Hagedorn spectrum and thermodynamics of SU(N) Yang-Mills theories. [1]

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Sestri Levante 16/09/2015

[1] M. Caselle, A.Nada, M. Panero, arXiv:1505.01106 JHEP 07 (2015) 143
M. Caselle, L. Castagnini, A Feo, F. Gliozzi, M.Panero, arXiv:1105.0359 JHEP 06 (2011) 142

Summary:

1. Introduction and motivations
2. Lattice Thermodynamics
3. Glueball gas and closed string model
4. Results for SU(2) and SU(3) YM theories in (3+1) dimensions
5. SU(N) YM theories in (2+1) dimensions
6. Lorentz invariance and universality of the Nambu-Goto effective string.
7. Conclusions

Introduction and motivations

One of the main features of SU(N) non-abelian gauge theories is the existence of a deconfinement phase transition, i.e. a temperature above which gluons are "deconfined", like the quark-gluon plasma (QGP) in Quantum Chromodynamics. Our goal is to study the thermodynamics of pure gauge theories in the confining phase when approaching the deconfinement transition

We choose to study the pure gauge sector of the theory, beacuse it retains most of the non trivial features of the full theory, without the problems that the regularization of fermions on the lattice induces. This choice allows much faster and precise Montecarlo simulations and more importantly In the confining phase the only degrees of freedom of the theory without quarks are the so-called **"glueballs"**. Looking at the thermodynamics of the theory in the confining phase we have a tool to explore the glueball spectrum of the theory.

Introduction and motivations

Our main result is that the thermodynamics of the model can only be described assuming a string-like description of glueballs (and thus a Hagedorn spectrum)
The fine details of the spectrum spectrum agree remarkably well with the predictions of the Nambu-Goto effective string This turns out to be an highly non trivial test of the effective string picture of confinement.

This analysis was performed in the 3+1 dimensional SU(3) model in 2009 in the pioneering work of Meyer[1]. Now, using the high precision lattice data for SU(3) of [2] and a new set of data on (3+1) SU(2) that we obtained in [3], we are in the position to refine the effective string analysis and test its predictive power.
The present results confirm our previous findings[4] for (2+1) dimensional SU(N) theories (with $N = 2, 3, 4, 5, 6$).

[1] H. Meyer, *High-Precision Thermodynamics and Hagedorn Density of States*, 2009

[2] Sz. Borsanyi *et al.*, *Precision SU(3) lattice thermodynamics for a large temperature range*, 2012

[3] M. Caselle, A.Nada, M. Panero,*Hagedorn spectrum and thermodynamics of SU(N) Yang-Mills theories*, arXiv:1505.01106

[4] M. Caselle *et al.*, *Thermodynamics of SU(N) Yang-Mills theories in 2+1 dimensions I - The confining phase*, 2011

Lattice regularization

Only a truly non-perturbative approach such as lattice regularization can describe the deconfinement transition and the confined phase of non-abelian gauge theories.

For SU(N) pure gauge theories on the lattice the dynamics are described by the standard Wilson action

$$S_W = \beta \sum_{p=sp,tp} (1 - \frac{1}{N} \mathrm{ReTr} U_p)$$

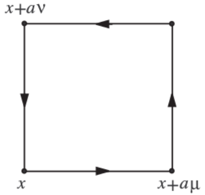
where U_P is the product of four U_μ SU(N) variables on the space-like or time-like plaquette P and $\beta = \frac{2N}{g^2}$.

The partition function is

$$Z = \int \prod_{x,\mu} \mathrm{d}U_\mu(x) e^{-S_W}$$

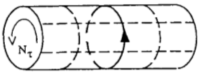
and the expectation value of an observable A

$$\langle A \rangle = \frac{1}{Z} \int \prod_{n,\mu} \mathrm{d}U_\mu(n)\, A(U_\mu(n))\, e^{-S_W}$$



Thermodynamic quantities

On a $N_t \times N_s^3$ lattice the volume is $V = (aN_s)^3$ (where a is the lattice spacing), while the temperature is determined by the inverse of the temporal extent (with periodic boundary conditions): $T = (aN_t)^{-1}$.



The thermodynamic quantities taken into account will be:

- the pressure p, that in the thermodynamic limit (i.e. for large and homogenous systems) can be written as

$$p \simeq \frac{T}{V} \log Z(T, V)$$

- the trace of the energy-momentum tensor Δ, that in units of T^4 is

$$\frac{\Delta}{T^4} = \frac{\epsilon - 3p}{T^4} = T \frac{\partial}{\partial T} \left(\frac{p}{T^4} \right)$$

Energy density $\epsilon = \Delta + 3p$ and entropy density $s = \frac{\epsilon + p}{T} = \frac{\Delta + 4p}{T}$ can be easily calculated.

Thermodynamics on the lattice

The pressure can be estimated by the means of the so-called "integral method"[1]:

$$p(T) \simeq \frac{T}{V} \log Z(T,V) = \frac{1}{a^4} \frac{1}{N_t\, N_s^3} \int_0^{\beta(T)} d\beta' \frac{\partial \log Z}{\partial \beta'}.$$

It can be written (relative to its $T = 0$ vacuum contribution) as

$$\frac{p(T)}{T^4} = -{N_t}^4 \int_0^{\beta} d\beta' \left[3(P_\sigma + P_\tau) - 6P_0\right]$$

where P_σ and P_τ are the expectation values of spacelike and timelike plaquettes respectively and P_0 is the expectation value at zero T.
The trace of energy-momentum tensor is simply

$$\frac{\Delta(T)}{T^4} = T \frac{\partial}{\partial T} \left(\frac{p}{T^4} \right) = -{N_t}^4\, T \frac{\partial \beta}{\partial T} \left[3(P_\sigma + P_\tau) - 6P_0\right].$$

ϵ and s can be obtained indirectly as linear combinations.

[1] J. Engels *et al.*, *Nonperturbative thermodynamics of SU(N) gauge theories*, 1990

Ideal glueball gas

The behaviour of the system is supposed to be dominated by a gas of non-interacting glueballs.

