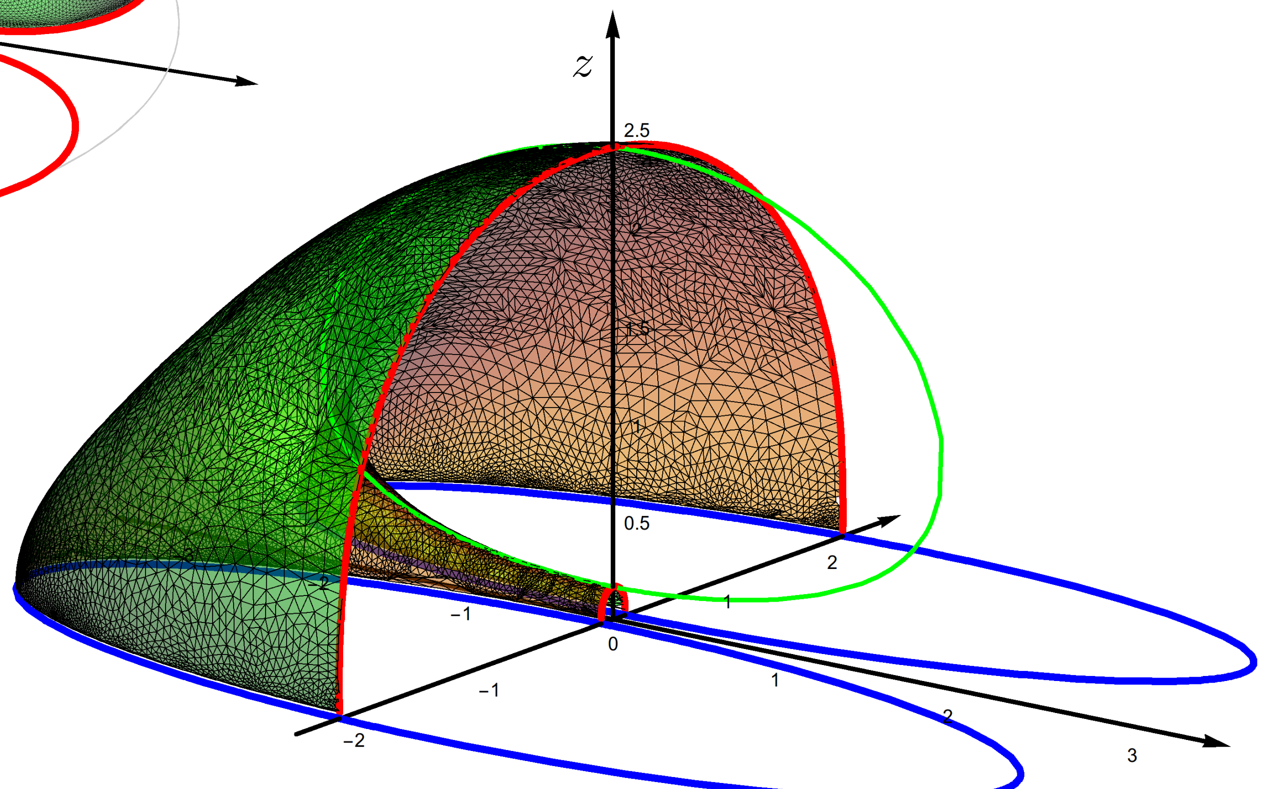
Minimal area surfaces in AdS(4)

[Fonda, Giomi, Salvio, E.T., (2014)]

- Pathwise connected domains A
  (also with non smooth ∂A)



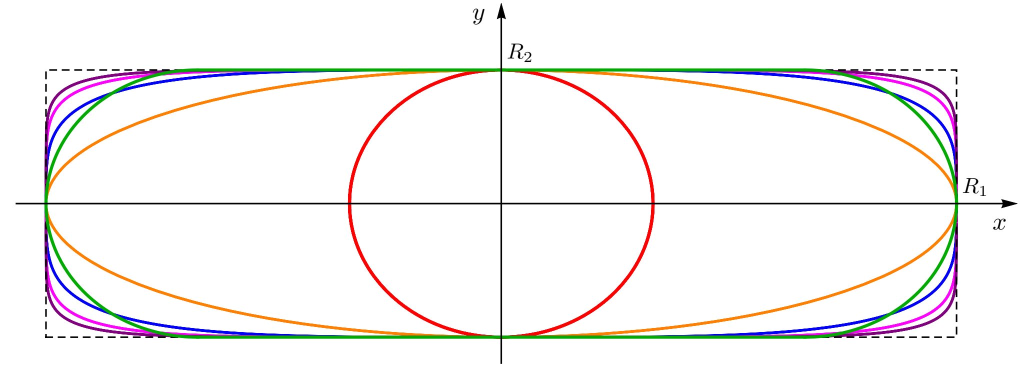


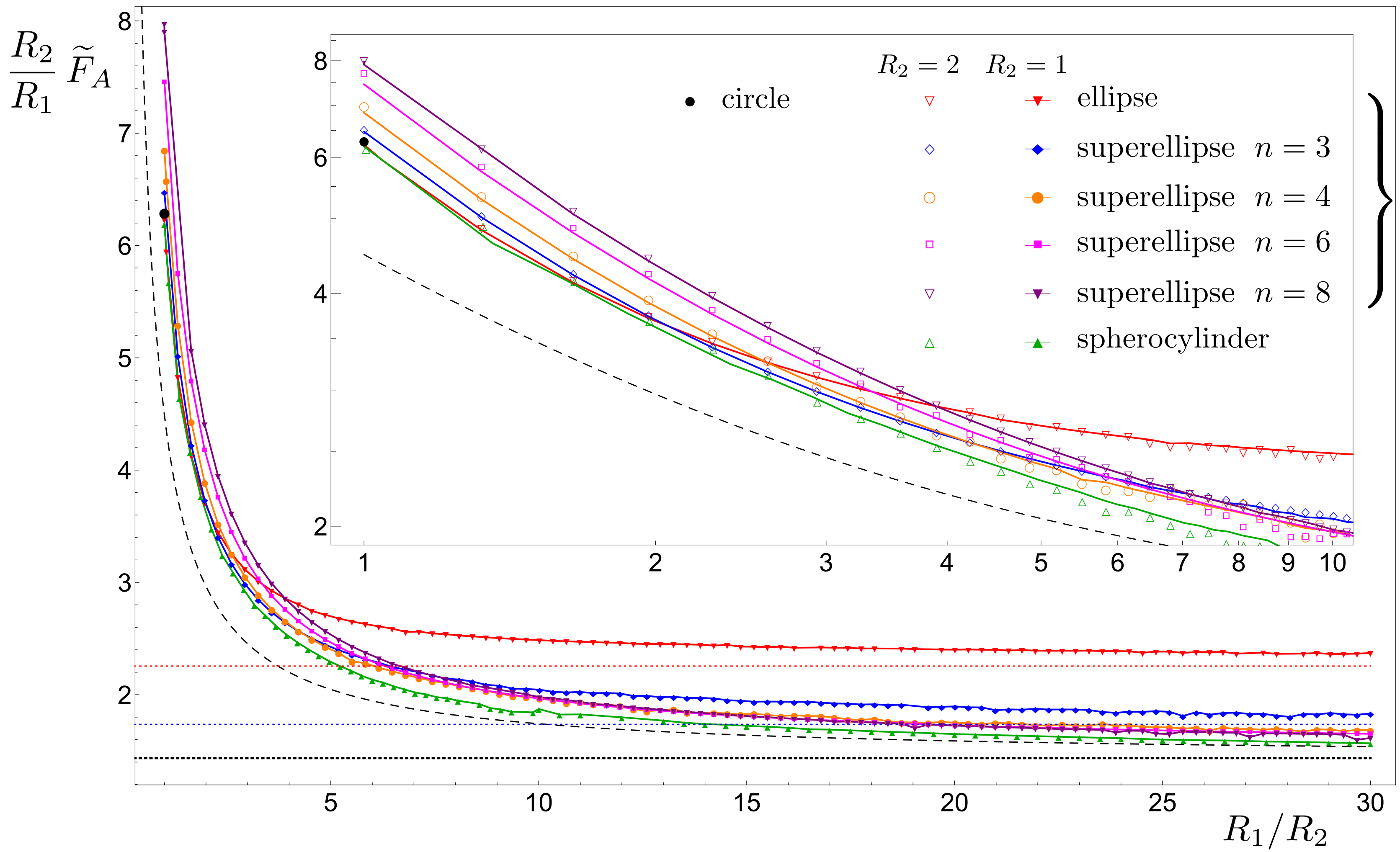


- Disjoint regions
  $(A = A_1 \cup A_2)$

HEE in AdS(4). From the disk to the infinite strip

$$\mathcal{A}_A = \frac{P_A}{\varepsilon} - F_A + o(1) \equiv \frac{P_A}{\varepsilon} - \widetilde{F}_A$$

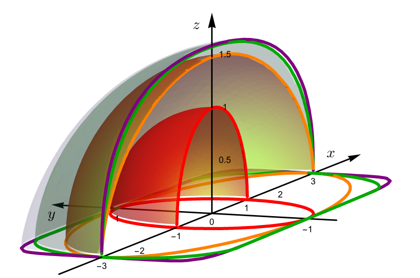




Superellipses:

$$\frac{|x|^n}{R_1^n} + \frac{|y|^n}{R_2^n} = 1$$

squircles: $R_1 = R_2$



HEE in AdS(4) & Willmore energy

- Willmore energy of a closed smooth surface $\Sigma_g \subset \mathbb{R}^3$ (extrinsic curvature $\widetilde{K}_{\mu\nu}$) [Willmore, (1965)]

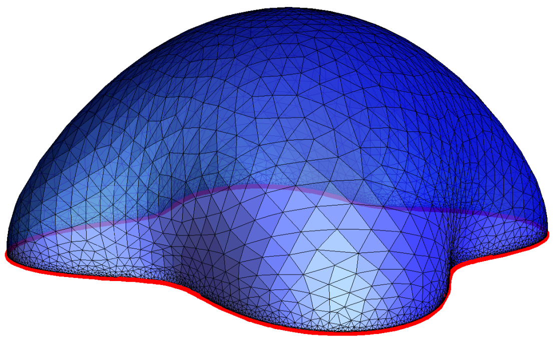
$$\mathcal{W}[\Sigma_g] \equiv \frac{1}{4}\int_{\Sigma_g} \left(\mathrm{Tr}\widetilde{K}\right)^2 d\tilde{A}$$

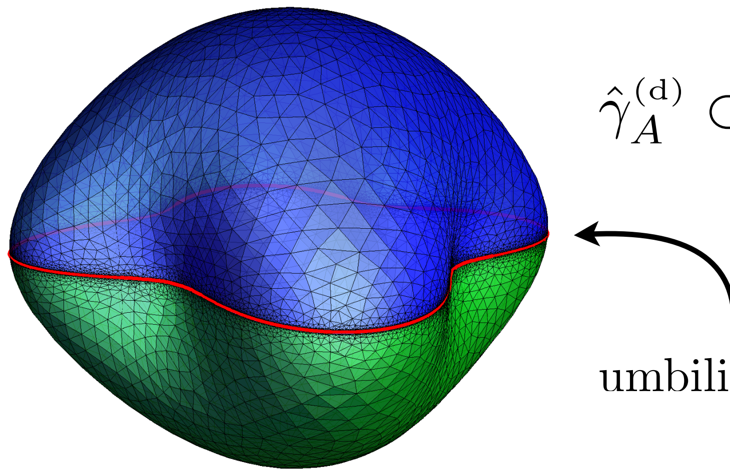
- Minimal area surface $\hat{\gamma}_A \subset \mathbb{H}^3$ has $\mathrm{Tr}K = 0$
  Consider $\hat{\gamma}_A \subset \mathbb{R}^3$

  [Babich, Bobenko, (1993)]
  [Alexakis, Mazzeo, (2010)]

$$F_A = \mathcal{W}[\hat{\gamma}_A] = \int_{\hat{\gamma}_A} \frac{(\tilde{n}^z)^2}{z^2}\, d\tilde{A} = \frac{1}{2}\mathcal{W}\left[\hat{\gamma}_A^{(\mathrm{d})}\right]$$

