Aspects of three-dimensional gauge theories

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[work in collaboration with F. Benini, K. Hristov]

Introduction

In recent years we have seen many progresses in the study of supersymmetric gauge theories in various dimensions

- dualities, exotic CFT's
- exact computation of partition functions of supersymmetric field theories on curved spaces
- several different types of Witten and superconformal indices

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  dominates the generating function for the chiral ring [Cremonesi,Hanany,AZ '13]

Today I will focus on a particular aspect, a two-faced story about 3d gauge theories and black holes.

Introduction

I. On the field theory side, consider 3d gauge theories on $S^2 \times S^1$ where susy is preserved by a twist on S^2

$$(\nabla_\mu - iA^R_\mu)\epsilon \equiv \partial_\mu \epsilon = 0\,, \qquad \int_{S^2} F^R = 1$$

[Witten '88]

The result becomes interesting when supersymmetric backgrounds for the flavor symmetry multiplets (A^F_μ, σ^F, D^F) are turned on:

$$u^F = A^F_t + i\sigma^F\,, \qquad q^F = \int_{S^2} F^F = iD^F$$

and the path integral, which can be exactly computed by localization, becomes a function of a set of magnetic charges q^F and chemical potentials u^F.

[Benini-AZ; arXiv 1504.03698]

Introduction

It can be re-interpreted as a twisted index: a trace over the Hilbert space $\mathcal{H}$ of states on a sphere in the presence of a magnetic background for the R and the global symmetries,

$$\mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{iJ_F A^F} e^{-\beta H}\right)$$

$Q^2 = H - \sigma^F J_F$

holomorphic in u^F

where J_F is the generator of the global symmetry.

Introduction

II. On the gravity side, consider BPS black holes in AdS_4.

- One of the success of string theory is the microscopic counting of asymptotically flat black holes made with D-branes [Vafa-Strominger'96]
- No similar result for AdS black holes

But AdS should be simpler and related to holography: counting of states in the dual CFT. People failed for AdS_5 black holes (states in N=4 SYM).

Introduction

There are many 1/4 BPS asymptotically AdS_4 static black holes

- solutions asymptotic to *magnetic* AdS_4 and with horizon $AdS_2 \times S^2$
- Characterized by a collection of magnetic charges $\int_{S^2} F$
- preserving supersymmetry via a twist

$$(\nabla_\mu - iA_\mu)\epsilon = \partial_\mu \epsilon \qquad \Longrightarrow \qquad \epsilon = \text{cost}$$

Various solutions with regular horizons, some embeddable in $AdS_4 \times S^7$.

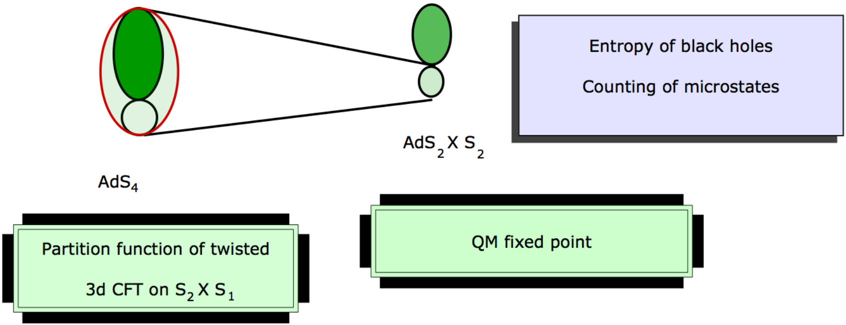
[Cacciatori, Klemm; Gnecchi, Dall'agata; Hristov, Vandoren];

Introduction

AdS black holes are dual to a twisted CFT on $S^2 \times S^1$

$$ds^2_{d+1} = \frac{dr^2}{r^2} + (r^2 ds^2_{M_d} + O(r)) \qquad A = A_{M_d} + O(1/r)$$

[A.Z. with Benini,Hristov]



[Thanks for many related discussions to A. Tomasiello]

Localization

Exact quantities in supersymmetric theories with a charge $Q^2 = 0$ can be obtained by a saddle point approximation

$$Z = \int e^{-S} = \int e^{-S+t\{Q,V\}} \underset{t\gg 1}{=} e^{-\bar{S}|_{class}} \times \frac{\det_{fermions}}{\det_{bosons}}$$

Very old idea that has become very concrete recently, with the computation of partition functions on spheres and other manifolds supporting supersymmetry.