The prediction of an ideal relativistic Bose gas can be used to describe the thermodynamics of such gas. Its partition function for 3 spatial dimensions is

$$\log Z = (2J+1)\frac{2V}{T}\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right)^2 K_2\left(k\frac{m}{T}\right)$$

where m is the mass of the glueball, J is its spin and K_2 is the modified Bessel function of the second kind of index 2.

Observables such as Δ and p thus can be easily derived:

$$p = \frac{T}{V}\log Z = 2(2J+1)\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right)^2 K_2\left(k\frac{m}{T}\right)$$

$$\Delta = \epsilon - 3p = 2(2J+1)\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right) K_1\left(k\frac{m}{T}\right)$$

Test with the SU(2) model

The SU(2) model is a perfect laboratory to test these results.

- It is easy to simulate: very precise results may be obtained with a reasonable amount of computing power
- The deconfinement transition is of second order and thus it is expected to coincide with the Hagedorn temperature
- The masses of several states of the glueball spectrum are known with remarkable accuracy
- The infrared physics of the model is very similar to that of the SU(3) theory, with one important difference: the gauge group is real and thus only C=1 glueball exist. The glueball spectrum contains only half of the states with respect to SU(3).

Lattice setup

N_s^4 at $T = 0$	$N_s^3 \times N_t$ at $T \neq 0$	n_β	β-range	n_{conf}
32^4	$60^3 \times 5$	17	$[2.25, 2.3725]$	1.5×10^5
40^4	$72^3 \times 6$	25	$[2.3059, 2.431]$	1.5×10^5
40^4	$72^3 \times 8$	12	$[2.439, 2.5124]$	10^5

Table: *

Setup of our simulations. The first two columns show the lattice sizes (in units of the lattice spacing a) for the $T = 0$ and finite-temperature simulations, respectively. In the third column, n_β denotes the number of β-values simulated within the β-range indicated in the fourth column. Finally, in the fifth column we report the cardinality n_{conf} of the configuration set for the $T = 0$ and finite-T simulations.

Scale setting

The SU(2) scale setting is fixed by calculating the string tension via the computation of Polyakov loop correlators with the multilevel algorithm.

The range of the parameter β which has been considered ($\beta \in [2.27, 2.6]$) covers approximately the temperature region analyzed in the finite temperature simulations.

The string tension is obtained with a two-parameter fit of potential

$$V = -\frac{1}{N_t} \log\langle PP \rangle$$

with the first order effective string prediction for the potential

$$V = \sigma r + V_0 - \frac{\pi}{12r}$$

Higher order effective string corrections turned out to be negligible within the precision of our data.

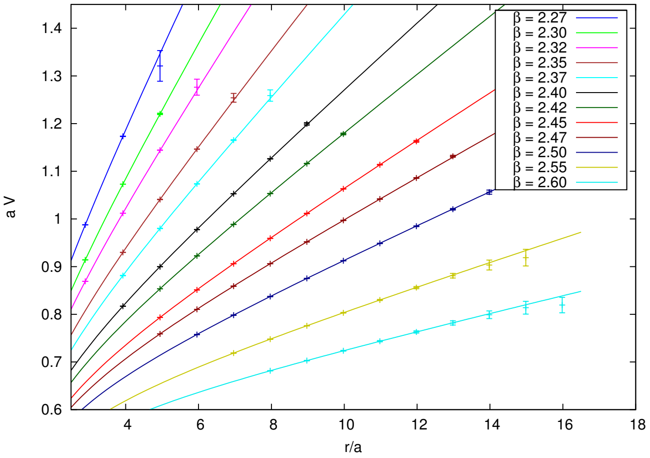
Scale setting

β	$r_{\min}/a$	σa^2	aV_0	χ^2_{red}
2.27	2.889	0.157(8)	0.626(14)	0.6
2.30	2.889	0.131(4)	0.627(30)	0.1
2.32	3.922	0.115(6)	0.627(32)	2.3
2.35	3.922	0.095(3)	0.623(20)	0.2
2.37	3.922	0.083(3)	0.621(18)	1.0
2.40	4.942	0.068(1)	0.617(10)	1.4
2.42	4.942	0.0593(4)	0.613(5)	0.1
2.45	4.942	0.0482(2)	0.608(4)	0.4
2.47	4.942	0.0420(4)	0.604(5)	0.3
2.50	5.954	0.0341(2)	0.599(2)	0.1
2.55	6.963	0.0243(13)	0.587(11)	0.2
2.60	7.967	0.0175(16)	0.575(16)	0.3

Table: *

Results for the string tension in units of the inverse squared lattice spacing at different values of the Wilson action parameter β (first column), calculated by fitting the potential V as a function of the tree-level improved interquark distance r to the Cornell form. V was extracted from Polyakov loop correlators on lattices of temporal extent $L_t = 32a$.

Scale setting

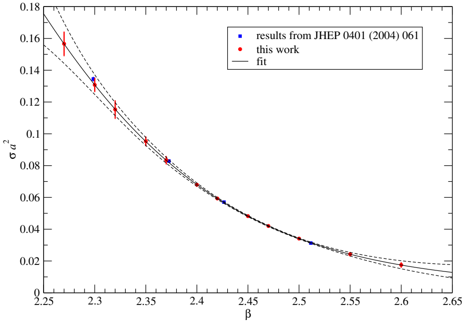


Scale setting

The values of the string tension are interpolated by a fit to

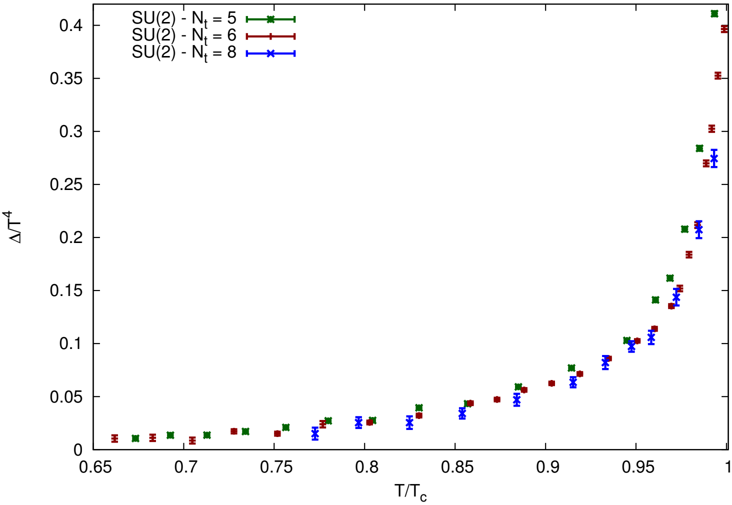
$$\log(\sigma a^2) = \sum_{j=0}^{n_{\mathrm{par}}-1} a_j(\beta - \beta_0)^j \qquad \text{with } \beta_0 = 2.35$$

which yields a χ^2_{red} of 0.01. It is presented below along with older data[1].



