

# The $(2, 0)$ Superconformal Bootstrap

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## $(2, 0)$ theories

Nahm's classification: **superconformal algebras** exist for  $d \leq 6$ .

In  $d = 6$ ,  $(\mathcal{N}, 0)$  algebras. Existence of  $T_{\mu\nu}$  multiplet requires  $\mathcal{N} \leq 2$ .

$(2, 0)$ : maximal susy in maximal  $d$ . No marginal couplings allowed.

*Interacting* models inferred from string/M-theory: **ADE catalogue**.

Central to many recent developments in QFT.

“Mothers” of many interesting QFTs in  $d < 6$ .

Key properties:

- Moduli space of vacua

$$\mathcal{M}_{\mathfrak{g}} = (\mathbb{R}^5)^{r_{\mathfrak{g}}} / W_{\mathfrak{g}}, \quad \mathfrak{g} = \{A_n, D_n, E_6, E_7, E_8\}.$$

- On  $\mathbb{R}^5 \times S^1$ , IR description as  $5d$  MSYM with gauge algebra  $\mathfrak{g}$ .

At **large  $n$** ,  $A_n$  and  $D_n$  theories described through AdS/CFT:

M-theory on  $AdS_7 \times S^4$  and  $AdS_7 \times \mathbb{RP}^4$ .

# The $(2, 0)$ theories as abstract CFTs

No intrinsic field-theoretic formulation yet.

No conventional Lagrangian (hard to imagine one from RG lore).

*Working hypothesis:* (at least) for correlators of local operators in  $\mathbb{R}^6$ , the  $(2, 0)$  theory is just another CFT, defined by a local operator algebra

$$\text{OPE : } \mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k c_{12k}(x)\mathcal{O}_k(0)$$

Can symmetry and basic consistency requirements *completely determine* the spectrum and OPE coefficients?

# Abstract CFT Framework

A general Conformal Field Theory hasn't much to do with “fields” (of the kind we write in Lagrangians).

We'll think more abstractly. A CFT is *defined* by its **local operators**,

$$\mathcal{A} \equiv \{\mathcal{O}_k(x)\},$$

and their correlation functions  $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$ .

**$\mathcal{A}$  is an algebra.** Operator Product Expansion (OPE),

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k c_{12k} (\mathcal{O}_k(0) + \dots),$$

where the  $\dots$  are fixed by conformal invariance. The sum **converges**.

**Caveat I:** This definition does *not* capture non-local observables, such as conformal defects. (E.g., Wilson lines in a conformal gauge theory.).

Reduce  $n$ pt to  $(n - 1)$ pt,

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \dots \mathcal{O}_n(x_n) \rangle = \sum_k c_{12k}(x_2) \langle \mathcal{O}_k(x_2) \dots \mathcal{O}_n(x_n) \rangle.$$

1pt functions are trivial,  $\langle \mathcal{O}_i(x) \rangle = 0$  except for  $\langle 1 \rangle \equiv 1$ .

$\mathcal{O}_{\Delta, \ell, f}(x)$  labeled by **conformal dimension**  $\Delta$ , Lorentz representation  $\ell$  and possibly flavor quantum number  $f$ .

The **CFT data**  $\{(\Delta_i, \ell_i, f_i), c_{ijk}\}$  completely specify the theory.

But not anything goes! Consistency conditions:

- **Associativity:**

$$(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}_3 = \mathcal{O}_1 (\mathcal{O}_2 \mathcal{O}_3).$$

- **Unitarity** (reflection positivity):

Lower bounds on  $\Delta$  for given  $\ell$ ;

$$c_{ijk} \in \mathbb{R}$$

**Caveat II:** In non-trivial geometries,  $\langle \mathcal{O} \rangle \neq 0 \rightarrow$  additional constraints.

In  $d = 2$ , **modularity**. In  $d > 2$ , harder to analyze, have been ignored so far.

# The bootstrap program

Old aspiration (1970s) Ferrara Gatto Grillo, Polyakov.

Associativity  $\equiv$  crossing symmetry of  $4pt$  functions

The diagram illustrates the crossing symmetry of 4-point functions. On the left, a sum over  $\mathcal{O}$  of a four-point function with external legs 1, 2, 3, and 4, where legs 1 and 2 meet at a vertex and legs 3 and 4 meet at another vertex, connected by a horizontal line labeled  $\mathcal{O}$ . This is equal to a sum over  $\mathcal{O}'$  of a four-point function where legs 1 and 3 meet at a top vertex and legs 2 and 4 meet at a bottom vertex, connected by a vertical line labeled  $\mathcal{O}'$ .

