Field Theory Description of Topological States of Matter

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Topological States of Matter

- System with bulk gap but non-trivial at energies below the gap
- global effects and global degrees of freedom:

→ massless edge states, exchange phases, ground-state degeneracies

- not described by symmetry breaking and Landau-Ginzburg approach
- quantum Hall effect is chiral (B field breaks T symmetry)
- quantum spin Hall effect is non-chiral (T symmetric)
- other systems: Chern Insulators, Topological Insulators, Topological Superconductors in d=1,2,3
- Ten-fold classification of non-interacting systems (Band Insulators)

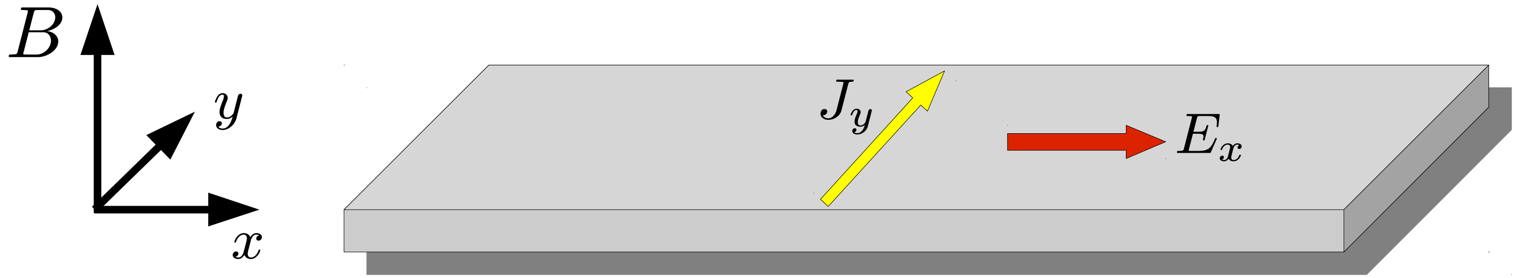
Topological Band Insulators have been observed in d=2 & 3

(Molenkamp et al. '07; Hasan et al. '08)

Quantum Hall Effect

2 dim electron gas at low temperature T ~ 10-100 mK

and high magnetic field B ~ 5-10 Tesla



Conductance tensor $J_i = \sigma_{ij} E_j, \quad \sigma_{ij} = R^{-1}_{ij}, \qquad i,j = x,y$

Plateaus: $\sigma_{xx} = 0, \ R_{xx} = 0$ no Ohmic conduction → gap

High precision & universality

$$\sigma_{xy} = R^{-1}_{xy} = \frac{e^2}{h}\nu, \quad \nu = 1(\pm 10^{-9}), 2, 3, \ldots \frac{1}{3}, \frac{2}{5}(\pm 10^{-6})$$