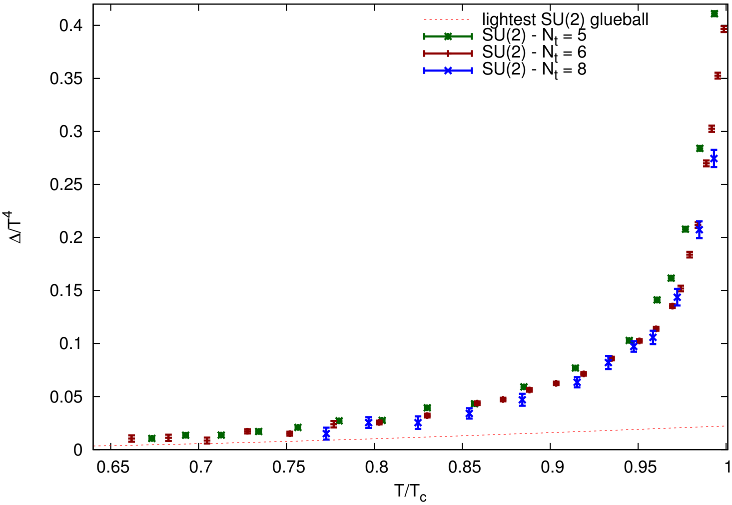
[1] B. Lucini, M. Teper, U. Wenger, *The high temperature phase transition in SU(N) gauge theories*, 2003

SU(2): trace of energy-momentum tensor



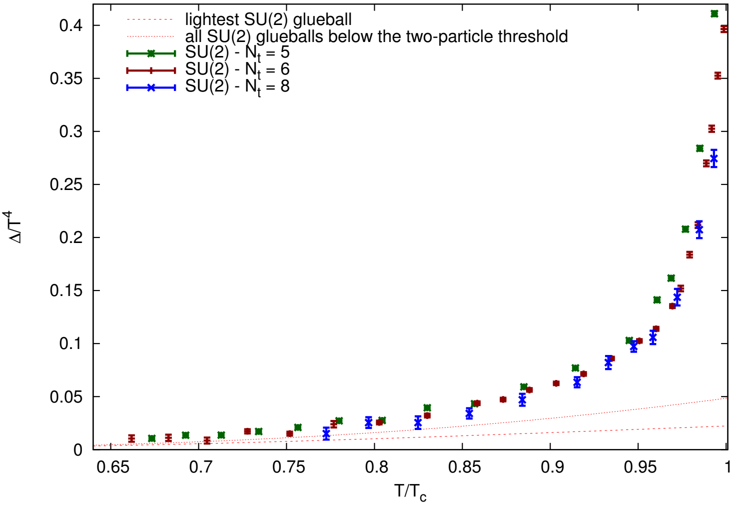
Despite the small values of N_t the data scale reasonably well.

SU(2): trace of energy-momentum tensor



Plot of the contribution of lowest glueball state $m_{0^{++}}$ compared with the data .

SU(2): trace of energy-momentum tensor



The contribution of all glueball states with mass $m < 2m_{0^{++}}$.

A few important observations

Usually the thermodynamics of the system is saturated by the first state (or, in some cases, the few lowest states) of the spectrum due to the exponential dependence on the mass.

The large gap between the $m_{0^{++}}$ and the $m < 2m_{0^{++}}$ curves and those between them and the data show that the spectrum must be of the Hagedorn type, i.e. that the number of states increases exponentially with the mass.

A Hagedorn spectrum is typically the signature of a string like origin of the spectrum.

The thermal behaviour of the model in the confining phase is thus a perfect laboratory to study the nature of this spectrum and of the underlying string model.

Effective string theory suggests that, with a very good approximation, this model should be a Nambu-Goto string. Let us see the consequences of this assumption.

Glueballs as rings of glue

A closed string model for the full glueball spectrum that follows the original work of Isgur and Paton[1,2] can be introduced to account for the values of thermodynamic quantities near the transition. In the closed-string approach glueballs are described in the limit of large masses as **"rings of glue"**, that is closed tubes of flux modelled by closed bosonic string states.

The **mass spectrum** of a closed strings gas in D spacetime dimensions is given by

$$m^2 = 4\pi\sigma \left(n_L + n_R - \frac{D-2}{12} \right)$$

where $n_L = n_R = n$ are the total contribution of left- and right-moving phonons on the string.

Every glueball state corresponds to a given phonon configuration, but associated to each fixed n there are multiple different states whose number is given by $\pi(n)$, i.e. the **partitions** of n.

[1] N. Isgur and J. Paton, *A Flux Tube Model for Hadrons in QCD*, 1985

[2] R. Johnson and M. Teper, *String models of glueballs and the spectrum of SU(N) gauge theories in (2+1)-dimensions*, 2002

The **density of states** $\rho(n)$ is expressed through the square of $\pi(n)$:

$$\rho(n) = \pi(n_L)\pi(n_R) = \pi(n)^2 \simeq 12\,(D-2)^{\frac{D-1}{2}} \left(\frac{1}{24n}\right)^{\frac{D+1}{2}} \exp\left(2\pi\sqrt{\frac{2(D-2)n}{3}}\right)$$

in D spacetime dimensions.

