# Hadronic Light-by-Light Scattering and the Muon $g-2$

Peter Stoffer

JHEP **09** (2014) 091 [arXiv:1402.7081 [hep-ph]]

in collaboration with G. Colangelo, M. Hoferichter and M. Procura

Helmholtz-Institut für Strahlen- und Kernphysik
University of Bonn

4th March 2015

Les Rencontres de Physique de la Vallée d'Aoste, La Thuile

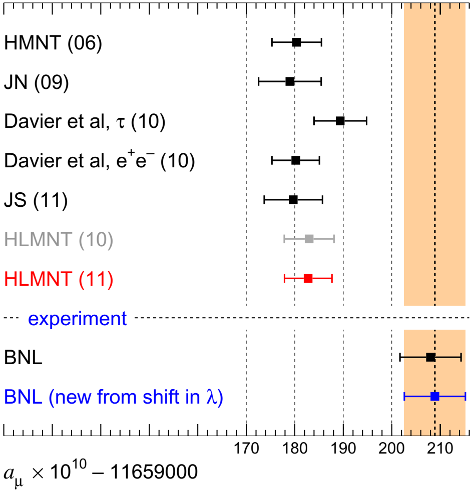
## Magnetic moment

- relation of spin and magnetic moment of a lepton:

$$\vec{\mu}_\ell = g_\ell \frac{e}{2m_\ell} \vec{s}$$

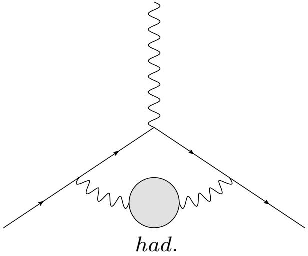
  $g_\ell$: Landé factor, gyromagnetic ratio
- Dirac's prediction: $g_e = 2$
- anomalous magnetic moment: $a_\ell = (g_\ell - 2)/2$
- helped to establish QED and QFT as the framework for elementary particle physics
- today: probing not only QED but entire SM

# $a_\mu$: comparison of theory and experiment



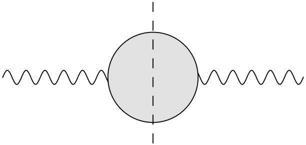

→ Hagiwara et al. 2012

# Leading hadronic contribution: $\mathcal{O}(\alpha^2)$



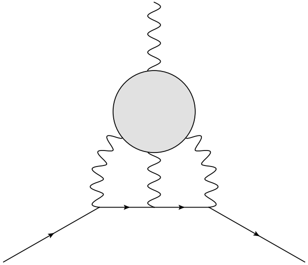

- problem: QCD is non-perturbative at low energies
- first principle calculations (lattice QCD) may become competitive in the future
- current evaluations based on dispersion relations and data

# Leading hadronic contribution: $\mathcal{O}(\alpha^2)$



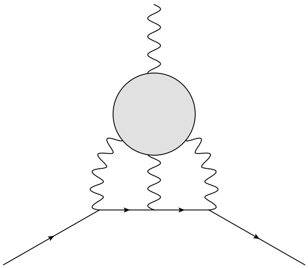
- basic principles: unitarity and analyticity
- direct relation to experiment: total hadronic cross section $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
- at present: dominant theoretical uncertainty
- can be systematically improved: dedicated $e^+e^-$ program (BaBar, Belle, BESIII, CMD3, KLOE2, SND)

# Higher order hadronic contributions: $\mathcal{O}(\alpha^3)$
# Hadronic light-by-light (HLbL) scattering



- hadronic matrix element of four EM currents
- up to now, only model calculations
- lattice QCD not yet competitive

# Higher order hadronic contributions: $\mathcal{O}(\alpha^3)$ Hadronic light-by-light (HLbL) scattering



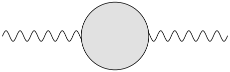
- uncertainty estimate based rather on consensus than on a systematic method
- will dominate theory error in a few years
- "dispersive treatment impossible"