Uniform density ground state: $\rho_o = \dfrac{eB}{hc}\nu$

Incompressible fluid

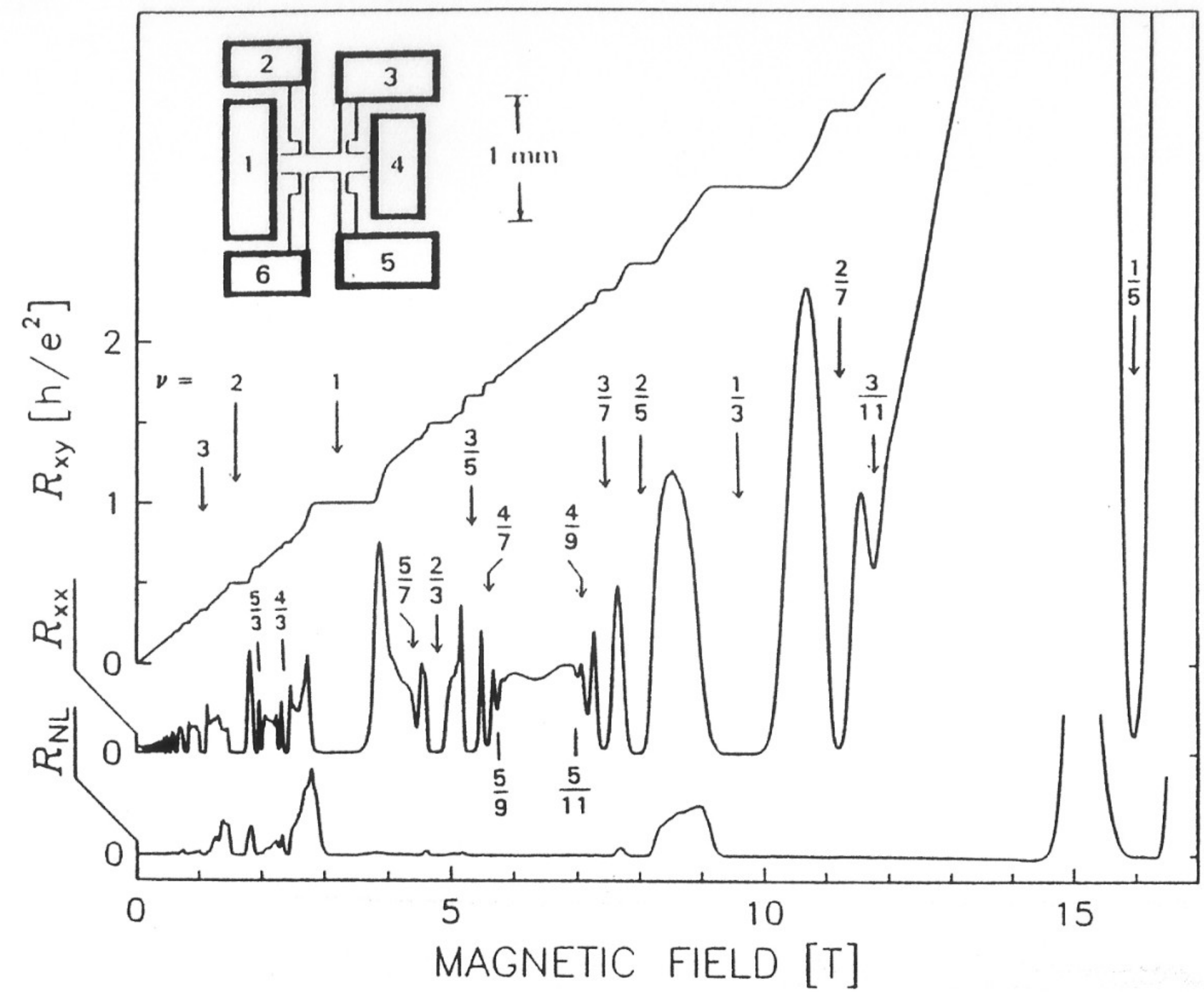
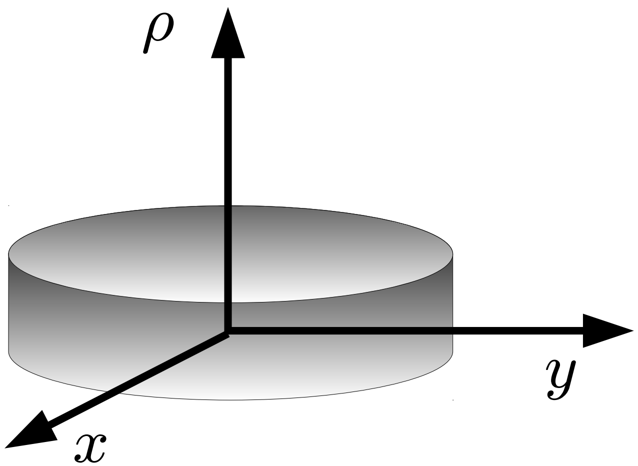