Spectral density

The **Hagedorn temperature**[1] is defined as

$$T_H = \sqrt{\frac{3\sigma}{\pi(D-2)}}$$

The spectral density as a function of the mass (i.e. $\hat{\rho}(m)\mathrm{d}m = \rho(n)\mathrm{d}n$) can be expressed as

$$\hat{\rho}(m) = \frac{(D-2)^{D-1}}{m} \left(\frac{\pi T_H}{3m}\right)^{D-1} e^{m/T_H}$$

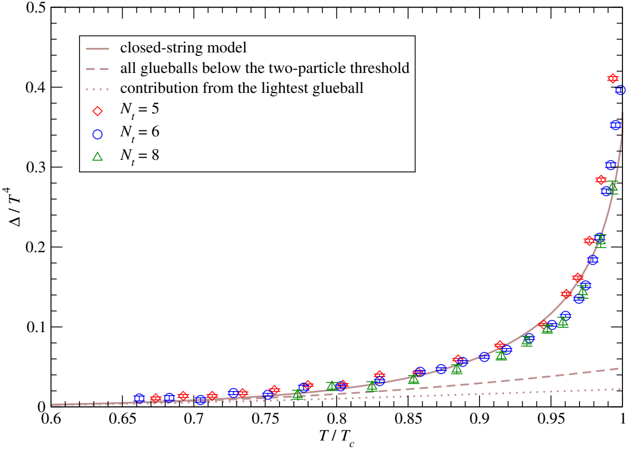
where the characteristic Hagedorn-like exponential spectrum appears and can be used to describe the glueball spectrum above an arbitrary mass threshold.

The trace of the energy-stress tensor can be integrated on masses bigger than $2m_{0^{++}}$ with the degeneracy $\hat{\rho}(m)$ and summed to the contribution of the mass states computed on the lattice.

$$\Delta = \sum_{m<2m_{0^{++}}} (2J+1)\Delta(m,T) + \int_{2m_{0^{++}}}^{\infty} \mathrm{d}m\, \hat{\rho}(m)\, \Delta(m,T)$$

[1] R. Hagedorn, Nuovo Cim. Suppl. **3**, 147 (1965)

SU(2): trace of energy-momentum tensor



SU(2) *vs.* SU(3)

The SU(3) case was studied for the first time in 2009 in the pioneering work of Meyer[1]. Now, using the high precision lattice data for SU(3) of [2] we are in the position to test the Hagedorn behaviour in a very stringent way.
There are two main diffeences between SU(2) and SU(3):

- SU(3) has a first order deconfining transition, so $T_c < T_H$.
- SU(3) has complex representations, thus glueballs have an additional quantum number C and the glueball spectrum contains twice the number of glueballs than in the SU(2) case

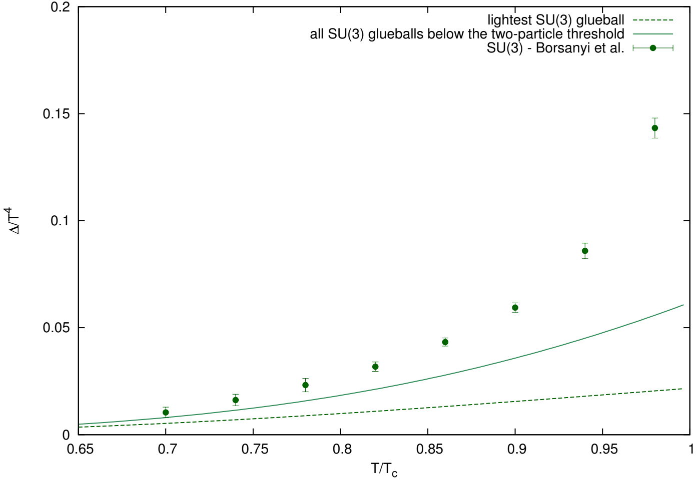
In principle we could consider in this case T_H as a free parameter, but in the effective string framework we may safely fix it at the expected Nambu-Goto value $T_H = \sqrt{3\sigma/2\pi} = 0.691..\sqrt{\sigma}$. Lorentz invariance of the effective string tells us that this should be a very good approximation of the exact result and that we should expect only small deviations from this value.
The relation between T_H and T_c is:

$$\frac{T_H}{T_c} = 1.098$$

[1] H. Meyer, *High-Precision Thermodynamics and Hagedorn Density of States*, 2009

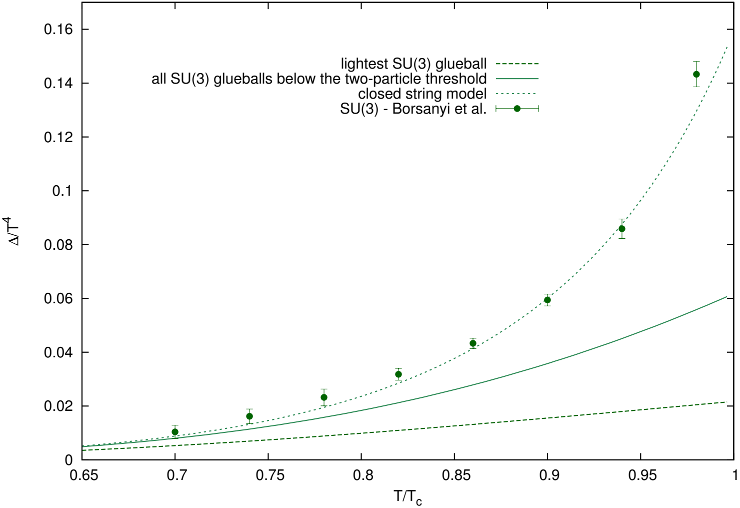
[2] Sz. Borsanyi *et al.*, *Precision SU(3) lattice thermodynamics for a large temperature range*, 2012

SU(3): trace of energy-momentum tensor



Also in this case the $m < 2m_{0^{++}}$ sector of the glueball spectrum is not enough to fit the behaviour of Δ/T^4 .

SU(3): trace of energy-momentum tensor



While including the whole Hagedorn spectrum we find again a remarkable agreement with no free parameter!