Vastly **over-constrained** system of equations for  $\{\Delta_i, c_{ijk}\}$ .

Classification and construction of CFTs reduced to an **algebraic** problem.

- Famous success story in  $d = 2$ , starting from BPZZ (1984).  
 $2d$  conformal symmetry is infinite dimensional,  $z \rightarrow f(z)$ .  
In some cases, *finite*-dimensional bootstrap problem (rational CFTs).  
Many exact solutions, partial classification.

# Bootstrapping in two steps

For  $d = 6$ ,  $\mathcal{N} = (2, 0)$  SCFTs (as well as  $d = 4$ ,  $\mathcal{N} \geq 2$  SCFTs) the crossing equations **split** into

- (1) Equations that depend only on the **intermediate BPS operators**. Captured by the  $2d$  chiral algebra.  
“Minibootstrap”
- (2) Equations that also include **intermediate non-BPS operators**.  
“Maxibootstrap”

(1) are tractable and determine an infinite amount of CFT data.

This is essential input to the full-fledged bootstrap (2), which can be studied numerically.

Beem Lemos Liendo Peelaers LR van Rees, Beem LR van Rees

# Meromorphy in $(2, 0)$ SCFTs

Fix a plane  $\mathbb{R}^2 \subset \mathbb{R}^6$ , parametrized by  $(z, \bar{z})$ .

**Claim** :  $\exists$  subsector  $\mathcal{A}_\chi = \{\mathcal{O}_i(z_i, \bar{z}_i)\}$  with **meromorphic**

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \dots \mathcal{O}_n(z_n, \bar{z}_n) \rangle = f(z_i).$$

**Rationale**:  $\mathcal{A}_\chi \equiv$  cohomology of a nilpotent  $\mathbb{Q}$ ,

$$\mathbb{Q} = \mathcal{Q} + \mathcal{S},$$

$\mathcal{Q}$  Poincaré,  $\mathcal{S}$  conformal supercharges.

$\bar{z}$  dependence is  $\mathbb{Q}$ -exact: cohomology classes  $[\mathcal{O}(z, \bar{z})]_{\mathbb{Q}} \rightsquigarrow \mathcal{O}(z)$ .

Analogous to the  $d = 4$ ,  $\mathcal{N} = 1$  chiral ring:

cohomology classes  $[\mathcal{O}(x)]_{\tilde{\mathcal{Q}}_{\tilde{\alpha}}}$  are  $x$ -independent.



## $\chi_6$ : 6d (2,0) SCFT $\longrightarrow$ 2d Chiral Algebra.

- Global  $\mathfrak{sl}(2) \rightarrow$  Virasoro, indeed  $T(z) := [\Phi_{(IJ)}(z, \bar{z})]_{\mathbb{Q}}$ , with  $\Phi_{(IJ)}$  the stress-tensor multiplet superprimary.

$$c_{2d} = c_{6d}$$

in normalizations where  $c_{6d}$  (free tensor)  $\equiv 1$ .

- All  $\frac{1}{2}$ -BPS operators ( $\Delta = 2R$ ) are in  $\mathbb{Q}$  cohomology.  
Generators of the  $\frac{1}{2}$ -BPS ring  $\rightarrow$  generators of the chiral algebra.
- Some semi-short multiplets also play a role.

## Chiral algebra for $(2, 0)$ theory of type $A_{N-1}$

One  $\frac{1}{2}$ -BPS generator each of dimension  $\Delta = 4, 6, \dots, 2N$



One chiral algebra generator each of dimension  $h = 2, 3, \dots, N$ .

Most economical scenario: these are **all** the generators.

Check: the superconformal index computed by Kim<sup>3</sup> is reproduced.

Plausibly a **unique** solution to crossing for this set of generators.

- The chiral algebra of the  $A_{N-1}$  theory is  $\mathcal{W}_N$ , with

$$c_{2d} = 4N^3 - 3N - 1.$$

(Similar proposals for  $D$  and  $E$  theories).

# Half-BPS 3pt functions of $(2, 0)$ SCFT

OPE of  $\mathcal{W}_g$  generators  $\Rightarrow$  half-BPS 3pt functions of SCFT.

Let us check the result at **large  $N$** .

$W_{N \rightarrow \infty}$  with  $c_{2d} \sim 4N^3 \rightarrow$  a *classical* Poisson algebra.