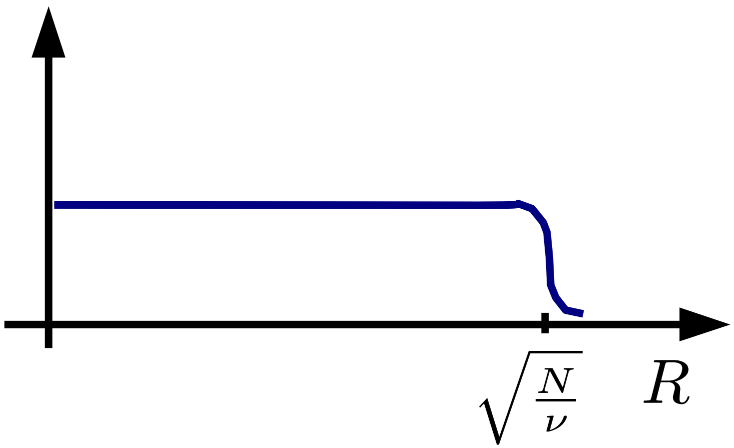
Figure 1. This figure shows a quantum Hall experiment trace. (Source: Goldman *et al.* [2].) The sample geometry is shown in the inset. The various resistances are defined as: Hall resistance $R_{xy} = V_{26}/I_{14}$; longitudinal resistance $R_{xx} = V_{23}/I_{14}$; and non-local resistance $R_{NL} = V_{26}/I_{35}$. Here, V_{jk} denotes the voltage difference between the leads j and k, and I_{jk} denotes the current from lead j to lead k. The experiment was performed at 40 mK.

Laughlin's quantum incompressible fluid

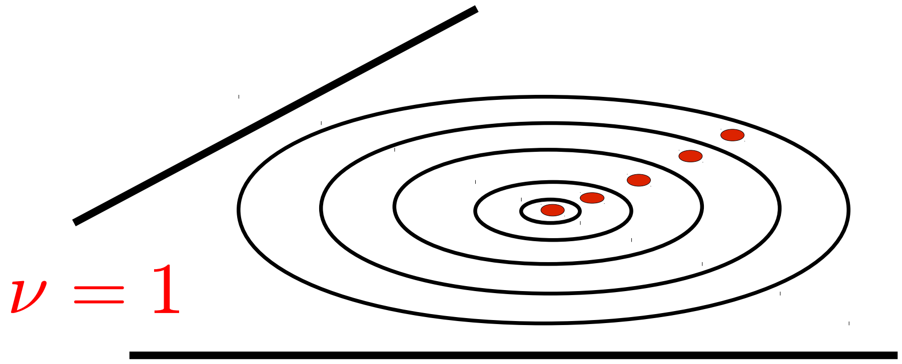
Electrons form a droplet of fluid:

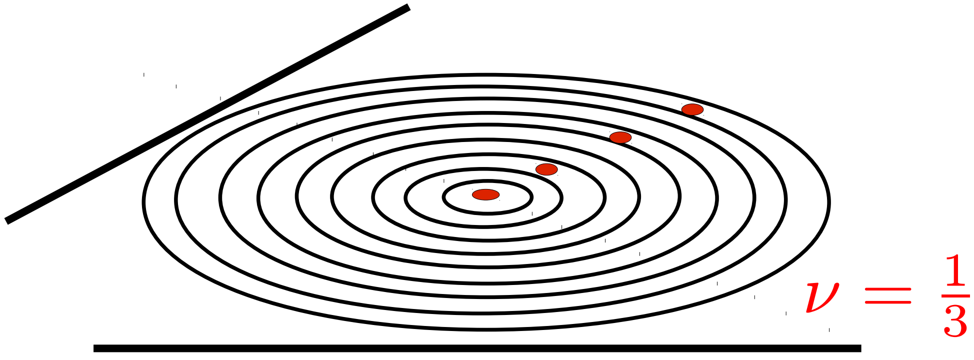
→ incompressible: gap → fluid: $\rho(x,y) = \rho_o = \mathrm{const.}$