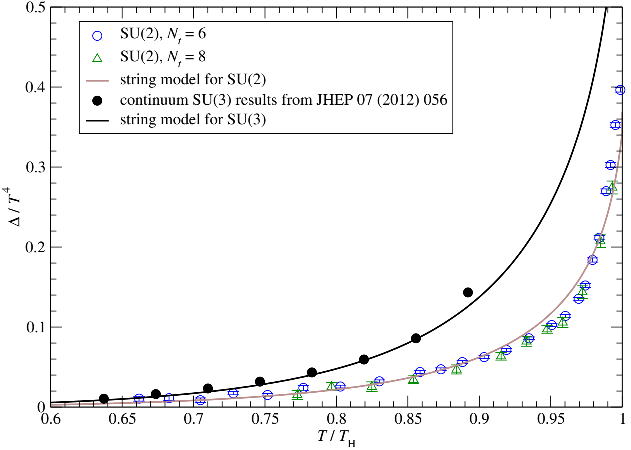
SU(2) *vs.* SU(3)

It is instructive to compare the SU(2) and SU(3) data

For $N = 3$ the closed flux tube has two possible orientations that account for the $\mathcal{C} = +1/-1$ sectors. Thus a further twofold degeneracy must be introduced in the string spectrum.

This doubling of the Hagedorn spectrum is clearly visible in the data

SU(2) *vs.* SU(3): results for trace of energy-momentum tensor



SU(N) Yang-Mills theories in $(2+1)$ dimensions

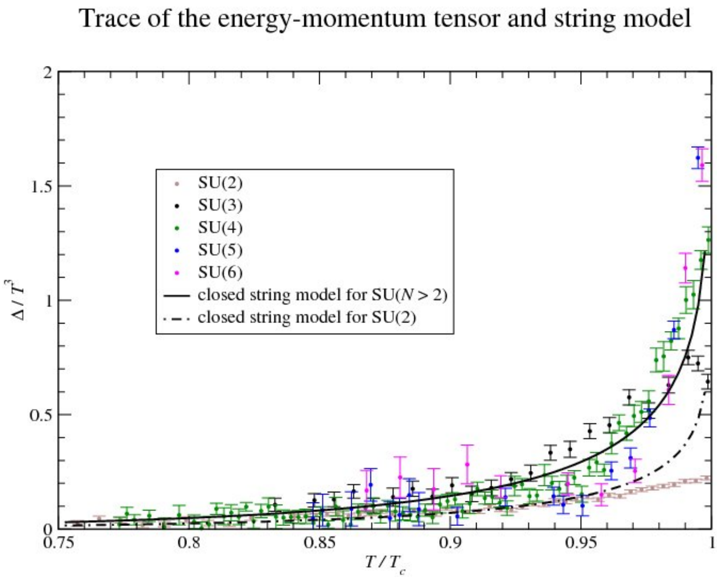
The same picture is confirmed by a study we performed a few years ago[1] in (2+1) dimensional SU(N) Yang-Mills theories for $N = 2, 3, 4, 6$.
Also in (2+1) dimensions we found that:

- a Hagedorn spectrum was mandatory to fit the thermodynamic data
- there was a jump between the SU(2) and the SU($N > 2$) case due to the doubling of the spectrum
- we had to fix the Hagedorn temperature to the Nambu-Goto value which, due to the different number of trensverse degrees of freedom is different from the (3+1) dimensional one: $T_H = \sqrt{3\sigma/\pi} = 0.977..\sqrt{\sigma}$

Moreover we found that in the vicinity of the critical point there was an excess of Δ/T^4 with respect to our predictions for $N = 4, 5$ and 6 and that this excess increases with N.
This could be understood as due to the k-string glueballs

[1] M. Caselle *et al.*, *Thermodynamics of SU(N) Yang-Mills theories in 2+1 dimensions I - The confining phase*, 2011

SU(N) Yang-Mills theories in $(2+1)$ dimensions



Summary of the first part of the Talk

- The thermodynamics of SU(2) and SU(3) Yang-Mills theories in $d = (3 + 1)$ is well described by a gas of non-interacting glueballs
- The agreement is obtained only assuming a Hagedorn spectrum for the glueballs
- The fine details of the spectrum, in particular the Hagedorn temperature, agree well with the predictions of the Nambu-Goto effective string.
- The results agree with previous findings in $d = (2 + 1)$ SU(N) Yang Mills theories with $N = 2, 3, 4, 5, 6$
- As N increases the data suggest the presence of extra states in the spectrum which could be k-glueballs states, which could be described by a k-string spectrum

Universality of the effective string action.

Why the Nambu-Goto string works so well?

- The main reason is that the Effective String action is strongly constrained by Lorentz invariance. The first few orders of the action are universal and coincide with those of the Nambu-Goto action. This explains why N.-G. describes so well the infrared regime of Wilson loops or Polyakov Loop correlators and the glueball spectrum.[1] [2] [3]

[1] M. Luscher and P. Weisz JHEP07(2004)014
[2] H. B. Meyer JHEP05(2006)066
[3] O. Aharony and M. Field JHEP01(2011)065

Lattice determination of the interquark potential.

In pure lattice gauge theories the interquark potential is usually extracted from two (almost) equivalent observables

- Wilson loop expectation values $< W(R,T) >$ ("zero temperature potential")

$$V(R) = \lim_{T\to\infty} -\frac{1}{T} \log < W(R,T) >$$

- Polyakov loop correlators $< P(0)P(R)^\dagger >$ ("finite temperature potential")

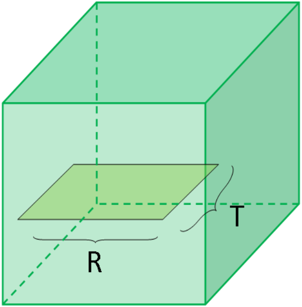
$$< P(0)P(R)^\dagger > \sim \sum_{n=0}^{\infty} c_n \ e^{-LE_n}$$

where L is the inverse temperature, i.e. the length of the lattice in the compactified imaginary time direction

$$E_0 = V(R) = - \lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^\dagger >$$

Wilson Loop.

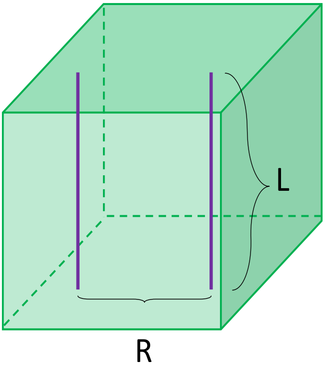
A Wilson loop of size $R \times T$



$$V(R) = \lim_{T \to \infty} -\frac{1}{T} \log < W(R,T) >$$

Polyakov loop correlator.