We can use results on universal Poisson algebra  $W_\infty[\mu]$ , with  $\mu = N$ .  
(Gaberdiel Hartman, Campoleoni Fredenhagen Pfenninger)

We find

$$C(k_1, k_2, k_3) = \frac{2^{2\alpha-2}}{(\pi N)^{\frac{3}{2}}} \Gamma\left(\frac{\alpha}{2}\right) \left( \frac{\Gamma\left(\frac{k_{123}+1}{2}\right) \Gamma\left(\frac{k_{231}+1}{2}\right) \Gamma\left(\frac{k_{312}+1}{2}\right)}{\sqrt{\Gamma(2k_1-1)\Gamma(2k_2-1)\Gamma(2k_3-1)}} \right)$$

$$k_{ijk} \equiv k_i + k_j - k_k, \quad \alpha \equiv k_1 + k_2 + k_3,$$

in precise agreement with calculation in **11d sugra on  $AdS_7 \times S^4$** !

(Corrado Florea McNees, Bastianelli Zucchini)

**$1/N$**  corrections in  $W_N$  OPE  $\Rightarrow$  quantum M-theory corrections.

**Universal** 4pt function of  $\Phi_{(IJ)}$ , superprimary of  $T_{\mu\nu}$  multiplet.

Unique structure in superspace.

Only input:  $6d$  Weyl anomaly coefficient  $c$ .

For ADE theories,

$$c = 4d_{\mathfrak{g}} h_{\mathfrak{g}}^{\vee} + r_{\mathfrak{g}} ,$$

but we keep it general.

# Double OPE expansion

$$\langle \Phi \Phi \Phi \Phi \rangle = \sum_{\mathcal{O} \in \Phi \times \Phi} f_{\Phi \Phi \mathcal{O}}^2 G_{\mathcal{O}}^{\Phi}$$

We **impose** the absence of **higher-spin currents**.

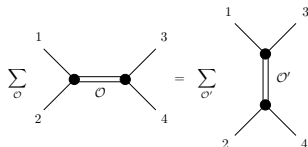
The  $\mathcal{O}$ s  $\in \Phi \times \Phi$  are:

- Infinite set  $\{\mathcal{O}_{\chi}\}$  of  $\mathbb{Q}$ -chiral BPS multiplets, fixed from  $\chi$ -algebra.
- Infinite tower of BPS multiplet  $\{\mathcal{D}, \mathcal{B}_1, \mathcal{B}_3, \dots\}$ , *not* in  $\chi$ -algebra.
- Infinite set of non-BPS multiplets  $\mathcal{L}_{\Delta, \ell}$ ,  $\mathfrak{so}(5)_R$  singlets.  
Bose symmetry  $\rightarrow \ell$  is *even*. Unitarity bound  $\Delta \geq \ell + 6$ .

Unfixed BPS multiplets correspond to long multiplets at threshold,

$$\lim_{\Delta \rightarrow \ell + 6} G_{\mathcal{L}_{\Delta, \ell}}^{\Phi} = G_{\mathcal{B}_{\ell-1}}^{\Phi} \quad (\mathcal{D} \equiv \mathcal{B}_{-1}),$$

# Bootstrap sum rule



When the dust settles, a *single* sum rule

$$\sum_{\text{long } \textcolor{blue}{super}\text{primaries}} f_{\Delta,\ell}^2 \mathcal{F}_{\Delta,\ell}(z, \bar{z}) + \textcolor{red}{\mathcal{F}^x}(z, \bar{z}; c) = 0$$

$z, \bar{z}$ : conformal cross ratios;

$\mathcal{F}_{\Delta,\ell} \equiv \mathcal{G}_{\Delta,\ell} - \mathcal{G}_{\Delta,\ell}^\times$ : *super*conformal block minus its crossing;

$\textcolor{red}{\mathcal{F}^x}(z, \bar{z}; c)$ : an explicitly known function (from minibootstrap).

The unknown CFT data to be constrained are:

- Set of (dimension, spin)  $\{(\Delta_i, \ell_i)\}$  of the intermediate multiplets.
- The (squared) OPE coefficients  $f_{\Delta_i, \ell_i}^2$ . *Non-negative* by unitarity.

# The numerical oracle (Rattazzi Rychkov Tonni Vichi)

$$\sum_{(\Delta_i, \ell_i)} f_{\Delta, \ell}^2 \mathcal{F}_{\Delta, \ell}(z, \bar{z}) + \mathcal{F}^{\text{known}}(z, \bar{z}; c) = 0$$

Use the sum rule to **constrain** the space of CFT data.