$\hat{\gamma}_A \subset \mathbb{R}^3$





$\hat{\gamma}_A^{(\mathrm{d})} \subset \mathbb{R}^3$

umbilic line

- Since $\mathcal{W}[\Sigma_g] \geqslant 4\pi$ (saturated only by round spheres) [Willmore, (1965)]
  HEE is maximised by the disk for a given perimeter P_A

HEE in asymptotically AdS(4) static spacetimes

[Fonda, Giomi, Seminara, Tonni, to appear]

- Take $ds^2|_{t=\text{const}} = g_{\mu\nu}\, dx^\mu dx^\nu$ with $g_{\mu\nu} = e^{2\varphi}\, \tilde{g}_{\mu\nu}$ and $\varphi = -\log(z) + \dots$
  The metric $\tilde{g}_{\mu\nu}$ is asymptotically flat

  - $\hat{\gamma}_A$ extremal area surface

$$\big(\text{Tr}\widetilde{K}\big)^2 = 4(\tilde{n}^\lambda \partial_\lambda \varphi)^2$$

$$F_A = \int_{\hat{\gamma}_A} \left[\frac{1}{2}\big(\text{Tr}\widetilde{K}\big)^2 + \widetilde{\nabla}^2\varphi - e^{2\varphi} - \tilde{n}^\mu \tilde{n}^\nu\, \widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \varphi \right] d\tilde{A}$$

  The unit vector $\tilde{n}^\mu$ is normal to the surface $\hat{\gamma}_A \subset \mathbb{R}^3$

- AdS_4: the previous formula with the Willmore energy is recovered

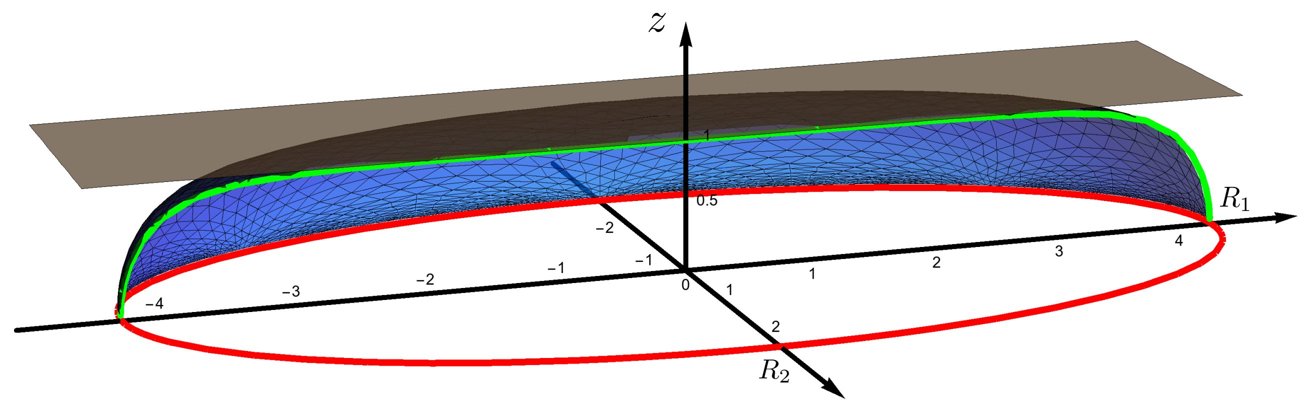
- Asymptotically AdS_4 black holes $\quad ds^2 = \dfrac{1}{z^2}\left(-f(z)\, dt^2 + \dfrac{dz^2}{f(z)} + d\boldsymbol{x}^2 \right)$
  $f(z) = 1 - Mz^3 + Q^2 z^4$

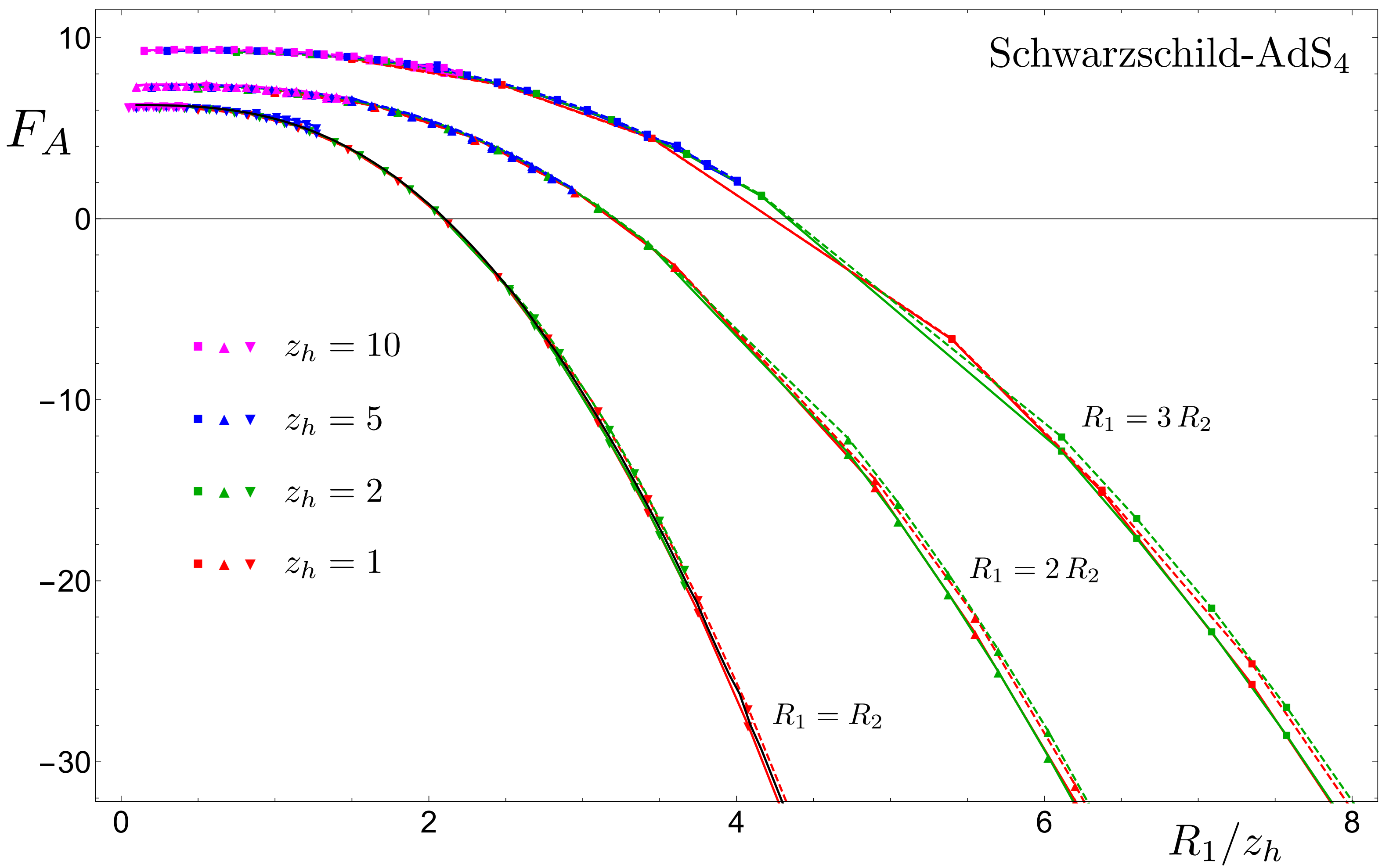
$$F_A = \int_{\hat{\gamma}_A} \frac{1}{z^2} \left[\left(1 + \frac{z f'(z)}{2f(z)} \right) (\tilde{n}^z)^2 + f(z) - \frac{z f'(z)}{2} - 1 \right] d\tilde{A}$$

- Also time dependent backgrounds (e.g. Vaidya) have been studied

HEE in asymptotically AdS(4) black holes. Ellipses

- Domains A delimited by ellipses





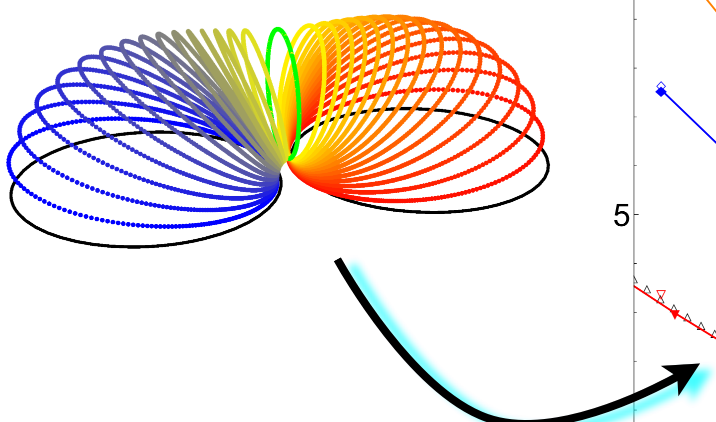
Holographic mutual information in AdS(4). Squircles

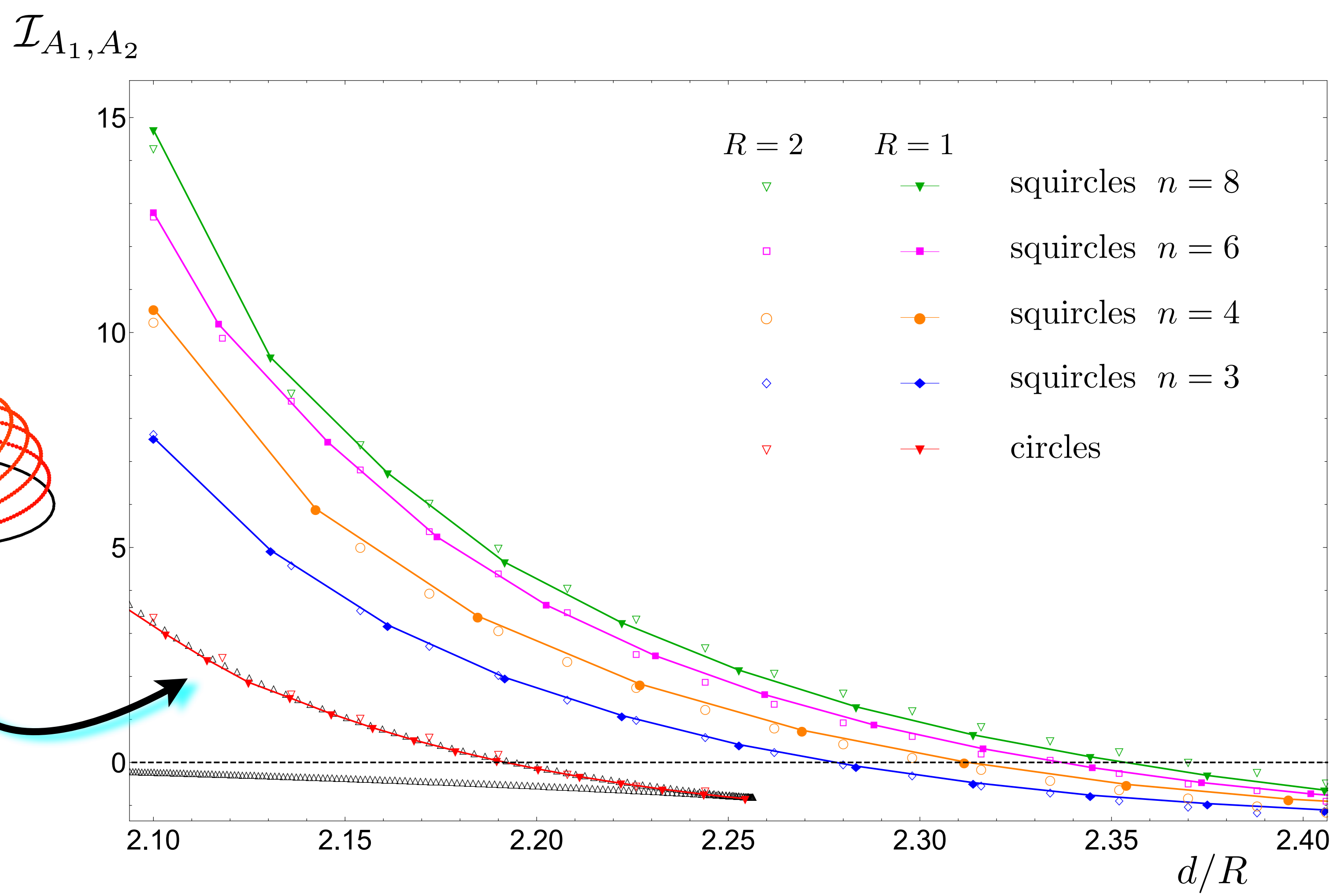
[Fonda, Giomi, Salvio, E.T., (2014)]

$$I_{A_1,A_2} \equiv S_{A_1} + S_{A_2} - S_{A_1 \cup A_2} \equiv \frac{\mathcal{I}_{A_1,A_2}}{4G_N}$$

$$\mathcal{I}_{A_1,A_2} = F_{A_1 \cup A_2} - F_{A_1} - F_{A_2} + o(1)$$

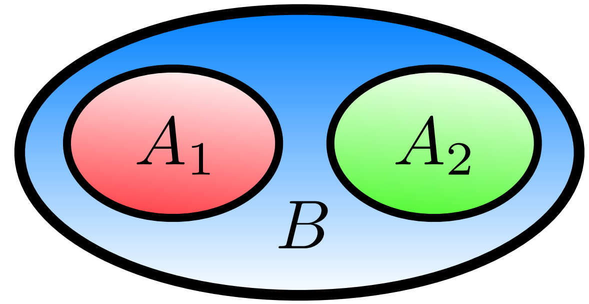
- Beyond a critical distance $\mathcal{I}_{A_1,A_2} = 0$ and the disconnected configuration is the minimal one