Expectation value of two Polyakov loops at distance R and Temperature $T = 1/L$



$$V(R) = -\lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^\dagger >$$

Wilson Loops.

In the Wilson loop framework confinement is equivalent to the well known area-perimeter-constant law:

$$< W(R,T) >= e^{-(\sigma RL + c(R+T)+k)}$$

which implies $V(R) = \sigma R + c$.
Confinement is usually associated to the creation (via a mechanism which still has to be understood) of a thin flux tube joining the quark antiquark pair. (Nielsen-Olesen, 't Hooft, Wilson, Polyakov, Nambu ....) However if we accept this picture we cannot neglect the quantum fluctuations of this flux tube. The area law is thus only the classical contribution to the interquark potential and we should expect quantum corrections to its form. The theory which describes these quantum fluctuations is known as "effective string theory".

Effective string action

The simplest choice for the effective string action is to describe the quantum fluctuations of the flux tube as free massless bosonic degrees of freedom

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X\right] ,$$

where:

- S_{cl} describes the usual ("classical") area-perimeter term.
- $X_i(\xi_0, \xi_1)$ $(i = 1, \ldots, d-2)$ parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand
- ξ_0, ξ_1 are the world-sheet coordinates.

The Lüscher term.

- The first quantum correction to the interquark potential is obtained summing over all the possible string configuration compatible with the Wilson loop (i.e. with Dirichlet boundary conditions along the Wilson loop).
- This is equivalent to the sum over all the possible surfaces borderd by the Wilson loop i.e. to the partition function

$$< W(R,T) >= \int e^{-\sigma RT - \frac{\sigma}{2}\int d^2\xi X^i(-\partial^2)X^i}$$

- The functional integration is a trivial gaussian integral, the result is

$$V(R) = \sigma R - \frac{(d-2)\pi}{24R} + c$$

- This quantum correction is known as "Lüscher term" and is universal i.e. it does not depend on the ultraviolet details of the gauge theory but only on the geometric properties of the flux tube.

The Lüscher term.

This correction is in remarkable agreement with numerical simulations. First high precision test in d=4 SU(3) LGT more than ten years ago. [1]

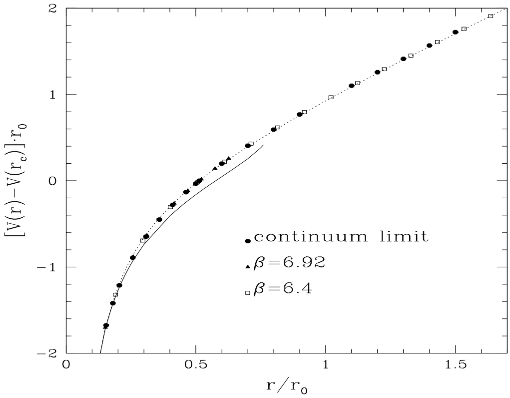


Figure: The static potential. The dashed line represents the bosonic string model and the solid line the prediction of perturbation theory.

[1] S. Necco and R. Sommer, Nucl.Phys. B622 (2002) 328

The Lüscher term.

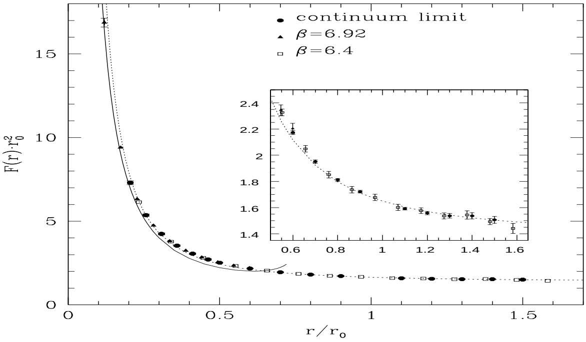


Figure: The force in the continuum limit and for finite resolution, where the discretization errors are estimated to be smaller than the statistical errors. The full line is the perturbative prediction. The dashed curve corresponds to the bosonic string model normalized by $r_0^2 F(r_0) = 1.65$.

The Nambu-Goto action.

- Evaluation of higher order quantum corrections requires further hypothesis on the nature of the flux tube. The simplest choice is the Nambu-Goto string in which quantum corrections are evaluated summing over all the possible surfaces bordered by the Wilson loop with a weight proportional to their area.

$$S = \sigma \int d^2\xi \sqrt{det(\eta_{\alpha\beta} + \partial_\alpha X \cdot \partial_\beta X)}$$

$$\sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8}(\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4}(\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right],$$

Interquark potential for the Nambu-Goto action.

- In the framework of the Nambu-Goto action one can evaluate exactly the energy of all the excited states of the flux tube:

$$E_n(R) = \sqrt{\sigma^2 R^2 + 2\pi\sigma\left(n - \frac{D-2}{24}\right)}$$

- In particular $E_0(R)$ corresponds to the interquark potential

$$V(R) = E_0(R) = \sqrt{\sigma^2 R^2 - 2\pi\sigma\frac{D-2}{24}} ,$$

$$V(R) \quad \sim \quad \sigma R - \frac{\pi(D-2)}{24R} - \frac{1}{2\sigma R^3}\left(\frac{\pi(D-2)}{24}\right)^2 + O(1/R^5) ,$$

The Nambu-Goto action.

High precision fit in the SU(2) case in 2+1 dimensions (A. Athenodorou, B. Bringoltz, M. Teper JHEP 1105:042 (2011))

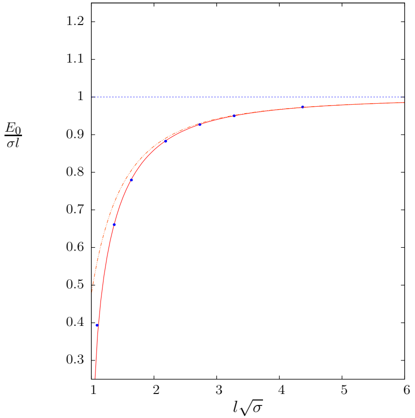
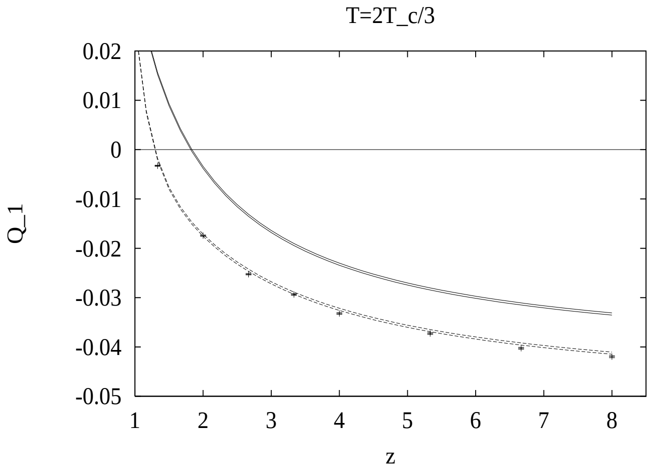


Figure 6: Energy of absolute ground state for SU(2) at $\beta = 5.6$. Compared to full Nambu-Goto (solid curve) and just the Lüscher correction (dashed curve).