For example, consider a **trial spectrum with**  $\Delta \geq \bar{\Delta}_\ell$  for operators of spin  $\ell$ .  
If there exists a linear functional  $\chi$  such that

$$\chi \cdot \mathcal{F}_{\Delta, \ell}(z, \bar{z}) \geq 0 \quad \text{when } \Delta \geq \bar{\Delta}_\ell$$

$$\chi \cdot \mathcal{F}^{\text{known}}(z, \bar{z}; c) = 1$$

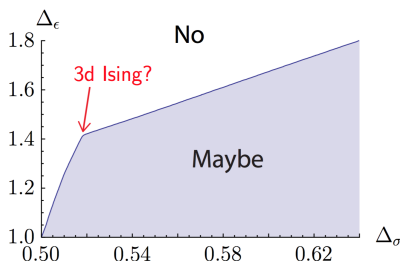
that trial spectrum is **ruled out** – oracle says NO.

If one cannot find such a  $\chi$ , oracle says MAYBE.

Implemented by linear programming or semi-definite programming.  
Surprisingly powerful!

# Scalar bound in general $d = 3$ CFT

[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi, PRD 86, 025022]



Exclusion plot in the subspace of  $d = 3$  CFT data  $(\Delta_\sigma, \Delta_\epsilon)$  with  $\sigma \times \sigma = \mathbf{1} + \epsilon + \dots$ , from the bootstrap of a single 4pt function  $\langle \sigma\sigma\sigma\sigma \rangle$ .

Two real **surprises**:

- $3d$  Ising appears to lie on the exclusion curve (i.e. it saturates the bound)
- $3d$  Ising appears to sit at a special kink on the exclusion curve.

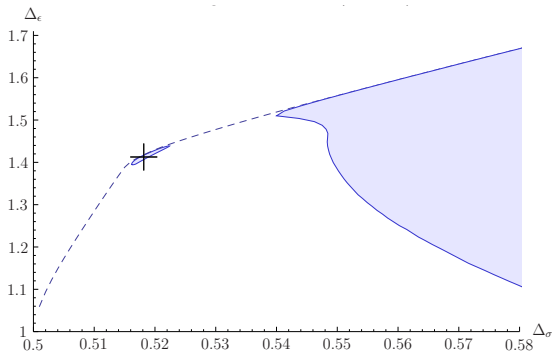


## Multiple Correlators [Kos, Poland, Simmons-Duffin, '14]

CFT<sub>3</sub> with  $\mathbb{Z}_2$  symmetry.  $\sigma$  odd,  $\epsilon$  even,  $\sigma \times \sigma = \mathbf{1} + \epsilon + \dots$

System of correlators  $\langle \sigma \sigma \sigma \sigma \rangle$ ,  $\langle \sigma \sigma \epsilon \epsilon \rangle$ ,  $\langle \epsilon \epsilon \epsilon \epsilon \rangle$ .

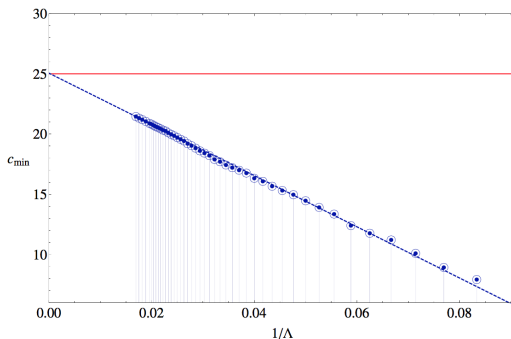
Allowed region assuming that **only one** odd scalar is relevant ( $\Delta_{\sigma'} \geq 3$ ):



**3d Ising gets cornered!**  $\Delta_{\sigma} = 0.518151(5)$ ,  $\Delta_{\epsilon} = 1.41263(5)$ ,  
most accurate to date [Simmons-Duffin '15]

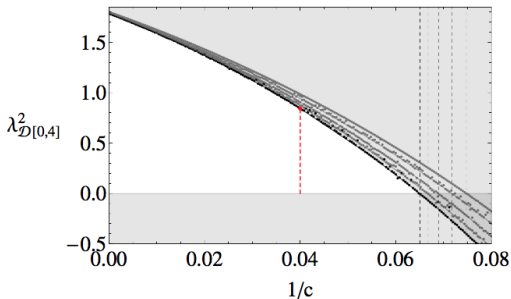
## A lower bound on $c$

There is a **minimum** anomaly  $c_{min}$  compatible with crossing and unitarity. The bound  $c_{min}$  increases as we increase the search space for the functional, parametrized by a cutoff  $\Lambda$ .



Extrapolating,  $c_{min} \rightarrow 25$ , the value of the  $A_1$  theory ( $\equiv$  two  $M5$ s)!  
(We are disallowing the free theory ( $c = 1$ ) by forbidding HS currents.)

For  $c < c_{min}$ , the oracle says NO. Why?