The background

Consider an $\mathcal{N} = 2$ gauge theory on $S^2 \times S^1$

$$ds^2 = R^2\big(d\theta^2 + \sin^2\theta \, d\varphi^2\big) + \beta^2 dt^2$$

with a background for the R-symmetry proportional to the spin connection:

$$A^R = -\frac{1}{2}\cos\theta \, d\varphi = -\frac{1}{2}\omega^{12}$$

so that the Killing spinor equation

$$D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon - iA_\mu^R \epsilon = 0 \qquad \Longrightarrow \qquad \epsilon = \text{const}$$

The partition function

The path integral for an $\mathcal{N}=2$ gauge theory on $S^2 \times S^1$ with gauge group G localizes on a set of BPS configurations specified by data in the vector multiplets $V = (A_\mu, \sigma, \lambda, \lambda^\dagger, D)$

- A magnetic flux on S^2, $\mathfrak{m} = \frac{1}{2\pi}\int_{S^2} F$ in the co-root lattice
- A Wilson line A_t along S^1
- The vacuum expectation value σ of the real scalar

Up to gauge transformations, the BPS manifold is

$$(u = A_t + i\sigma, \mathfrak{m}) \in \mathcal{M}_{\text{BPS}} = \big(H \times \mathfrak{h} \times \Gamma_{\mathfrak{h}}\big)/W$$

The partition function

The path integral reduces to a the saddle point around the BPS configurations

$$\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}} \int du d\bar{u}\, \mathcal{Z}^{\text{cl }+\text{1-loop}}(u, \bar{u}, \mathfrak{m})$$

- The integrand has various singularities where chiral fields become massless
- There are fermionic zero modes

The two things nicely combine and the path integral reduces to an r-dimensional contour integral of a meromorphic form

$$\boxed{\frac{1}{|W|} \sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}} \oint_C Z_{\text{int}}(u, \mathfrak{m})}$$

The partition function

- In each sector with gauge flux $\mathfrak{m}$ we have a a meromorphic form

$$Z_{\text{int}}(u,\mathfrak{m}) = Z_{\text{class}} Z_{\text{1-loop}}$$

$$\boxed{Z^{\text{CS}}_{\text{class}} = x^{k\mathfrak{m}}} \qquad x = e^{iu}$$

$$\boxed{Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\rho\in\mathfrak{R}} \left[\frac{x^{\rho/2}}{1-x^{\rho}}\right]^{\rho(\mathfrak{m})-q+1}} \qquad q = \text{R charge}$$

$$\boxed{Z^{\text{gauge}}_{\text{1-loop}} = \prod_{\alpha\in G} (1-x^{\alpha})\,(i\,du)^{r}}$$

- Supersymmetric localization selects a particular contour of integration C and picks some of the residues of the form $Z_{\text{int}}(u,\mathfrak{m})$.

[Jeffrey-Kirwan residue - similar to Benini,Eager,Hori,Tachikawa '13; Hori,Kim,Yi '14]

A Simple Example: SQED

The theory has gauge group $U(1)$ and two chiral Q and $\tilde{Q}$

$$Z = \sum_{\mathfrak{m}\in\mathbb{Z}} \int \frac{dx}{2\pi i\, x} \left(\frac{x^{\frac{1}{2}} y^{\frac{1}{2}}}{1-xy}\right)^{\mathfrak{m}+\mathfrak{n}} \left(\frac{x^{-\frac{1}{2}} y^{\frac{1}{2}}}{1-x^{-1}y}\right)^{-\mathfrak{m}+\mathfrak{n}}$$

	$U(1)_g$	$U(1)_A$	$U(1)_R$
Q	1	1	1
$\tilde{Q}$	-1	1	1

Consistent with duality with three chirals with superpotential XYZ

$$Z = \left(\frac{y}{1-y^2}\right)^{2\mathfrak{n}-1} \left(\frac{y^{-\frac{1}{2}}}{1-y^{-1}}\right)^{-\mathfrak{n}+1} \left(\frac{y^{-\frac{1}{2}}}{1-y^{-1}}\right)^{-\mathfrak{n}+1}$$