Entanglement between disjoint regions: Negativity

- $\rho = \rho_{A_1 \cup A_2}$ is a mixed state



ρ^{T_2} is the *partial transpose* of ρ

$$\langle e_i^{(1)} e_j^{(2)} |\, \rho^{T_2} \,| e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} |\, \rho \,| e_k^{(1)} e_j^{(2)} \rangle$$

$(|e_i^{(k)}\rangle$ base of $\mathcal{H}_{A_k})$

[Peres, (1996)] [Zyczkowski, Horodecki, Sanpera, Lewenstein, (1998)] [Eisert, (2001)] [Plenio, (2005)] [Vidal, Werner, (2002)]

- *Trace norm*

$$||\rho^{T_2}|| = \mathrm{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = 1 - 2 \sum_{\lambda_i < 0} \lambda_i$$

λ_j eigenvalues of ρ^{T_2}

$\mathrm{Tr}\, \rho^{T_2} = 1$

Logarithmic negativity

$$\mathcal{E}_{A_2} = \ln ||\rho^{T_2}|| = \ln \mathrm{Tr}|\rho^{T_2}|$$

- *Bipartite* system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ in *any* state ρ $\longrightarrow$ $\mathcal{E}_1 = \mathcal{E}_2$

Replica approach to Negativity

[Calabrese, Cardy, E.T., (2012)]

- A *parity effect* for $\mathrm{Tr}(\rho^{T_2})^n$

$$\mathrm{Tr}(\rho^{T_2})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e}$$

$$\mathrm{Tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$$

- Analytic continuation on the *even* sequence $\mathrm{Tr}(\rho^{T_2})^{n_e}$ (*make 1 an even number*)

$$\mathcal{E} = \lim_{n_e \to 1} \log \left[\mathrm{Tr}(\rho^{T_2})^{n_e} \right]$$

$$\lim_{n_o \to 1} \mathrm{Tr}(\rho^{T_2})^{n_o} = \mathrm{Tr}\, \rho^{T_2} = 1$$

- Pure states $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system ($\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$)

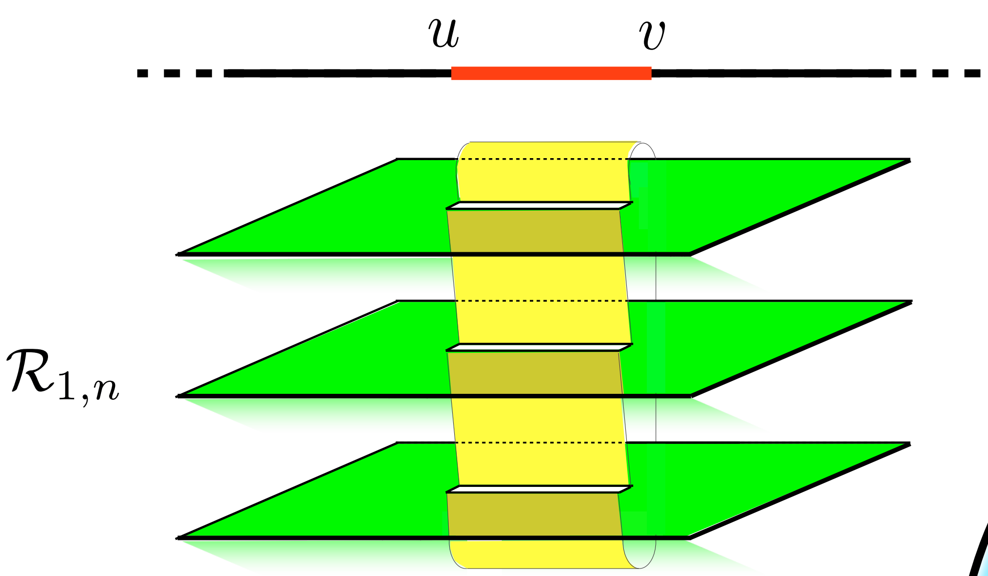
Schmidt decomposition

$$\mathrm{Tr}(\rho^{T_2})^n = \begin{cases} \mathrm{Tr}\, \rho_2^n & n = n_o \quad \text{odd} \\ \left(\mathrm{Tr}\, \rho_2^{n/2}\right)^2 & n = n_e \quad \text{even} \end{cases}$$

- Taking $n_e \to 1$ we have $\mathcal{E} = 2 \log \mathrm{Tr} \rho_2^{1/2}$ (Renyi entropy 1/2)

2D CFT: Renyi entropies as correlation functions

- One interval ($N = 1$): the Renyi entropies can be written as a two point function of *twist fields* on the sphere [Calabrese, Cardy, (2004)]



[Holzhey, Larsen, Wilczek, (1994)]

$$S_A = \frac{c}{3} \log \frac{\ell}{\epsilon} + c'_1$$

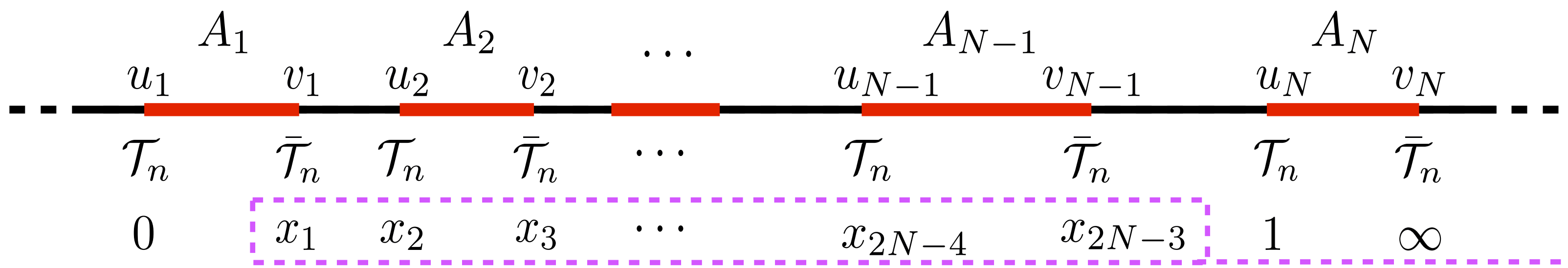
$$\mathrm{Tr}\rho_A^n = \frac{\mathcal{Z}_{1,n}}{\mathcal{Z}^n} = \langle \mathcal{T}_n(u) \bar{\mathcal{T}}_n(v) \rangle = \frac{c_n}{|u-v|^{2\Delta_n}}$$

$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

- Twist fields have been largely studied in the 1980s
  [Zamolodchikov, (1987)] [Dixon, Friedan, Martinec, Shenker, (1987)]
  [Knizhnik, (1987)] [Bershadsky, Radul, (1987)]
- Integrable field theories [Cardy, Castro-Alvaredo, Doyon, (2008)] [Doyon, (2008)]

2D CFT: Renyi entropies for many disjoint intervals

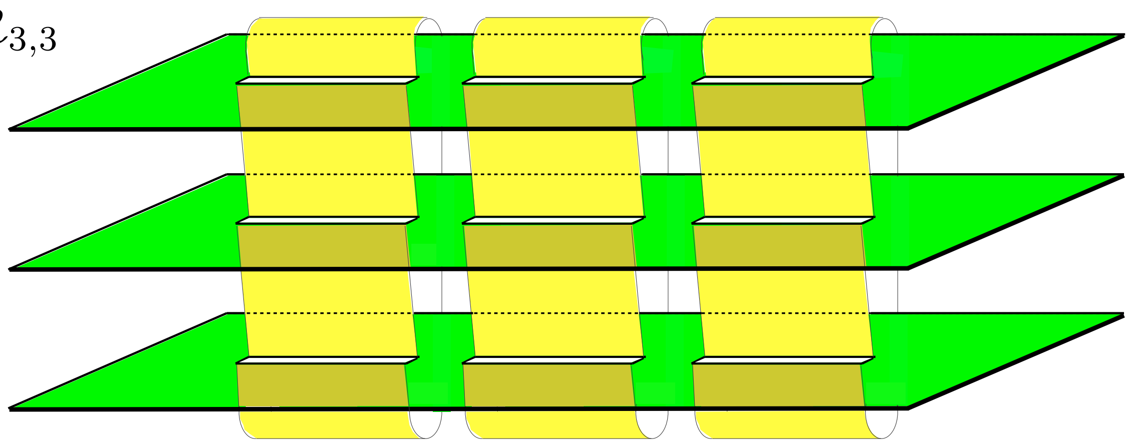
- N disjoint intervals $\Longrightarrow$ $2N$ point function of twist fields



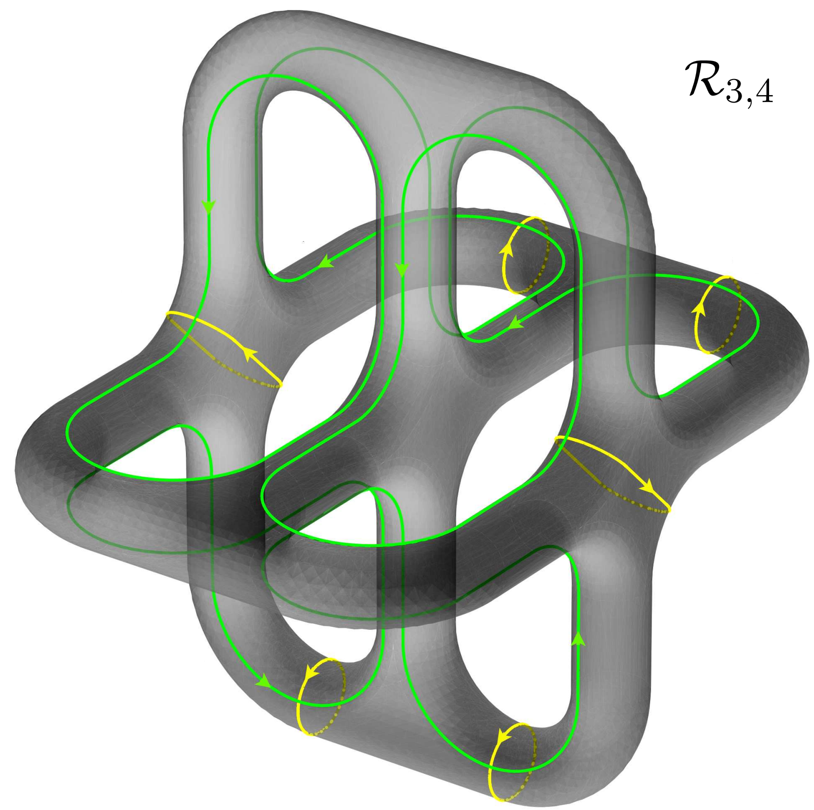
$$\mathrm{Tr}\rho_A^n = \frac{\mathcal{Z}_{N,n}}{\mathcal{Z}^n} = \langle \prod_{i=1}^{N} \mathcal{T}_n(u_i)\bar{\mathcal{T}}_n(v_i)\rangle = c_n^N \left| \frac{\prod_{i<j}(u_j-u_i)(v_j-v_i)}{\prod_{i,j}(v_j-u_i)} \right|^{2\Delta_n} \mathcal{F}_{N,n}(\boldsymbol{x})$$