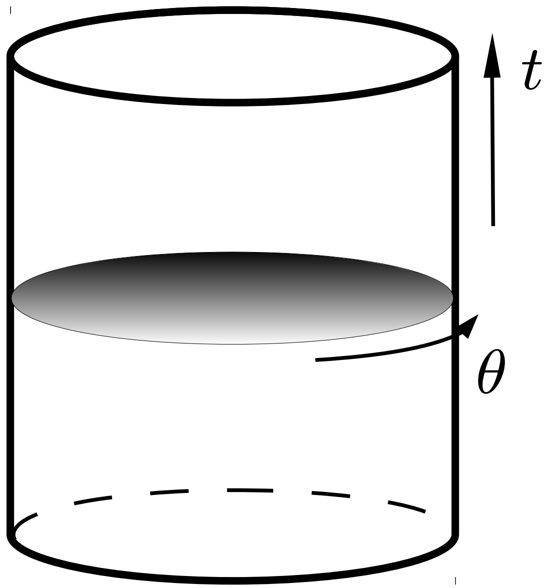
filling fraction: $\nu = \frac{N}{\mathcal{D}_A} = \frac{N}{\Phi/\Phi_o} = 1, 2, \ldots \frac{1}{3}, \frac{1}{5}, \ldots$ $\Phi_o = \frac{hc}{e}$

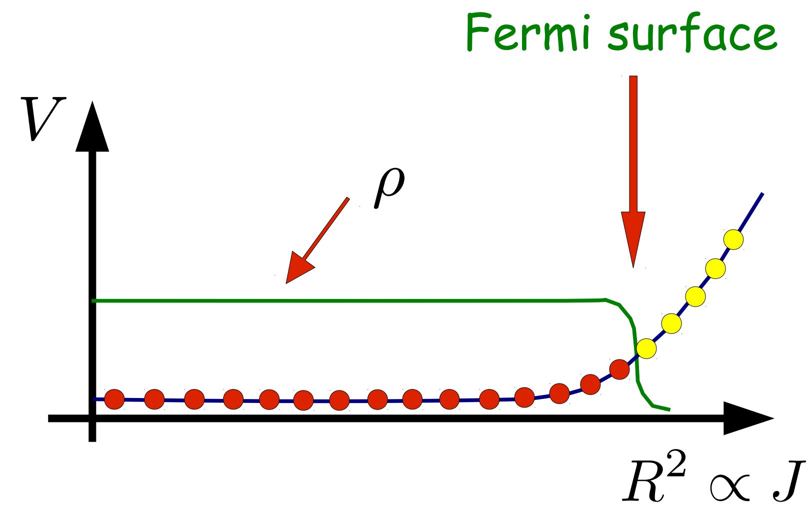




Edge excitations

The edge of the droplet can fluctuate: massless (1+1)-dimensional edge waves





edge ~ Fermi surface: linearize energy $\varepsilon(k) = \frac{v}{R}(k - k_F), \quad k \in \mathbb{Z}$

relativistic field theory in 1+1 dimensions with chiral excitations (X.G.Wen)

- conformal field theory of edge excitations (chiral Luttinger liquid)
- CFT modelling describes nonperturbative quantum effects
- experimental predictions for conduction and tunneling

Effective field theory

Quantum field theory in a nutshell:

- Take a massive phase and fix a maximal energy scale Λ
- Guess the low-energy degrees of freedom (fields) and symmetries
- Write the action compatible with them, as a power series in the fields and their derivatives ($1/\Lambda$ expansion). Ex. Landau-Ginzburg:

$$S[J] = \int (\partial_\mu \phi)^2 + a\,\phi + b\,\phi^2 + c\,\phi^4 + \cdots + \phi J \qquad a, b, c, \ldots \text{ to be fitted}$$

→ Successful examples: LG, SC, FL, SM, SYM, AdS/CFT, you name it

→ Successful if leading terms are simple: universality

- Topological states need effective theories beyond Landau-Ginzburg, Higgs etc.