The Nambu-Goto action.

High precision fit in the 2+1 dimensional Ising gauge model (M. Caselle, M. Hasenbusch, M. Panero JHEP 0301 (2003) 057)



Interquark potential via Polyakov Loop correlators and the Hagedorn temperature.

- In this case we have different boundary conditions in the two directions (space R and inverse temperature L).
- The novel feature of this observable is that by exchanging R and L (the so called "open-closed string transformation") we can study the finite temperature behaviour of the string tension.

$$V(R) = \sigma(T)R, \quad \sigma(T) = \sigma_0\sqrt{1 - \frac{(d-2)\pi T^2}{3\sigma_0}}$$

where T is now the temperature and σ_0 the zero temperature string tension

- From this expression we may deduce a "Nambu-Goto" prediction for the critical temperature:

$$\frac{T_c}{\sqrt{\sigma_0}} = \sqrt{\frac{3}{(d-2)\pi}}$$

which turns out to be in remarkable agreement with LGT results both in d=3 and d=4.

Universality of effective string corrections.

The reason of the success of the Nambu-Goto approximation is that the effective String action is strongly constrained by Lorentz invariance[1] [2] [3]. The first few orders of the action are universal and coincide with those of the Nambu-Goto action.

The most general action for the effective string can be written as a low energy expansion in the number of derivatives of the transverse fields ("physical gauge").

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + c_2 (\partial_\alpha X \cdot \partial^\alpha X)^2 + c_3 (\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right] + S_b \,,$$

where:

- S_{cl} describes the usual ("classical") perimeter-area term.
- S_b is the boundary contribution characterizing the open string
- $X_i(\xi_0, \xi_1)$ $(i = 1, \dots, d-2)$ parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand
- ξ_0, ξ_1 are the world-sheet coordinates.
- In the Nambu-Goto case $c_2 = \frac{1}{8}$ and $c_3 = -\frac{1}{4}$

[1] M. Luscher and P. Weisz JHEP07(2004)014

[2] H. B. Meyer JHEP05(2006)066

[3] O. Aharony and M. Field JHEP01(2011)065

Effective string and spacetime symmetries.

- Symmetries of the action must hold in the low energy regime.
- String vacuum is not Poincaré invariant.

$\Longrightarrow$ Poincaré symmetry is broken spontaneously.

$ISO(D-1,1) \rightarrow SO(D-2) \otimes ISO(1,1)$. $\Longrightarrow$ $3(D-2)$ Goldstone bosons?

Only $D-2$ tranverse fluctuations of the string, where are the remaining Goldstone bosons?

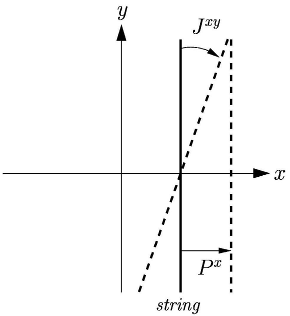
Goldstone's theorem states that there is a massless mode for each broken symmetry generator, but this counting cannot be naively extended to the case of spontaneously broken spacetime symmetries[1] .

[1] I. Low and A.V. Manohar, "Spontaneously broken spacetime symmetries and Goldstone's theorem" Phys.Rev.Lett. 88 (2002) 101602

Effective string and spacetime symmetries.

The remaining $2(D-2)$ Lorentz transformations are realized non-linearly and induce a set of recurrence relations among different terms in the action.! [1]

$$\delta_\epsilon^{bj} X_i = \epsilon \left(-\delta_{ij}\xi_b - X_j \partial_b X_i\right)$$



[1] I. Low and A.V. Manohar, "Spontaneously broken spacetime symmetries and Goldstone's theorem" Phys.Rev.Lett. 88 (2002) 101602

Non-linear realization and long-string expansion.

A few rules to construct the most general effective string action:

- Broken translations:
  $X^i \to X^i + a^i$. $\Longrightarrow$ Only field derivatives in the effective action.
- Broken rotation in the plane $(1,2)$:

$$\delta_\epsilon^{bj} X_i = \epsilon \left(-\delta_{ij} \xi_b - X_j \partial_b X_i \right)$$

Number of derivatives minus number of fields (weight) preserved.

Fields and coordinates rescaling $\Longrightarrow$ Derivative expansion:

$$\partial_a X^i \longrightarrow \frac{1}{\sqrt{\sigma} R} \partial_a X^i .$$

Variations by broken rotation mix orders $\Longrightarrow$ Recurrence relations.

$ISO(1,1)$ and $SO(D-2)$ invariance $\Longrightarrow$ Contraction of indices.

Effective string action is strongly constrained!