- $\mathcal{Z}_{N,n}$ partition function of $\mathcal{R}_{N,n}$, a particular Riemann surface of genus $g=(N-1)(n-1)$ obtained through replication

$\mathcal{R}_{3,3}$



$\mathcal{R}_{3,4}$



N intervals: free compactified boson & Ising model

- $\mathcal{R}_{N,n}$ is $$y^n = \prod_{\gamma=1}^{N}(z - x_{2\gamma-2}) \left[\prod_{\gamma=1}^{N-1}(z - x_{2\gamma-1}) \right]^{n-1}$$ $g = (N-1)(n-1)$ [Enolski, Grava, (2003)]

- Partition function for a generic Riemann surface studied long ago in string theory [Zamolodchikov, (1987)] [Alvarez-Gaume, Moore, Vafa, (1986)] [Dijkgraaf, Verlinde, Verlinde, (1988)]

  Riemann theta function with characteristic $$\Theta[\boldsymbol{e}](\boldsymbol{0}|\Omega) = \sum_{\boldsymbol{m}\in\mathbb{Z}^p} \exp\left[\mathrm{i}\pi(\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \Omega \cdot (\boldsymbol{m}+\boldsymbol{\varepsilon}) + 2\pi\mathrm{i}(\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \boldsymbol{\delta} \right]$$

- Free compactified boson $(\eta \propto R^2)$ [Coser, Tagliacozzo, E.T., (2013)]

$$\mathcal{F}_{N,n}(\boldsymbol{x}) = \frac{\Theta(\boldsymbol{0}|T_\eta)}{|\Theta(\boldsymbol{0}|\tau)|^2} \qquad T_\eta = \begin{pmatrix} \mathrm{i}\,\eta\,\mathcal{I} & \mathcal{R} \\ \mathcal{R} & \mathrm{i}\,\mathcal{I}/\eta \end{pmatrix}$$

$\tau = \mathcal{R} + \mathrm{i}\,\mathcal{I}$ period matrix $(g \times g)$

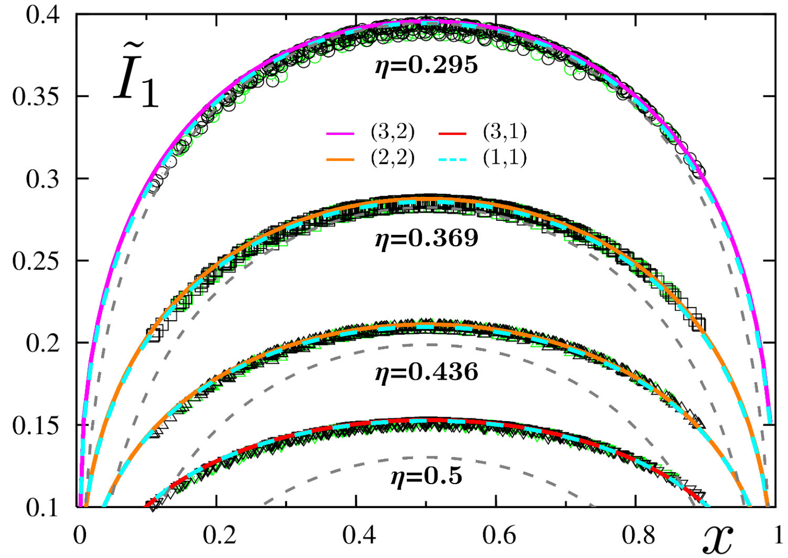
- Ising model $$\mathcal{F}^{\mathrm{Ising}}_{N,n}(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{e}} |\Theta[\boldsymbol{e}](\boldsymbol{0}|\tau)|}{2^g\, |\Theta(\boldsymbol{0}|\tau)|}$$

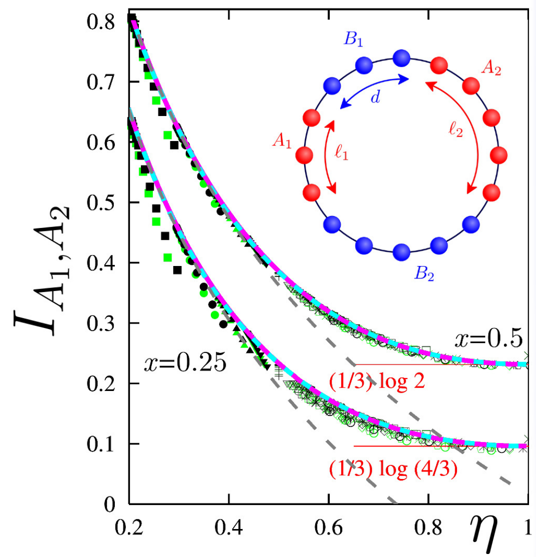
Nasty n dependence

- Two intervals case: [Caraglio, Gliozzi, (2008)] [Furukawa, Pasquier, Shiraishi, (2009)] [Calabrese, Cardy, E.T., (2009), (2011)] [Fagotti, Calabrese, (2010)] [Alba, Tagliacozzo, Calabrese, (2010), (2011)]

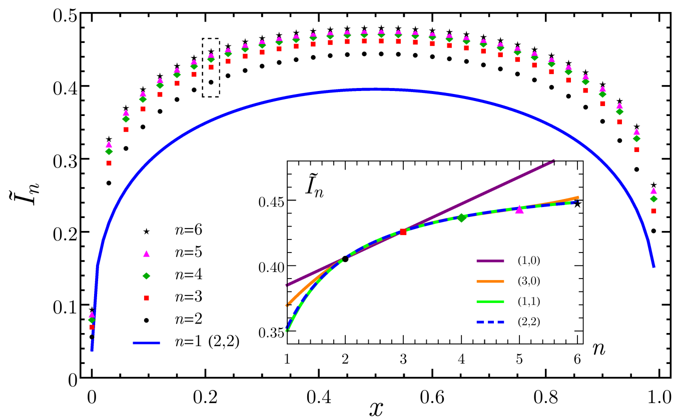
Two disjoint intervals

- Mutual information in XXZ model
  (exact diagonalization) [Furukawa, Pasquier, Shiraishi, (2009)]

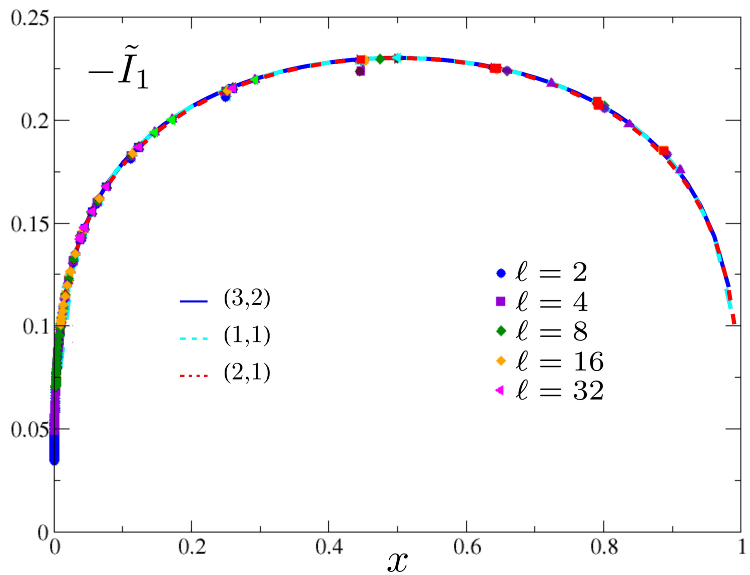




Rational interpolation:
an example



- Mutual information in critical Ising chain
  (Tree Tensor Network) [Alba, Tagliacozzo, Calabrese, (2010)]



- Rational interpolation:

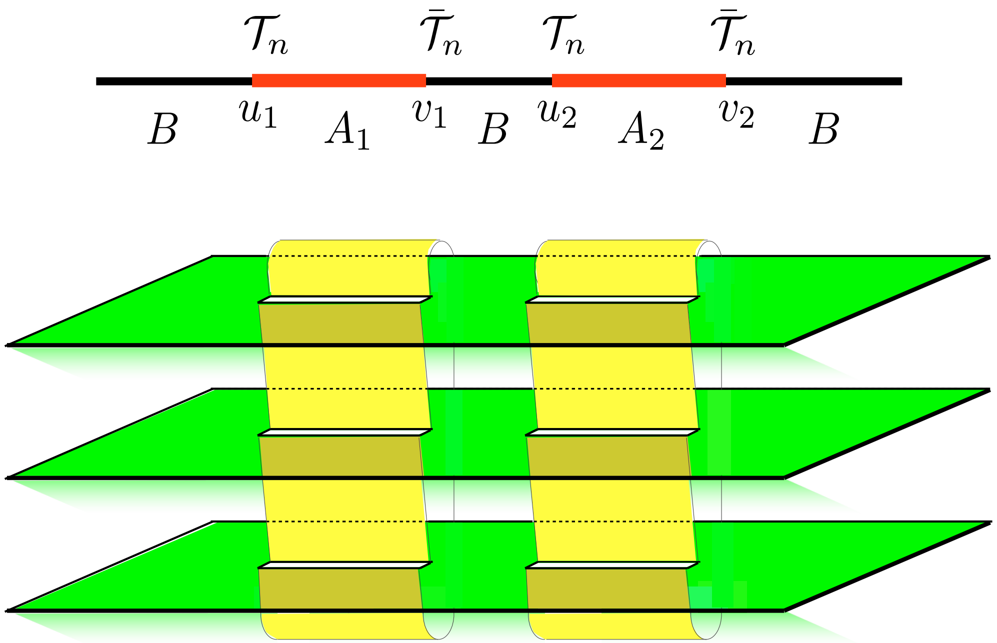
[De Nobili, Coser, E.T., (2015)]

$$W^{(n)}_{(p,q)}(x) \equiv \frac{a_0(x) + a_1(x)n + \cdots + a_p(x)n^p}{b_0(x) + b_1(x)n + \cdots + b_q(x)n^q}$$

Method first employed for Riemann theta functions in $2+1$ dimensions [Agón, Headrick, Jafferis, Kasko, (2014)]

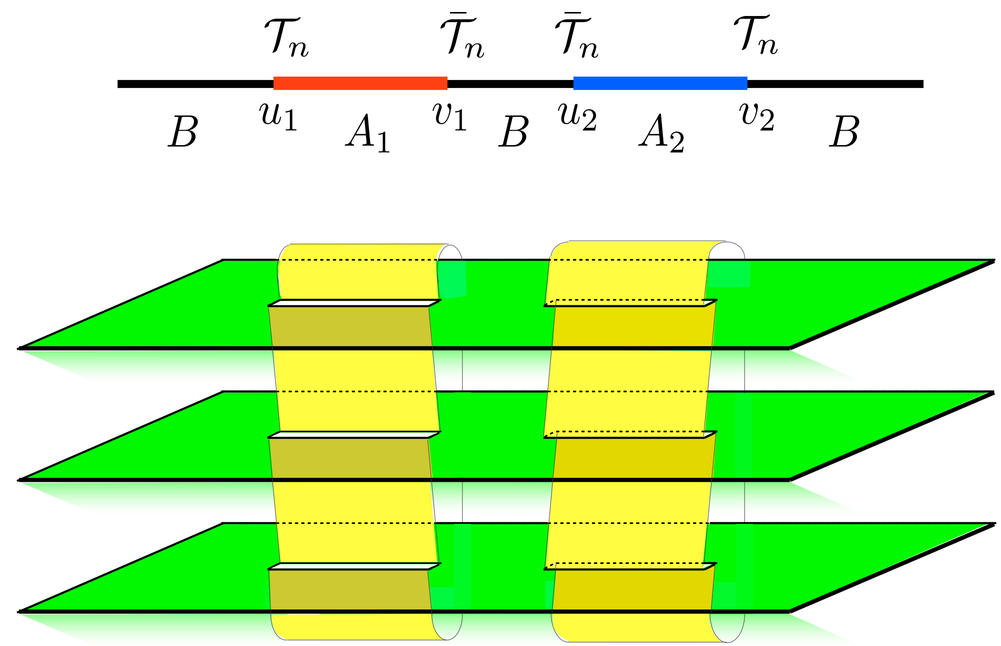
Partial transposition: two disjoint intervals

$$\mathrm{Tr}\rho^n_{A_1\cup A_2}$$



$$\mathrm{Tr}\rho^n_A = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$

$$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$$



$$\mathrm{Tr}(\rho^{T_2}_A)^n = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle$$