→ Topological gauge theories and anomalies

Bulk & boundary

Chern-Simons bulk effective action

$$S\left[A\right]=\frac{\nu}{4\pi}\int dx^{3}\,\varepsilon^{\mu\nu\rho}A_{\mu}\partial_{\nu}A_{\rho}=\frac{\nu}{4\pi}\int AdA$$

Laughlin state $\nu=1,\frac{1}{3},\frac{1}{5},\cdots$

- no local degrees of freedom: only global effects

$$\rho=\frac{\delta S}{\delta A_0}=\frac{\nu}{2\pi}\mathcal{B}$$ Density

$$J^{i}=\frac{\delta S}{\delta A_i}=\frac{\nu}{2\pi}\varepsilon^{ij}\mathcal{E}^{j}$$ Hall current

- Hall current is topological, i.e. robust
- Introduce Wen's hydrodynamic matter field a_μ and current $j^{\mu}=\varepsilon^{\mu\nu\rho}\partial_{\nu}a_{\rho}$

$$S[A]=\int -\frac{1}{4\pi\nu}ada+Ada=S_{\rm matt}[a]+(\text{e.m. coupl.})$$

- Sources of a_μ field are anyons (Aharonov-Bohm phases $\frac{\theta}{\pi}=\nu=\frac{1}{3},\cdots$)
- Gauge invariance requires a boundary action:

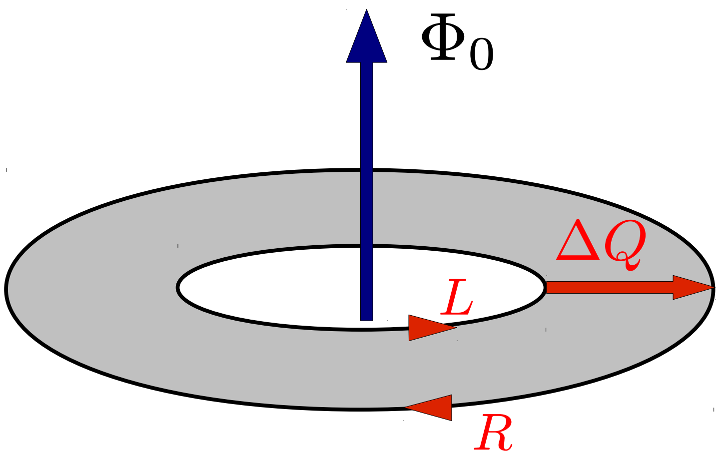
$$S_{\rm matt}[a]\to S_{\rm matt}[a]+S_{CFT}\left[\varphi\right],\quad \partial_{\mu}\varphi=a_{\mu}|_{b}$$ ➡ massless edge states

- Bulk topological theory is tantamount to conformal field theory on boundary

Boundary CFT and chiral anomaly

- edge states are chiral fermions/bosons
- chiral anomaly: boundary charge is not conserved
- bulk (B) and boundary (b) compensate:

$$\partial_i J^i + \partial_t \rho = 0, \ \rightarrow \ \oint dx J_B + \partial_t Q_b = 0$$



- adiabatic flux insertion (Laughlin)

$$\Phi \rightarrow \Phi + \Phi_0,$$

$$Q_R \rightarrow Q_R + \Delta Q_b = \nu, \qquad \Delta Q_b = \int_{-\infty}^{+\infty} dt \oint dx \ \partial_t \rho_R = \nu \int F_R = \nu \, n$$

chiral anomaly

- Anomaly inflow — Index theorem: exact quantization of Hall current
- edge chiral anomaly = response of topological bulk to e.m. background
- chiral edge states cannot be gapped ⟷ topological phase is stable
- anomalous response extended to other systems in D=1,2,3,......