- The terms with only first derivatives coincide with the Nambu-Goto action to all orders in the derivative expansion.
- The first allowed correction to the Nambu-Goto appears at a very large order and, in SU(N) gauge models gives a contribution to the interquark potential of the order $1/R^5$ in $d = 3 + 1$ and $1/R^7$ in $d = (2 + 1)$ which are almost negligible[1] [2] [3].
(This is not the case for the (2+1) U(1) model, but this is another story...) [4]
- The effective string action is much more predictive than typical effective models in particle physics!
- The fact that the first deviations from the Nambu-Goto string are of such high order explains why it works so well both in describing the interquark potential and the glueball spectrum

[1] M. Luscher and P. Weisz JHEP07(2004)014
[2] H. B. Meyer JHEP05(2006)066
[3] O. Aharony and M. Field JHEP01(2011)065
[4] M. Caselle, M.Panero, R. Pellegrini, D. Vadacchino, JHEP 1501 (2015) 105

Conclusions

- The thermodynamics of SU(2) and SU(3) Yang-Mills theories in $d = (3 + 1)$ is well described by a gas of non-interacting glueballs
- The agreement is obtained only assuming a Hagedorn spectrum for the glueballs
- The fine details of the spectrum, in particular the Hagedorn temperature, agree well with the predictions of the Nambu-Goto effective string.
- The special role played by the Nambu-Goto string can be understood in the framework of the effective string approach to the infrared regime of confining gauge theories and is a direct consequence of the non-linear realization of the Lorentz invariance of these theories.
- The results agree with previous findings in $d = (2 + 1)$ SU(N) Yang Mills theories with $N = 2, 3, 4, 5, 6$
- As N increases the data suggest the presence of extra states in the spectrum which could be k-glueballs states, whcih could be described by a k-string spectrum
- Overall the behaviour of thermodynamic observables in the confining regime of Yang-Mills theories turns out to be an highly non trivial test of the effective string picture of confinement.

Acknowledgements

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Geometrical description.

A more intuitive geometrical description of this result is obtained using the original string action, without fixing the physical gauge.
The effective action is given by the most general mapping:

$$X^\mu : \mathcal{M} \to \mathbb{R}^D, \qquad \mu = 0, \cdots, D-1$$

- $\mathcal{M}$: two-dimensional surface describing the worldsheet of the string
- $\mathbb{R}^D$: (flat) D dimensional target space $\mathbb{R}^D$ of the gauge theory.

Main Result [1] :

- The first few terms of the action compatible with Poincaré and parity invariance are suitable combinations of geometric invariants constructed from the induced metric $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$.
- These terms can be classified according to their weight, i.e. the difference between the number of derivatives minus the number of fields X^μ

[1] O. Aharony and Z. Komargodski, JHEP **1305** (2013) 118

Geometrical description.

- The only term of weight zero is the Nambu-Goto action

$$S_{NG} = \sigma \int d^2\xi \sqrt{g} \ ,$$

 where $g \equiv \det(g_{\alpha\beta})$.

- This term has a natural geometric interpretation: it measures the area swept out by the worldsheet in space-time.
- Fixing the physical gauge one finds (choosing an euclidean metric)

$$S = \sigma \int d^2\xi \sqrt{det(\eta_{\alpha\beta} + \partial_\alpha X \cdot \partial_\beta X)}$$

$$\sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8}(\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4}(\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right] ,$$

Geometrical description.

- At weight two, two new contributions appear:

$$S_{2,\mathcal{R}} = \gamma \int d^2\xi \sqrt{g}\mathcal{R} \ ,$$

$$S_{2,K} = \alpha \int d^2\xi \sqrt{g} K^2,$$

where $\mathcal{R}$ denotes the Ricci scalar constructed from the induced metric, and $K \equiv \Delta(g)X$ is the extrinsic curvature, where $\Delta(g)$ is the Laplacian in the space with metric $g_{\alpha\beta}$.

However both these terms can be neglected!

- $\mathcal{R}$ is topological in two dimensions and, since in the long string limit in which we are interested we do not expect topologically changing fluctuations, its contribution is constant and can be neglected.
- In ordinary Yang-Mills theories K^2 only gives exponentially suppressed corrections and can be neglected.

Evaluation of the Lüscher term.

- The gaussain integration gives:

$$\int e^{-\frac{\sigma}{2}\int d^2\xi X^i(-\partial^2)X^i} \propto \left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} \quad .$$

- The determinant must be evaluated with Dirichlet boundary conditions. The spectrum of $-\partial^2$ with Dirichlet boundary conditions is:

$$\lambda_{mn} = \pi^2 \left(\frac{m^2}{T^2} + \frac{n^2}{R^2} \right)$$

corresponding to the normalized eigenfunctions

$$\psi_{mn}(\xi) = \frac{2}{\sqrt{RT}} \sin \frac{m\pi\tau}{T} \sin \frac{n\pi\varsigma}{R} \quad .$$

Evaluation of the Lüscher term.

- The determinant can be regularized with the ζ-function technique: defining

$$\zeta_{-\partial^2}(s) \equiv \sum_{mn=1}^{\infty} \lambda_{mn}^{-s}$$

the regularized determinant is defined through the analytic continuation of $\zeta'_{-\partial^2}(s)$ to $s = 0$:

$$\det(-\partial^2) = \exp\left[-\zeta'_{-\partial^2}(0)\right] \quad .$$

- The result is

$$\left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} = \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}} \quad .$$

where $\eta(\tau)$ is the Dedekind function

$$\eta(\tau) = q^{1/24}\Pi_{n=1}^{\infty}(1-q^n)$$

with $q \equiv e^{2\pi i\tau}$ and $\tau = iT/R$.

Derivation of the Nambu-Goto action.

- The Nambu-Goto action is given by the area of the world–sheet:

$$S = \sigma \int_0^T d\tau \int_0^R d\varsigma \sqrt{g} \quad ,$$

where g is the determinant of the two–dimensional metric induced on the world–sheet by the embedding in R^d:

$$g = \det(g_{\alpha\beta}) = \det\ \partial_\alpha X^\mu \partial_\beta X^\mu \quad (\alpha, \beta = \tau, \varsigma,\ \mu = 1, \dots, d)$$

- Choosing the "physical gauge"

$$X^1 = \tau \quad X^2 = \varsigma$$

g may be expressed as a function of the transverse degrees of freedom only:

$$\begin{aligned} g &= 1 + \partial_\tau X^i \partial_\tau X^i + \partial_\varsigma X^i \partial_\varsigma X^i \\ &\quad + \partial_\tau X^i \partial_\tau X^i \partial_\varsigma X^j \partial_\varsigma X^j - (\partial_\tau X^i \partial_\varsigma X^i)^2 \ (i = 3, \dots, d) \ . \end{aligned}$$

- Expanding we find:

$$S \sim \sigma R T + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8} (\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4} (\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right] ,$$