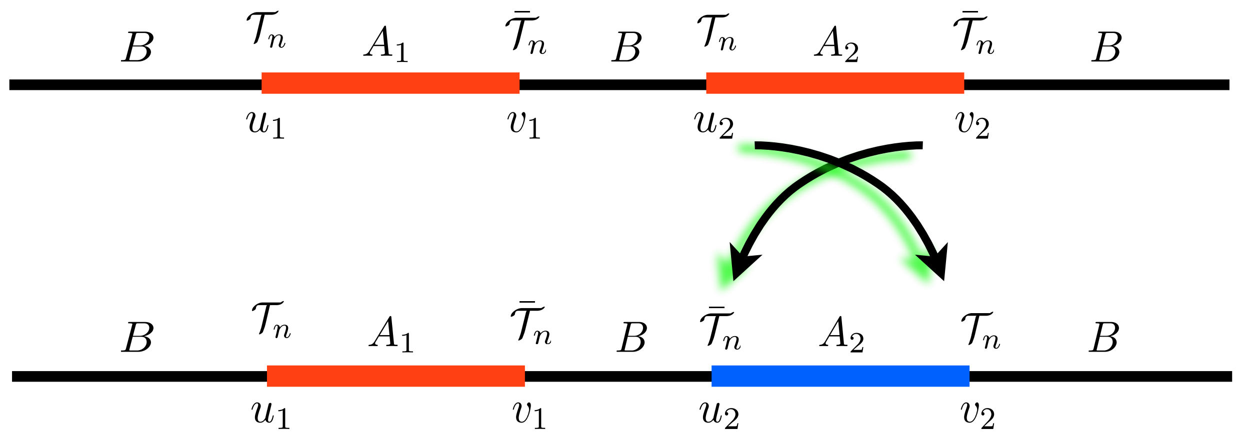
- The partial transposition exchanges $\mathcal{T}_n$ and $\bar{\mathcal{T}}_n$

[Calabrese, Cardy, E.T., (2012)]

Partial Transpose in 2D CFT: two disjoint intervals

$\mathrm{Tr}\rho^n_{A_1\cup A_2}$

$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$



$$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n = c_n^2\left[\ell_1\ell_2(1-y)\right]^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\mathcal{G}_n(y)$$

- $\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$ is obtained from $\mathrm{Tr}\rho^n_{A_1\cup A_2}$ by exchanging two twist fields

$$\mathcal{G}_n(y) = (1-y)^{\frac{c}{3}\left(n-\frac{1}{n}\right)}\mathcal{F}_n\left(\frac{y}{y-1}\right)$$

- $$\mathcal{E}(y) = \lim_{n_e\to 1}\mathcal{G}_{n_e}(y) = \lim_{n_e\to 1}\left[\mathcal{F}_n\left(\frac{y}{y-1}\right)\right]$$

- Two adjacent intervals: $\mathrm{Tr}(\rho_A^{T_2})^n = \langle\mathcal{T}_n(-\ell_1)\bar{\mathcal{T}}_n^2(0)\mathcal{T}_n(\ell_2)\rangle$

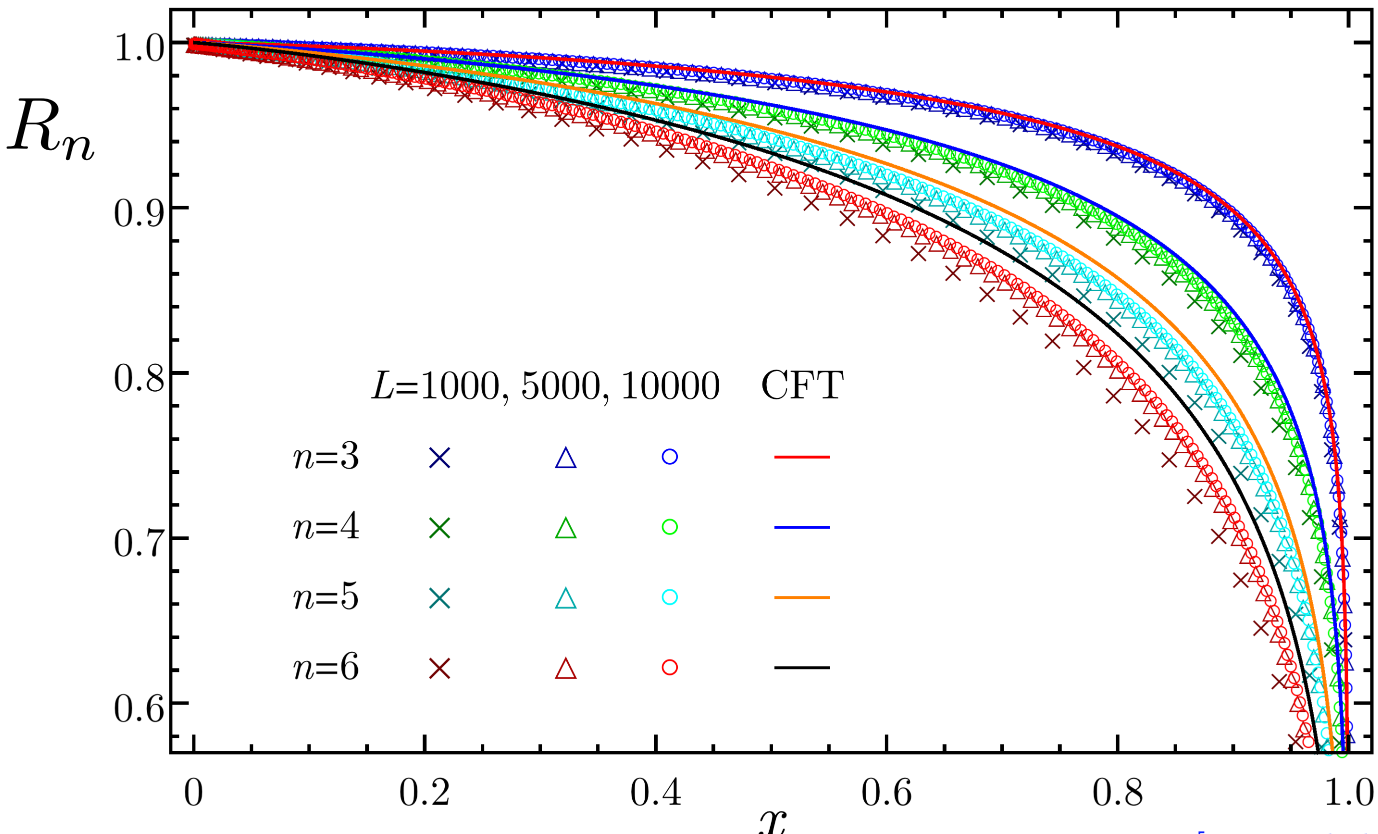
Two disjoint intervals: periodic harmonic chains

- Previous numerical results for $\mathcal{E}$: Ising (DMRG) and harmonic chains [Wichterich, Molina-Vilaplana, Bose, (2009)] [Marcovitch, Retzker, Plenio, Reznik, (2009)]

- Non compact free boson [Calabrese, Cardy, E.T., (2012)]

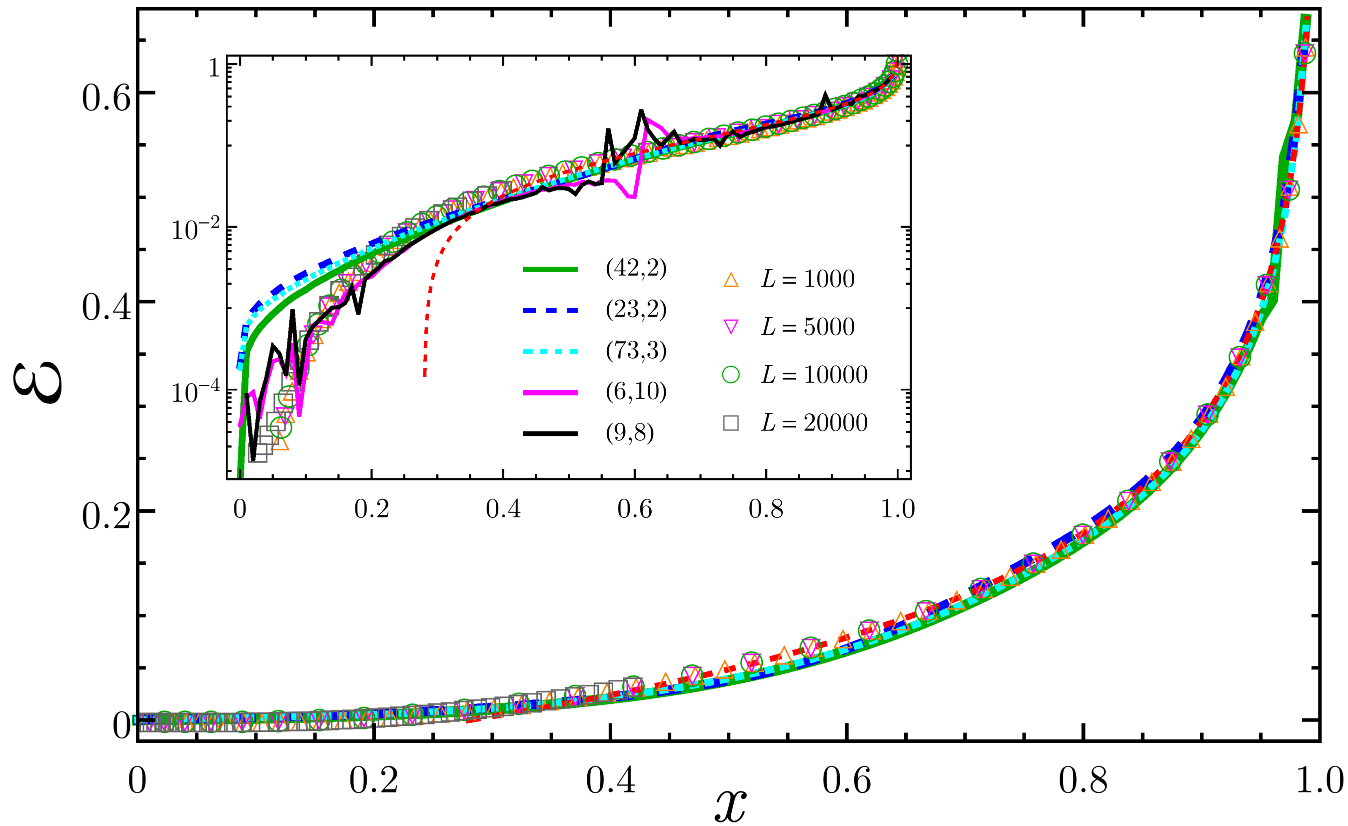
$$R_n = \frac{\mathrm{Tr}(\rho_A^{T_2})^n}{\mathrm{Tr}\,\rho_A^n}$$

$$R_n = \left[\frac{(1-x)^{\frac{2}{3}(n-\frac{1}{n})} \prod_{k=1}^{n-1} F_{\frac{k}{n}}(x)\, F_{\frac{k}{n}}(1-x)}{\prod_{k=1}^{n-1} \mathrm{Re}\big(F_{\frac{k}{n}}(\frac{x}{x-1})\, \bar{F}_{\frac{k}{n}}(\frac{1}{1-x})\big)} \right]^{\frac{1}{2}}$$