Ten-fold classification

space dim. d

QHE → A; Top. Ins. → AII

A, AIII: period 2

AI – CI: period 8 (Bott)

class\\δ	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$+$	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$+$	$+$	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	$+$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	$-$	$+$	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	$-$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	$-$	$-$	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	$-$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$+$	$-$	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

- Study T, C, P symmetries of quadratic fermionic Hamiltonians (A. Kitaev; Ludwig et al. 09)
- Matches classes of disordered systems/random matrices/Clifford algebras
- Does it extend to interacting systems? YES - NO - ???

→ study field theory anomalies

Classification by chiral anomalies: $\mathbb{Z}$ classes

	Cartan\\d	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$U(1)$ anomaly	A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
	AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
gravitational & mixed anomalies	AI	$\mathbb{Z}^{\spadesuit}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\spadesuit}$	0	0	0	$\cdots$
	BDI	$\mathbb{Z}_2$	$\mathbb{Z}^{\clubsuit}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\clubsuit}$	0	0	$\cdots$
	D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\heartsuit}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\heartsuit}$	0	$\cdots$
	DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\diamondsuit}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\diamondsuit}$	$\cdots$
	AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\spadesuit}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
	CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\clubsuit}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
	C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\heartsuit}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
	CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{\diamondsuit}$	0	0	0	$2\mathbb{Z}$	$\cdots$

- $d = \text{even}$ boundary anomaly, bulk Chern-Simons theory (as in QHE)
- $d = \text{odd}$ bulk anomaly, bulk theta term, ex. d=3 $U(1)$ gauge theory (AIII)

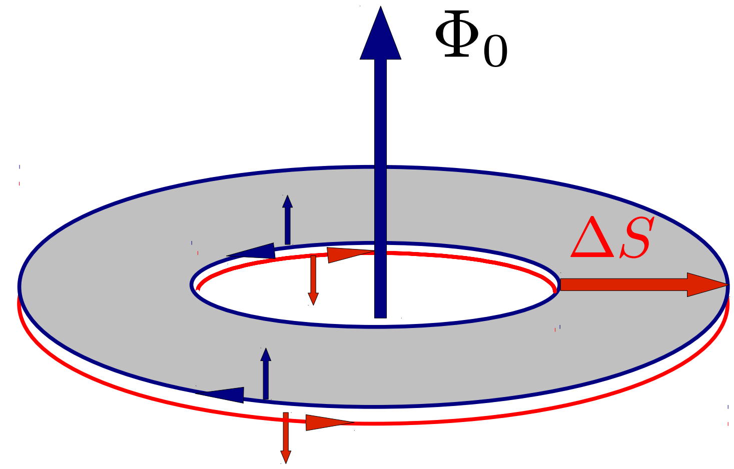
$$S[A] = \frac{\theta}{32\pi^2}\int F\wedge F = \frac{\theta}{4\pi^2}\int dx^4\, E\cdot B$$

magneto-electric effect

- gravitational anomaly = non-conservation of the thermal current

(A.Ludwig, Furusaki, J. Moore, S. Ryu, Schnyder '08-12)

Non-chiral topological states



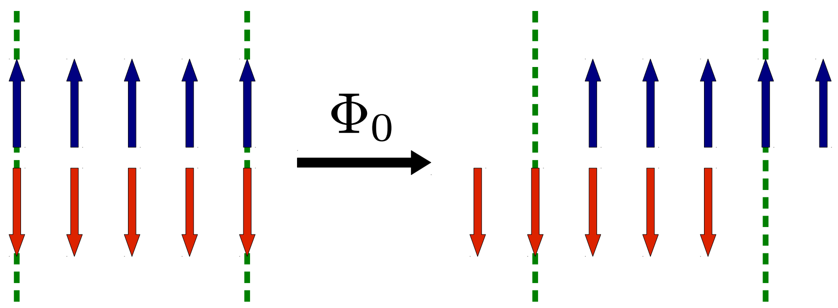
Quantum Spin Hall Effect

- take two $\nu = 1$ Hall states of spins ↑↓
- system is Time-reversal invariant:

$$\mathcal{T}: \ \psi_{k\uparrow} \to \psi_{-k\downarrow} \ , \qquad \psi_{k\downarrow} \to -\psi_{-k\uparrow}$$

- non-chiral CFT with $U(1)_Q \times U(1)_S$ symmetry
- adding flux pumps spin → $U(1)_S$ anomaly

(L. Fu, C. Kane, E. Mele 06)



$$\Delta Q = \Delta Q^{\uparrow} + \Delta Q^{\downarrow} = 1 - 1 = 0$$

$$\Delta S = \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) = 1$$

$$\Delta S = \Delta Q^{\uparrow} = \nu^{\uparrow}$$

- in Topological Insulators $U(1)_S$ is explicitly broken by spin-orbit interaction
- no currents $\sigma_H = \sigma_{sH} = 0$
- but T symmetry keeps $\mathbb{Z}_2$ symmetry of $(-1)^{2S}$ (Kramers theorem)

Topological Insulators with T symmetry

class\δ	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Top. Ins. → AII

- stability of TI ⟷ stability of non-chiral edge states
- T symmetry forbids mass term with odd number of free fermions

$$\mathcal{T} : H_{\text{int.}} = m \int \psi_\uparrow^\dagger \psi_\downarrow + h.c. \ \rightarrow \ -H_{\text{int.}}$$

$\mathbb{Z}_2$ classification (free fermions)

Topological insulators and $\mathbb{Z}_2$ anomaly

- from Spin QHE to Topological Insulator: $U(1)_S \rightarrow \mathbb{Z}_2$ "spin parity" $(-1)^{2S}$
- $U(1)_S$ anomaly reduces to discrete $\mathbb{Z}_2$ anomaly:

→ no anomalous currents & actions; study partition function

→ this is not unique and undergoes discrete transformations

- study transformations under half-flux insertions $\frac{\Phi_0}{2}$ (I. Fu, C. Kane, '07)

$$Z_{AA} \leftrightarrow Z_{PA}, \quad Z_{AP} \leftrightarrow Z_{PP}, \qquad \text{resp. } NS,\ R,\ \widetilde{NS},\ \widetilde{R}$$

- amount to change of Levin-Stern index

$$(-1)^{2\Delta S} = -1, \qquad 2\Delta S = \frac{\nu^{\uparrow}}{e^*}$$

- Partition function of edge CFTs can be found for general interacting systems

→ $\mathbb{Z}_2$ spin parity anomaly characterizes d=2 Topological Insulators in general interacting theories

(A. C., E. Randellini, 13-14)

Conclusion

- Many topological states of matter exist and are actively investigated both theoretically and experimentally
- Effective field theories of massless edge excitations and their anomalies describe universal properties and characterize interacting systems
- Theoretical problems both practical and highly technical:
  - study signatures and observables of topological phases
  - study discrete anomalies (gravitational) in 2d, 3d, K-theory,....

(S. Ryu et al.; G. Moore; E. Witten)

- Technological applications of topologically protected excitations:
  - quantum information and computation
  - conduction without dissipation
  - quantum devices, quantum sensors, etc

Readings

- M. Franz, L. Molenkamp eds., “Topological Insulators”, Elsevier (2013)
- X. L. Qi, S. C. Zhang, Rev. Mod. Phys. 83 (2011) 1057
- C. K. Chiu, J. C. Y. Teo, A. P. Schnyder, S. Ryu, arXiv:1505.03535 (to appear in RMP)