Aharony and Giveon-Kutasov dualities

The twisted index can be used to check dualities: for example, $U(N_c)$ with $N_f = N_c$ flavors is dual to a theory of chiral fields M_{ab}, T and $\tilde{T}$, coupled through the superpotential $W = T\tilde{T}\det M$

$$Z_{N_f=N_c} = \left(\frac{y}{1-y^2}\right)^{(2\mathfrak{n}-1)N_c^2}\left(\frac{\xi^{\frac{1}{2}}y^{-\frac{N_c}{2}}}{1-\xi y^{-N_c}}\right)^{N_c(1-\mathfrak{n})+\mathfrak{t}}\left(\frac{\xi^{-\frac{1}{2}}y^{-\frac{N_c}{2}}}{1-\xi^{-1}y^{-N_c}}\right)^{N_c(1-\mathfrak{n})-\mathfrak{t}}$$

Aharony and Giveon-Kutasov dual pairs for generic (N_c, N_f) have the same partition function.

Refinement and other dimensions

We can add refinement for angular momentum on S^2.

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We can go up and down in dimension

- In a $(2,2)$ theory in 2d on S^2 we are computing amplitudes in gauged linear sigma models [also Cremonesi-Closset-Park '15]
- In a $\mathcal{N}=1$ theory on $S^2 \times T^2$ we are computing an elliptically generalized twisted index

[also Closset-Shamir '13;Nishioka-Yaakov '14;Yoshida-Honda '15]

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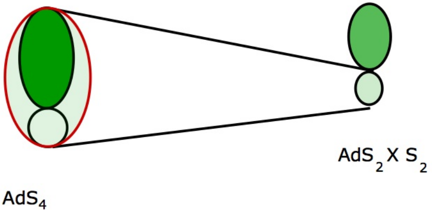
The index adds to and complete the list of existing tools (superconformal indices, sphere partition functions) for testing dualities.

The AdS_4 black hole

We focus on a BPS black hole with metric

$$ds^2 = e^{2f(r)}(-dt^2 + dr^2) + e^{2g(r)+2h(r)}(d\theta^2 + \sin\theta^2 d\phi^2)$$

asymptotically AdS and with horizon AdS2 $\times S^2$



$$ds^2 = \frac{-dt^2 + dr^2 + (d\theta^2 + \sin\theta^2 d\phi^2)}{r^2} \qquad ds^2 = \frac{-dt^2 + dr^2}{r^2} + (d\theta^2 + \sin\theta^2 d\phi^2)$$

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embeddable in $\mathrm{AdS}_4 \times S^7$ and with four magnetic charges on S^2

$$\mathfrak{n}_1\,,\ \mathfrak{n}_2\,,\ \mathfrak{n}_3\,,\ \mathfrak{n}_4\,, \qquad \mathfrak{n}_i = \int_{S^2} F^{(i)}\,, \qquad \sum \mathfrak{n}_i = 2$$

under the abelian vectors $U(1)^4 \subset SO(8)$ that come from the reduction on S^7.

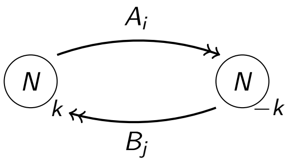
The metric is analytically known and the entropy is (for $\mathfrak{n}_1 = \mathfrak{n}_2 = \mathfrak{n}_3$)

$$\sqrt{-1 + 6\mathfrak{n}_1 - 6\mathfrak{n}_1^2 + (-1 + 2\mathfrak{n}_1)^{3/2}\sqrt{-1 + 6\mathfrak{n}_1}}$$

[Cacciatori, Klemm]

The dual field theory

The dual field theory to $\mathrm{AdS}_4 \times S^7$ is known: is the ABJM theory with gauge group $U(N) \times U(N)$



with quartic superpotential

$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$$

defined on twisted $S^2 \times \mathbb{R}$ with magnetic fluxes $\mathfrak{n}_i$ for the R/global symmetries

$$SU(2)_A \times SU(2)_B \times U(1)_B \times U(1)_R \subset SO(8)$$

.