[De Nobili, Coser, E.T., (2015)]

Two disjoint intervals: periodic harmonic chains



- Analytic continuation for $x \sim 1$
  [Calabrese, Cardy, E.T., (2012)]

$$\mathcal{E} = -\frac{1}{4}\log(1-x) + \log K(x) + \text{cnst}$$

- Analytic continuation $n_e \to 1$ for $0 < x < 1$ not known
- $\mathcal{E}(x)$ for $x \sim 0$ vanishes faster than any power
- Numerical extrapolations (rational interpolation method) [De Nobili, Coser, E.T., (2015)]

Two disjoint intervals: Ising model

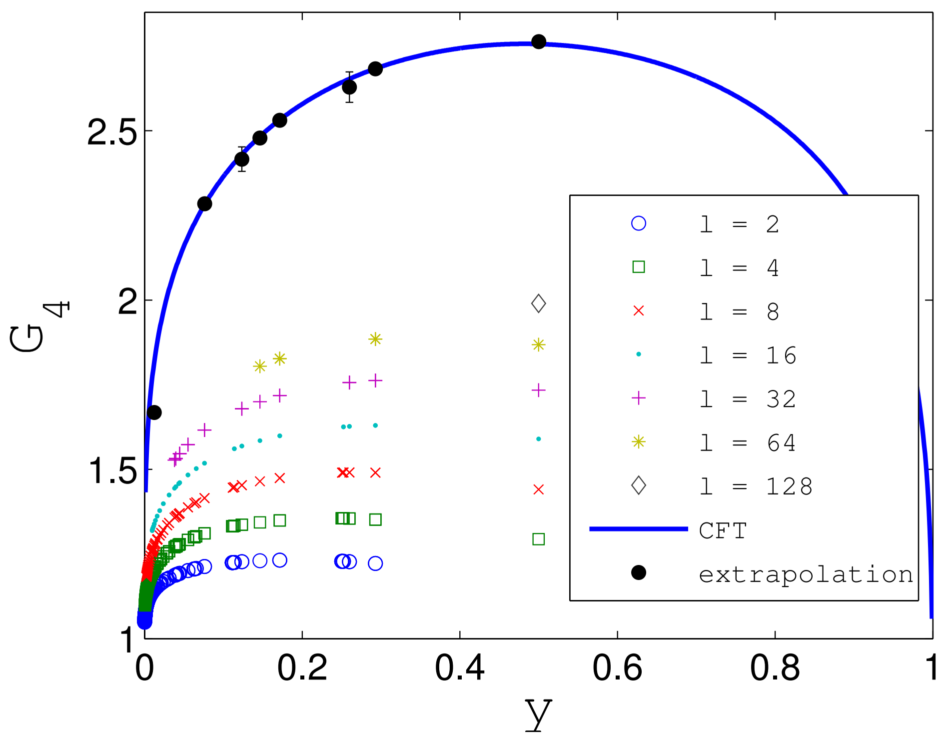
[Alba, (2013)] [Calabrese, Tagliacozzo, E.T., (2013)]

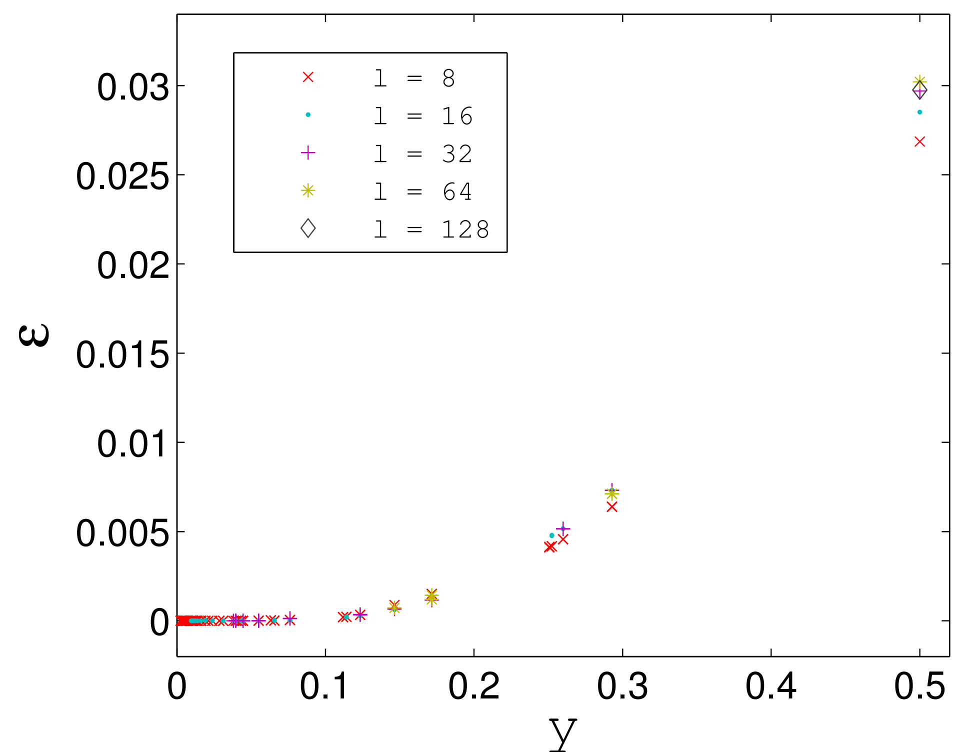
- CFT

$$\mathcal{G}_n(y) = (1-y)^{(n-1/n)/6} \, \frac{\sum_{\mathbf{e}} |\Theta[\mathbf{e}](\mathbf{0}|\tau(\frac{y}{y-1}))|}{2^{n-1}\prod_{k=1}^{n-1} |F_{k/n}(\frac{y}{y-1})|^{1/2}} \qquad 0 < y < 1$$

- Tree tensor network:

[Calabrese, Tagliacozzo, E.T., (2013)]





XY spin chain: two disjoint blocks

- XY spin chain with periodic b.c.

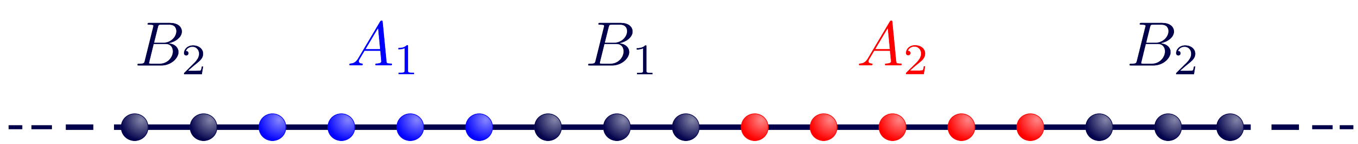
$$H_{XY} = -\frac{1}{2}\sum_{j=1}^{L}\left(\frac{1+\gamma}{2}\,\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{2}\,\sigma_j^y\sigma_{j+1}^y + h\,\sigma_j^z\right)$$

Ising model in a transverse field for $\gamma=1$, XX spin chain for $\gamma=0$

- Jordan-Wigner transformation $\quad c_j = \Big(\prod_{m<j}\sigma_m^z\Big)\frac{\sigma_j^x - \mathrm{i}\sigma_j^z}{2} \qquad c_j^\dagger = \Big(\prod_{m<j}\sigma_m^z\Big)\frac{\sigma_j^x + \mathrm{i}\sigma_j^z}{2}$

then introduce Majorana fermions $a_{2j} = c_j + c_j^\dagger$ and $a_{2j-1} = \mathrm{i}(c_j - c_j^\dagger)$.

- Two disjoint blocks



→ The string $P_{B_1} \equiv \prod_{j\in B_1}(\mathrm{i}a_{2j-1}a_{2j})$ enters in a crucial way
[Alba, Tagliacozzo, Calabrese, (2010)] [Igloi, Peschel, (2010)] [Fagotti, Calabrese, (2010)]

→ Rényi entropies can be written through 4 fermionic Gaussian operators
[Fagotti, Calabrese, (2010)]

$$\mathrm{Tr}\rho_A^n = \mathrm{Tr}\left(\frac{\rho_A^{\mathbf{1}} + P_{A_2}\rho_A^{\mathbf{1}}P_{A_2}}{2} + \langle P_{B_1}\rangle\,\frac{\rho_A^{B_1} - P_{A_2}\rho_A^{B_1}P_{A_2}}{2}\right)^n \qquad \rho_A^{B_1} \equiv \frac{\mathrm{Tr}_B\big(P_{B_1}|\Psi\rangle\langle\Psi|\big)}{\langle P_{B_1}\rangle}$$

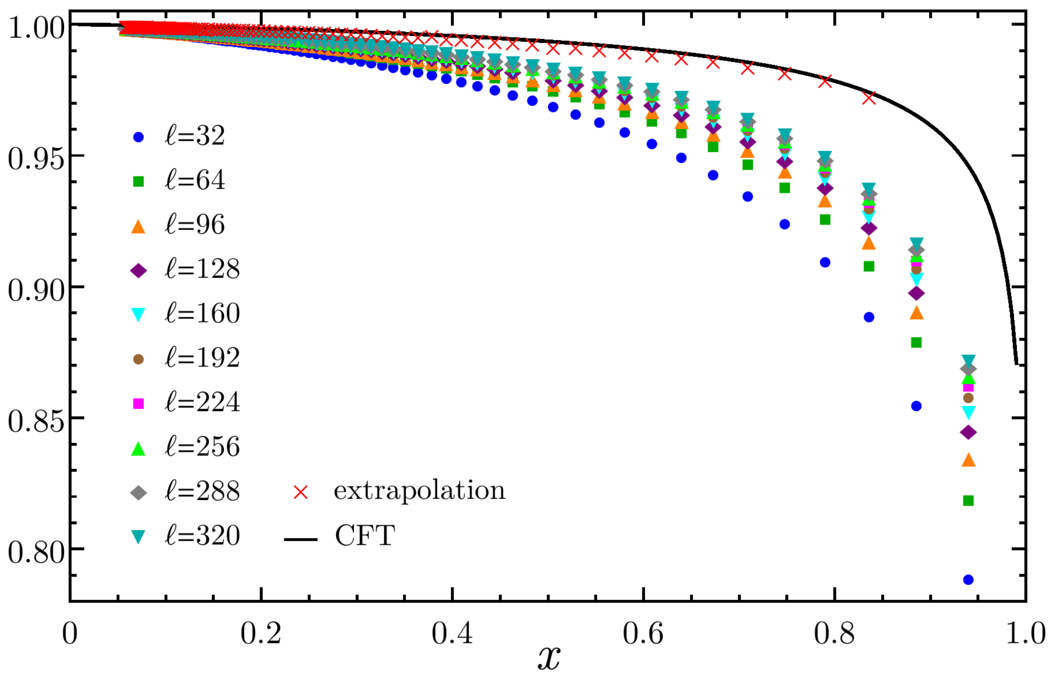
XY spin chain: partial transpose of two disjoint blocks