For  $c < c_{min}$ , solutions to crossing have  $\lambda_D^2 < 0$ , violating unitarity.

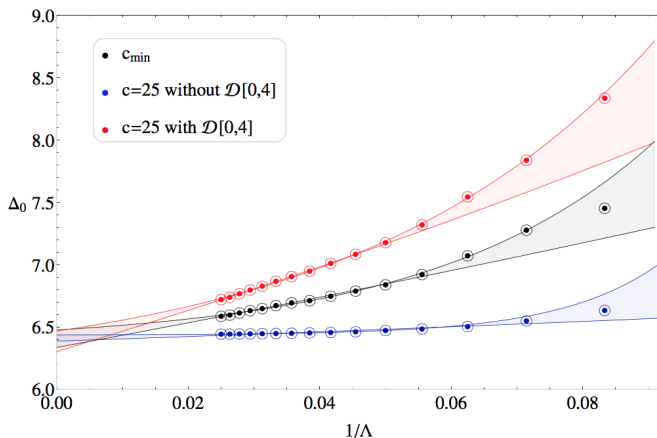
$\lambda_D^2 = 0$  precisely at  $c = c_{min}$ .

Agrees with conjecture of [Batthacharyya and Minwalla](#) about  $\frac{1}{4}$ BPS partition function of  $A_1$  theory:  $\mathcal{D}$  multiplet absent!

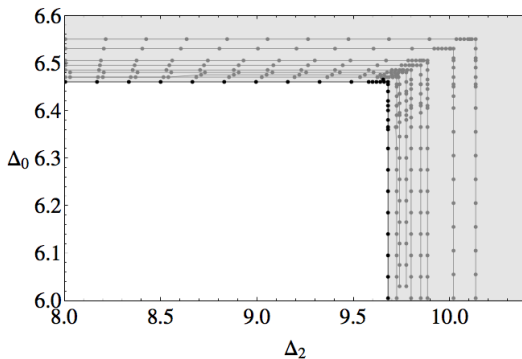
# Bootstrapping the $A_1$ theory

For  $c = c_{\min} \rightarrow 25$ ,  $\exists$  **unique** unitary solution to crossing.

Claim: The  $A_1$  theory can be completely bootstrapped!



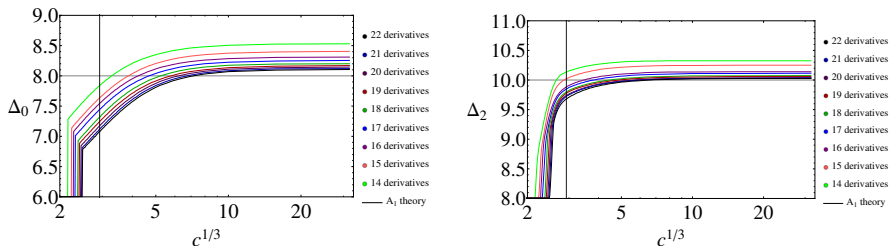
Upper bounds on the dimension of the leading-twist unprotected scalar, under different assumptions. Perfectly consistent.



Exclusion region in  $(\Delta_0, \Delta_2)$  plane for  $c = 25$  ( $A_1$  value).

The corner values are conjectured to be the **true** leading-twist dimensions of the physical  $A_1$  theory.

# General $c$



Bounds for the leading-twist unprotected operators of spin  $\ell = 0, 2$ .

For  $c \rightarrow \infty$ , they appear to be saturated by  $AdS_7 \times S^4$  sugra, including  $1/c$  corrections.

For large  $c$ , leading-twist unprotected operators are double-traces of the form  $\mathcal{O}_s = \mathcal{O}_{14} \partial^s \mathcal{O}_{14}$ , with  $\Delta_s = 8 + s - O(1/c)$ .

**Summary:** Both at small and large  $c$  the bootstrap bounds appear to be saturated by **physical**  $(2, 0)$  theories.

# Outlook

The  $(2, 0)$  theories can be successfully studied by bootstrap methods.

- Exact results from the chiral algebra, e.g.  $\frac{1}{2}$  BPS  $3pt$  functions.

Systematic  $1/N$  expansion and its M-theory interpretation?

A derivation of the AGT correspondence?

Codimension-two defects  $\Rightarrow$  Toda vertex operators?

- Numerical results for the non-protected spectrum.

$A_1$  theory in principle completely cornered by bootstrap equations.  
Beginning of a systematic algorithm to solve it.

$A_{n>1}$  theories need input on BPS spectrum and multiple correlators.

Precision numerics? Multiple correlators?

Further analytic insights?