The dual field theory

The ABJM twisted index is

$$Z = \frac{1}{(N!)^2} \sum_{\mathfrak{m},\widetilde{\mathfrak{m}}\in\mathbb{Z}^N} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} x_i^{k\mathfrak{m}_i} \tilde{x}_i^{-k\widetilde{\mathfrak{m}}_i} \times \prod_{i\neq j}^{N} \left(1-\frac{x_i}{x_j}\right)\left(1-\frac{\tilde{x}_i}{\tilde{x}_j}\right) \times$$
$$\times \prod_{i,j=1}^{N} \left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j} y_1}}{1-\frac{x_i}{\tilde{x}_j} y_1}\right)^{\mathfrak{m}_i-\widetilde{\mathfrak{m}}_j-\mathfrak{n}_1+1} \left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j} y_2}}{1-\frac{x_i}{\tilde{x}_j} y_2}\right)^{\mathfrak{m}_i-\widetilde{\mathfrak{m}}_j-\mathfrak{n}_2+1}$$
$$\left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i} y_3}}{1-\frac{\tilde{x}_j}{x_i} y_3}\right)^{\widetilde{\mathfrak{m}}_j-\mathfrak{m}_i-\mathfrak{n}_3+1} \left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i} y_4}}{1-\frac{\tilde{x}_j}{x_i} y_4}\right)^{\widetilde{\mathfrak{m}}_j-\mathfrak{m}_i-\mathfrak{n}_4+1}$$

where $\mathfrak{m},\widetilde{\mathfrak{m}}$ are the gauge magnetic fluxes and y_i are fugacities for the three independent $U(1)$ global symmetries ($\prod_i y_i = 1$)

The dual field theory

Strategy:

- Re-sum geometric series in $\mathfrak{m}, \widetilde{\mathfrak{m}}$.

$$Z = \int \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} \frac{f(x_i, \tilde{x}_i)}{\prod_{j=1}^N (e^{iB_i} - 1) \prod_{j=1}^N (e^{i\widetilde{B}_j} - 1)}$$

- Find the zeros of denominator $e^{iB_i} = e^{i\tilde{B}_j} = 1$ at large N

- Evaluate the residues at large N

$$Z \sim \sum_I \frac{f(x_i^{(0)}, \tilde{x}_i^{(0)})}{\det \mathbb{B}}$$

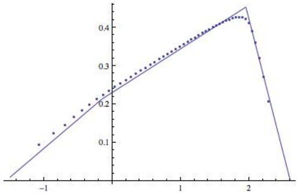
The large N limit

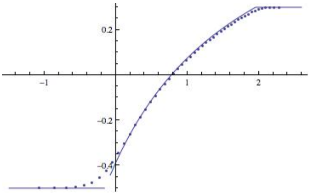
Step 2: solve the large N Limit of algebraic equations giving the positions of poles

$$1 = x_i^k \prod_{j=1}^{N} \frac{\left(1 - y_3 \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_4 \frac{\tilde{x}_j}{x_i}\right)}{\left(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i}\right)} = \tilde{x}_j^k \prod_{i=1}^{N} \frac{\left(1 - y_3 \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_4 \frac{\tilde{x}_j}{x_i}\right)}{\left(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i}\right)}$$

with an ansatz

$$\log x_i = i\sqrt{N} t_i + v_i \,, \qquad \log \tilde{x}_i = i\sqrt{N} t_i + \tilde{v}_i$$





The large N limit

Step 3: plug into the partition function. The final result is surprisingly simple

$$\mathbb{R}\mathrm{e} \log Z = -\frac{1}{3} N^{2/3} \sqrt{2k\Delta_1\Delta_2\Delta_3\Delta_4} \sum_a \frac{\mathfrak{n}_a}{\Delta_a} \qquad y_i = e^{i\Delta_i}$$

This function can be extremized with respect to the Δ_i and

$$\boxed{\mathbb{R}\mathrm{e} \log Z|_{crit}(\mathfrak{n}_i) = \mathrm{BH\,Entropy}(\mathfrak{n}_i)}$$

The large N limit

The twisted index depends on Δ_i because we are computing the trace

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F e^{i\Delta_i J_i} \equiv \mathrm{Tr}_{\mathcal{H}}(-1)^R$$

where $R = F + \Delta_i J_i$ is a possible R-symmetry of the system.

Here an extremization is at work: symmetry enhancement at the horizon AdS_2

$$\mathrm{QM}_1 \to \mathrm{CFT}_1$$

- R is the exact R-symmetry at the superconformal point
- Natural thing to extremize: in even dimensions central charges are extremized, in odd partition functions...

Conclusions

We related the entropy of a class of AdS_4 black holes to a microscopic counting of states.

We also gave a general formula for the topologically twisted path integral of 2d (2,2), 3d $\mathcal{N} = 2$ and 4d $\mathcal{N} = 1$ theories.

- Higher genus $S^2 \to \Sigma$? Include Witten index
- 2d theories, learn about Calabi-Yaus's and sigma-models?
- Extremization of the index is a general principle?