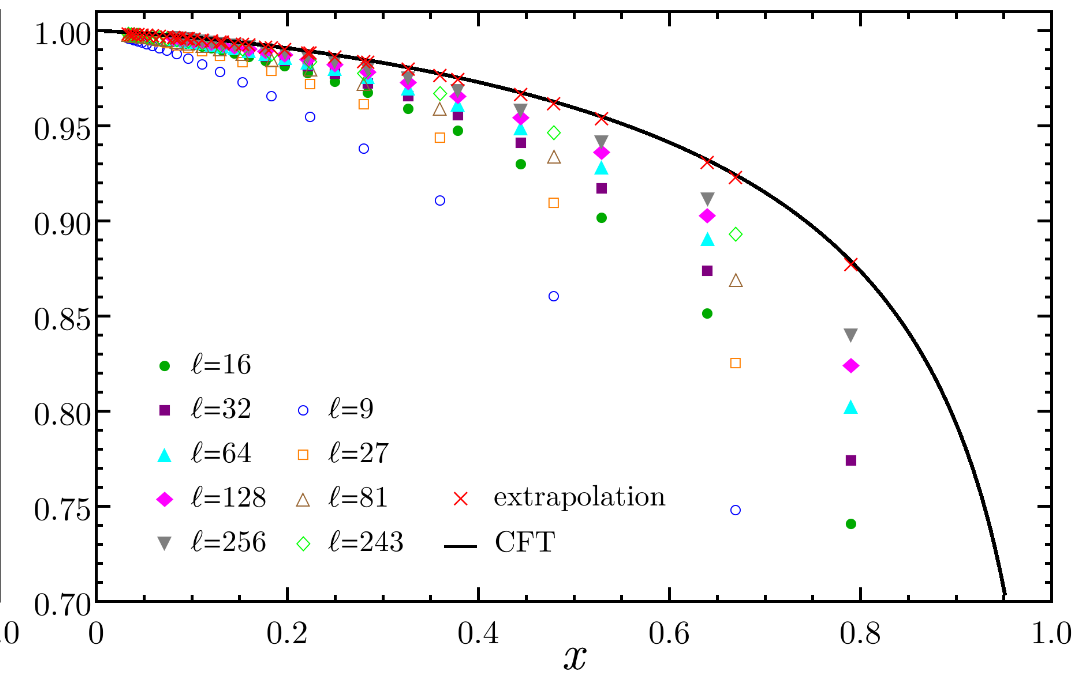
- Free fermion: $\rho_A^{T_2}$ is a sum of 2 fermionic Gaussian operators [Eisler, Zimboras, 1502.01369]

- XY spin chain: $\mathrm{Tr}\left(\rho_A^{T_2}\right)^n$ can be written in terms of 4 fermionic Gaussian operators [Coser, E.T., Calabrese, 1503.09114]

$$\mathrm{Tr}\left(\rho_A^{T_2}\right)^n = \mathrm{Tr}\left(\frac{\tilde{\rho}_A^{\mathbf{1}} + P_{A_2}\tilde{\rho}_A^{\mathbf{1}}P_{A_2}}{2} + \langle P_{B_1}\rangle\,\frac{\tilde{\rho}_A^{B_1} - P_{A_2}\tilde{\rho}_A^{B_1}P_{A_2}}{2\mathrm{i}}\right)^n$$

- CFT predictions have been checked for Ising chain and XX chain (e.g. $\mathrm{Tr}(\rho_A^{T_2})^n / \mathrm{Tr}\rho_A^n$ for $n = 4$)





Free fermion: partial transpose of two disjoint intervals

[Coser, E.T., Calabrese, 1508.00811]

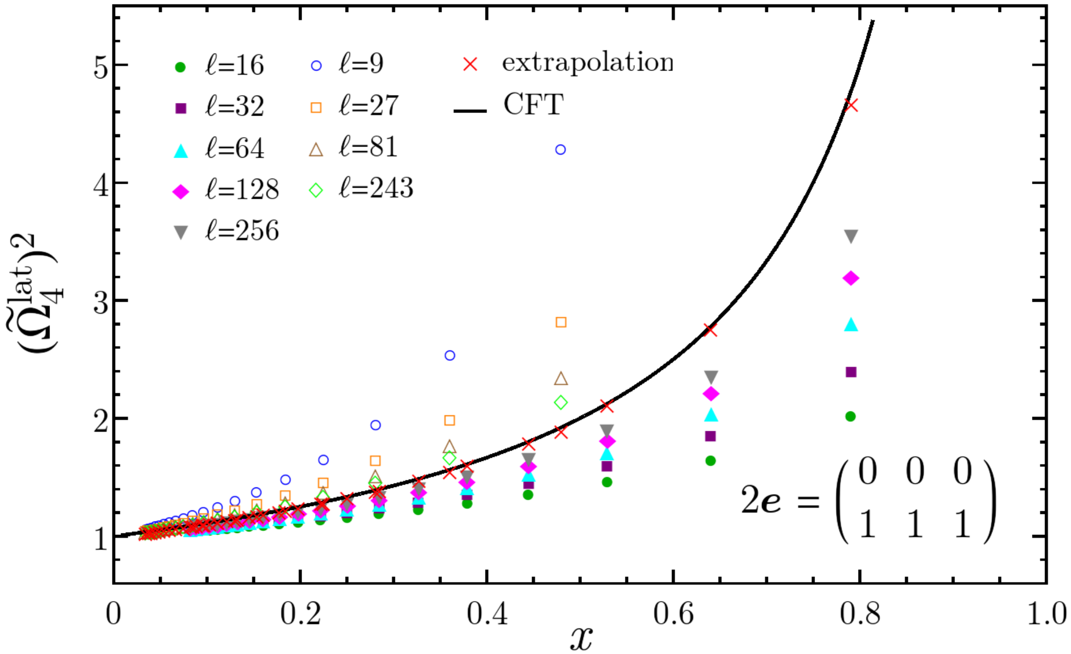
- CFT expression:

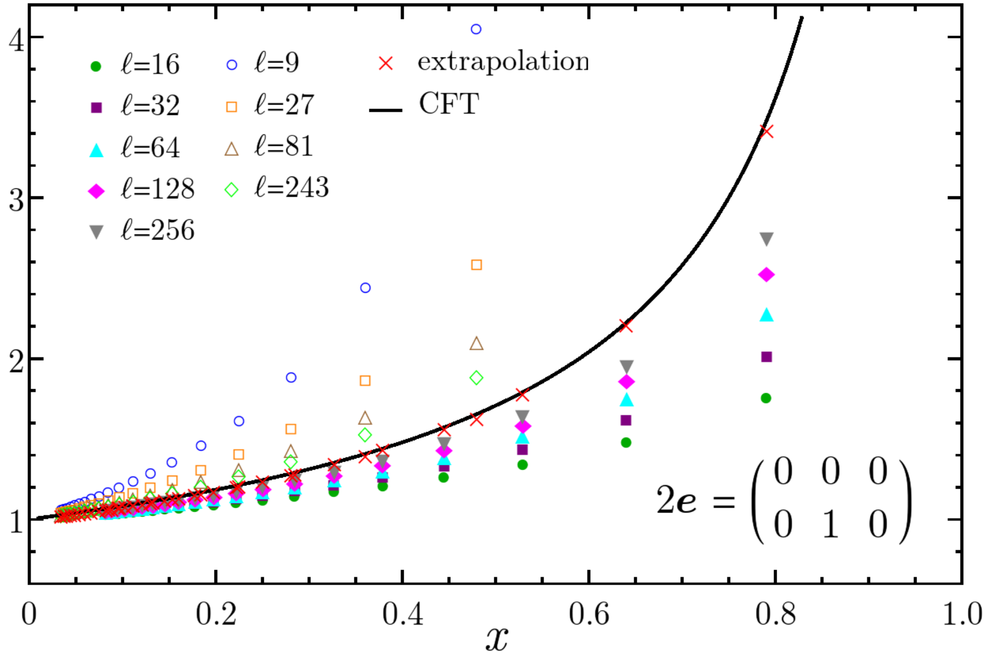
$$\mathrm{Tr}\big(\rho_A^{T_2}\big)^n = c_n^2 \left(\frac{1-x}{\ell_1 \ell_2}\right)^{2\Delta_n} \frac{1}{2^{n/2-1}} \sum_{\boldsymbol{\delta}} \cos\left[\frac{\pi}{4}\left(1+\sum_{i=1}^{n-1}(-1)^{2\sum_{j=i}^{n-1}\delta_j}\right)\right] \left|\frac{\Theta[\boldsymbol{e}](\tilde{\tau})}{\Theta(\tilde{\tau})}\right|^2$$

where $\tilde{\tau} \equiv \tau\big(x/(x-1)\big)$ and the sum is over the characteristics $\boldsymbol{e} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\delta} \end{pmatrix}$

- Same result for the compact boson at selfdual radius

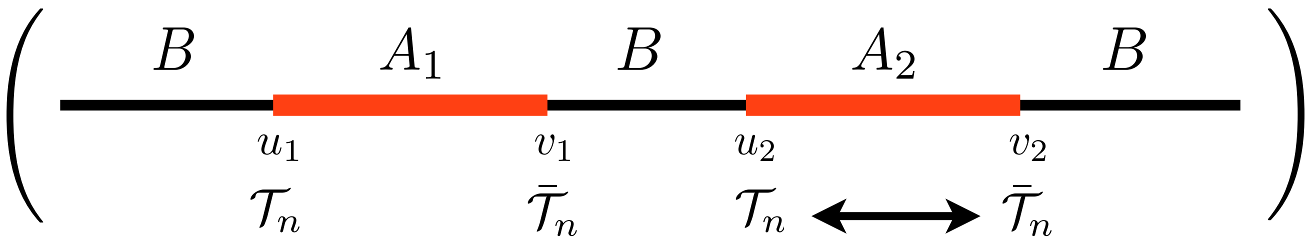
- The lattice counterpart of each term in the sum can be found





Partial Transposition for bipartite systems: pure states

$\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$

- $\lim_{B \to \emptyset}$ 

$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n^2(u_2) \bar{\mathcal{T}}_n^2(v_2) \rangle$$

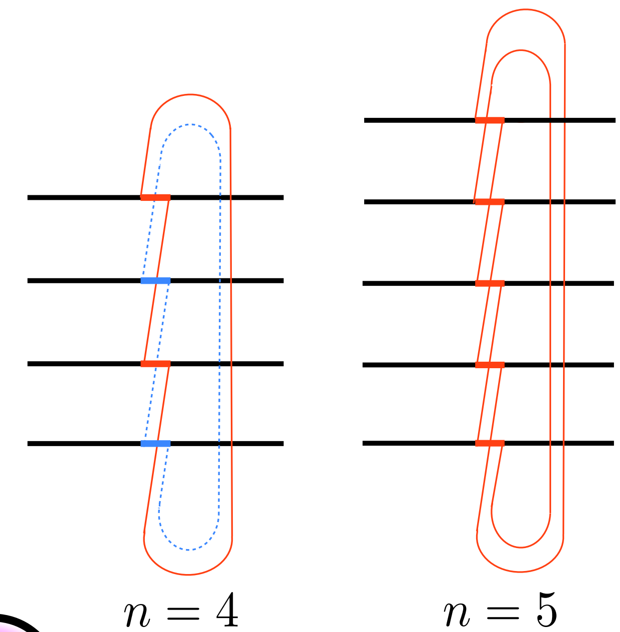
Partial Transposition = exchange two twist fields

- $\mathcal{T}_n^2$ connects the j-th sheet with the $(j+2)$-th one

  Even $n = n_e \implies$ decoupling

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} = \left(\langle \mathcal{T}_{n_e/2}(u_2) \bar{\mathcal{T}}_{n_e/2}(v_2) \rangle \right)^2 = \left(\mathrm{Tr}\, \rho_{A_2}^{n_e/2} \right)^2$$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} = \langle \mathcal{T}_{n_o}(u_2) \bar{\mathcal{T}}_{n_o}(v_2) \rangle = \mathrm{Tr}\, \rho_{A_2}^{n_o}$$



- Two dimensional CFTs

$$\Delta_{\mathcal{T}^2_{n_o}} = \frac{c}{12}\left(n_o - \frac{1}{n_o} \right) = \Delta_{\mathcal{T}_{n_o}}$$

$$\Delta_{\mathcal{T}^2_{n_e}} = \frac{c}{6}\left(\frac{n_e}{2} - \frac{2}{n_e} \right)$$

$$\mathcal{E} = \frac{c}{2} \ln \ell + \mathrm{const}$$

One interval at finite temperature: a naive approach

[Calabrese, Cardy, E.T., (2014)]

- Logarithmic negativity $\mathcal{E}$ of one interval at finite $T = 1/\beta$

- A naive approach: compute $\langle \mathcal{T}_n^2(u) \bar{\mathcal{T}}_n^2(v) \rangle_\beta$ through the conformal map relating the cylinder to the complex plane

$$\mathcal{E}_{\text{naive}} = \frac{c}{2} \ln \left(\frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta} \right) + 2 \ln c_{1/2}$$

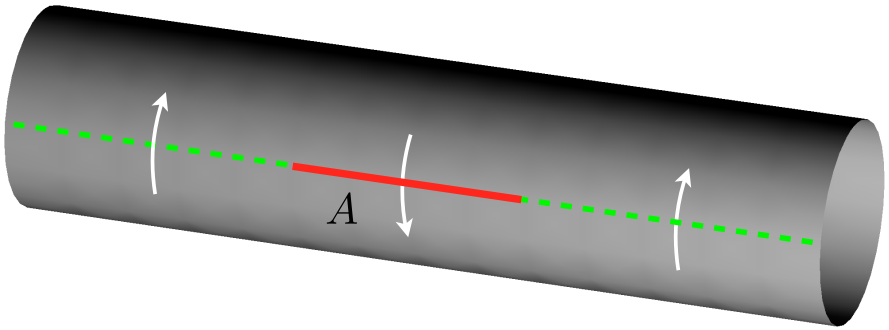
Problems:

→ The Rényi entropy $n = 1/2$ is not an entanglement measure at finite T

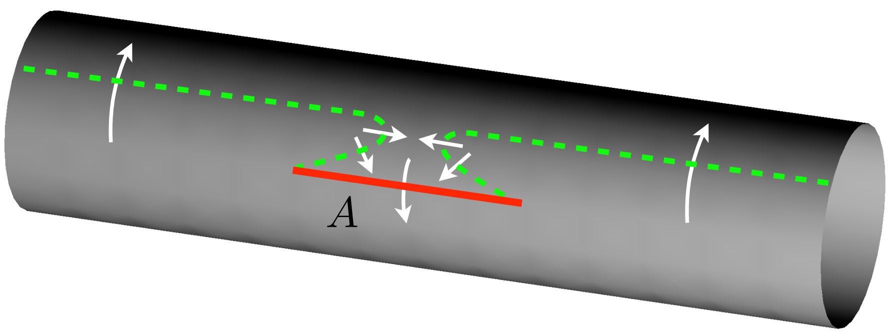
→ $\mathcal{E}_{\text{naive}}$ is an increasing function of T, linearly divergent at high T
Entanglement should decrease as the system becomes classical

One interval at finite temperature in the infinite line

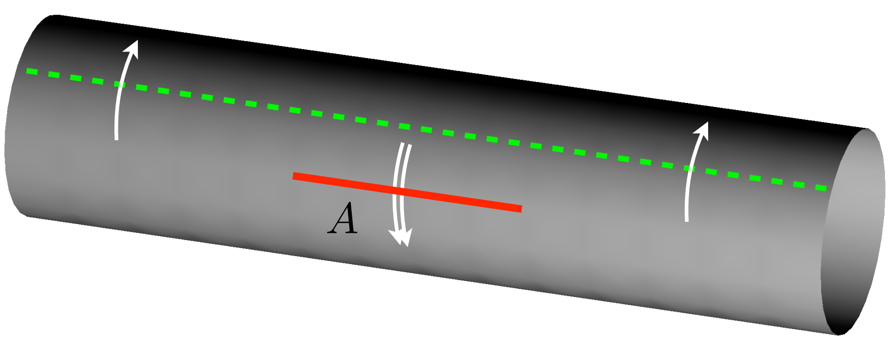
(connection to the $(j+1)$-th cylinder following the arrows)



Single copy of $\rho_\beta^{T_A} \implies \mathrm{Tr}\big(\rho_\beta^{T_A}\big)^n$



Deformation of the cut along B

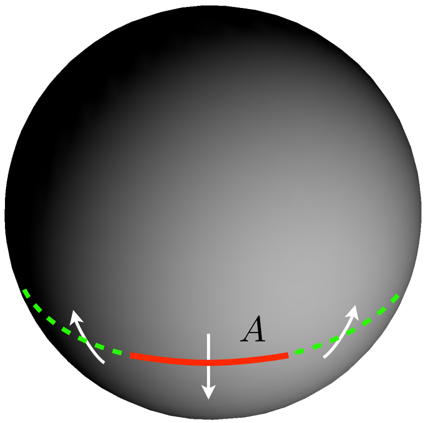


A cut remains connecting consecutive copies

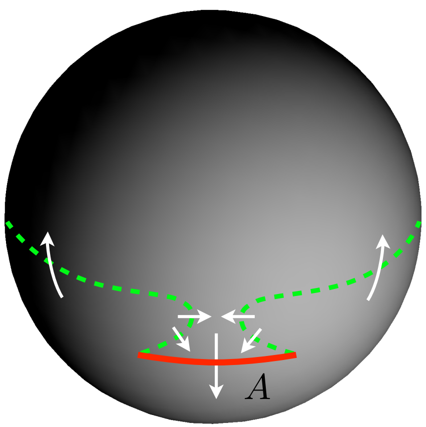
$\implies$ No factorization for even n

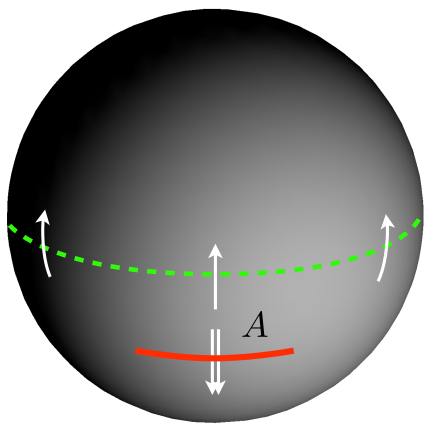
(The double arrow indicates the connection to the $(j+2)$-th copy)

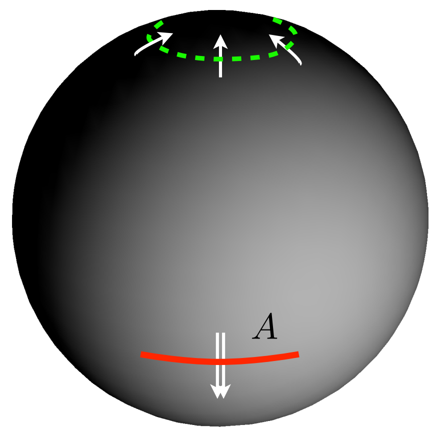
Deforming the cut at zero temperature



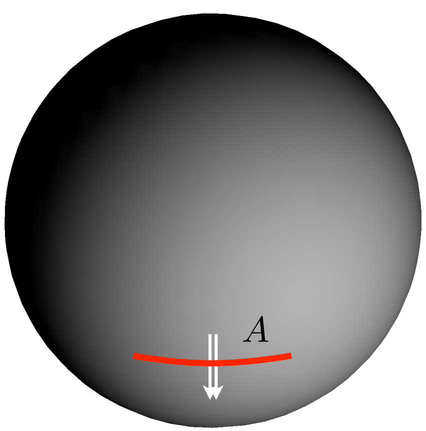
Single copy of $\left(|\psi\rangle\langle\psi|\right)^{T_A} \implies \mathrm{Tr}\left[\left(|\psi\rangle\langle\psi|\right)^{T_A}\right]^n$







Deformation of the cut along B

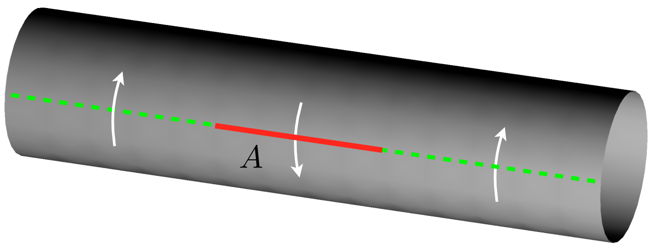


The cut connecting consecutive copies shrinks to a point

Only the connection to the $j \pm 2$ copies along A remains

$\implies$ Factorization for even n

One interval at finite temperature in the infinite line



Two auxiliary twist fields at $\mathrm{Re}(w) = \pm L$, then $L \to \infty$

$$\mathcal{E}_A = \lim_{L\to\infty} \lim_{n_e\to 1} \ln \langle \mathcal{T}_{n_e}(-L)\bar{\mathcal{T}}^2_{n_e}(-\ell)\mathcal{T}^2_{n_e}(0)\bar{\mathcal{T}}_{n_e}(L)\rangle_\beta$$

- Conformal map the cylinder into the plane $z = e^{2\pi w/\beta}$

$$\langle \mathcal{T}_n(z_1)\bar{\mathcal{T}}^2_n(z_2)\mathcal{T}^2_n(z_3)\bar{\mathcal{T}}_n(z_4)\rangle = \frac{c_n\, c_n^{(2)}}{z_{14}^{2\Delta_n}\, z_{23}^{2\Delta_n^{(2)}}} \frac{\mathcal{F}_n(x)}{x^{\Delta_n^{(2)}}} \qquad \mathcal{F}_n(1) = 1 \qquad \mathcal{F}_n(0) = \frac{C^2_{\mathcal{T}_n\bar{\mathcal{T}}^2_n\bar{\mathcal{T}}_n}}{c_n^{(2)}}$$

$x \to e^{-2\pi\ell/\beta}$ when $L \to \infty$

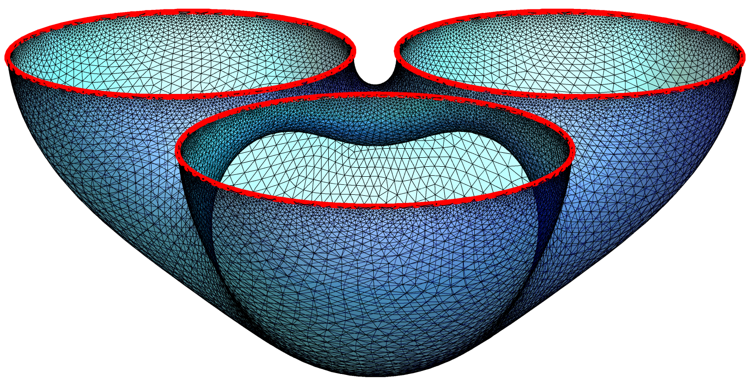
$$f(x) \equiv \lim_{n_e\to 1} \ln[\mathcal{F}_{n_e}(x)]$$

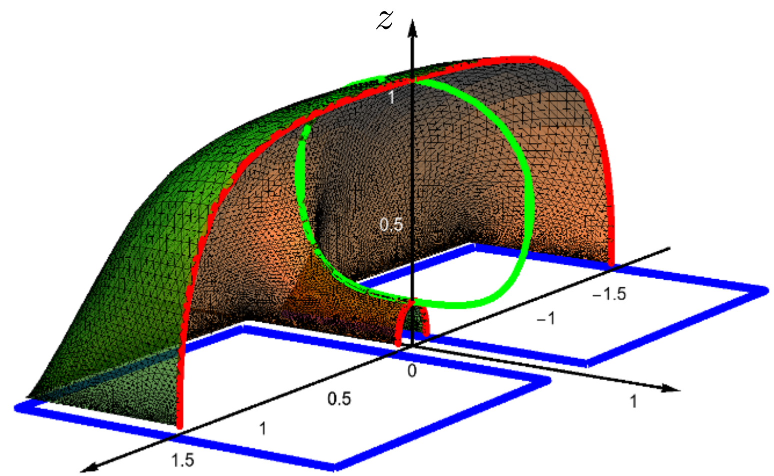
$$\mathcal{E}_A = \frac{c}{2}\ln\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi\ell}{\beta}\right)\right] - \frac{\pi c\ell}{2\beta} + f(e^{-2\pi\ell/\beta}) + 2\ln c_{1/2}$$

- $\mathcal{E}_A = \mathcal{E}_B$
- $\mathcal{E}$ depends on the full operator content of the model
- large T linear divergence of $\mathcal{E}_{\mathrm{naive}}$ is canceled
- semi infinite systems $\mathrm{Re}(w) < 0$ (BCFT) have been also studied

Conclusions & open issues

- Shape dependence of holographic entanglement entropy in AdS_4/CFT_3





- Entanglement for mixed states.
  Entanglement negativity in QFT (1+1 CFTs): $\mathrm{Tr}(\rho^{T_2})^n$ and $\mathcal{E}$
  → free boson, Ising model, finite temperature, free fermion

- Some open issues:
  → Analytic continuations
  → Higher dimensions
  → Interactions
  → Negativity in AdS/CFT

Thank you!