## $a_\mu$: theory vs. experiment

- theory error completely dominated by hadronic effects
- discrepancy between Standard Model and experiment $\sim 3\sigma$
- hint to new physics?
- new experiments (FNAL, J-PARC) aim at reducing the experimental error by a factor of 4

# Leading hadronic contribution: $\mathcal{O}(\alpha^2)$

Photon hadronic vacuum polarisation function:

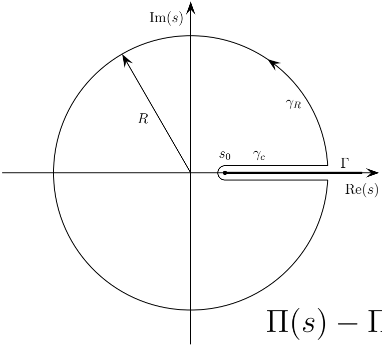
 $= -i(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)$

Unitarity of the $S$-matrix implies the optical theorem:

$$\mathrm{Im}\Pi(s) = \frac{s}{e(s)^2}\sigma_{\mathrm{tot}}(e^+e^- \to \gamma^* \to \mathrm{hadrons})$$

# Dispersion relation

Causality implies analyticity:



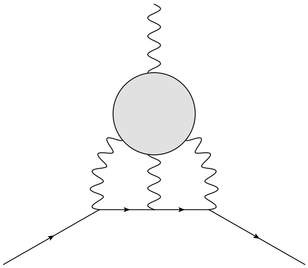

Cauchy integral formula:

$$\Pi(s) = \frac{1}{2\pi i} \oint_\gamma \frac{\Pi(s')}{s' - s} ds'$$

Deform integration path:

$$\Pi(s) - \Pi(0) = \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{Im}\Pi(s')}{(s' - s - i\epsilon)s'} ds'$$

# How to improve HLbL calculation?



- "dispersive treatment impossible": no!
- relate HLbL to experimentally accessible quantities
- make use of unitarity, analyticity, gauge invariance and crossing symmetry

# The HLbL tensor

- object in question: $\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$
- a priori 138 Lorentz structures
- gauge invariance: 95 linear relations
  $\Rightarrow$ (off-shell) basis: 43 independent structures
- six dynamical variables, e.g. two Mandelstam variables
  $$s = (q_1 + q_2)^2, \quad t = (q_1 + q_3)^2$$
  and the photon virtualities $q_1^2$, $q_2^2$, $q_3^2$, $q_4^2$
- complicated analytic structure

# HLbL tensor: Lorentz decomposition

Problem: find a decomposition

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

with the following properties:

- Lorentz structures $T_i^{\mu\nu\lambda\sigma}$ manifestly gauge invariant
- scalar functions $\Pi_i$ free of kinematic singularities and zeros

# HLbL tensor: Lorentz decomposition

Recipe by Bardeen, Tung (1968) and Tarrach (1975):

- apply gauge projectors to the 138 initial structures:

$$I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^\lambda q_3^\sigma}{q_3 \cdot q_4}$$

- remove poles taking appropriate linear combinations
- Tarrach: no kinematic-free basis of 43 elements exists
- extend basis by additional structures taking care of remaining kinematic singularities

# HLbL tensor: Lorentz decomposition

Solution for the Lorentz decomposition:

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures manifestly gauge invariant
- crossing symmetry manifest
- scalar functions $\Pi_i$ free of kinematics
  $\Rightarrow$ ideal quantities for a dispersive treatment

# Master formula: contribution to $(g-2)_\mu$

$$a_\mu^{\mathrm{HLbL}} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known integration kernel functions
- $\hat{\Pi}_i$: linear combinations of the scalar functions $\Pi_i$
- five loop integrals can be performed with Gegenbauer polynomial techniques

## Mandelstam representation

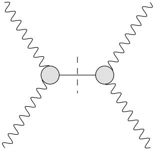
- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$

# Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$

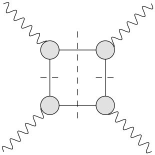
one-pion intermediate state: 

# Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$

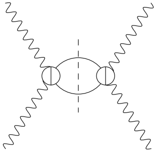
two-pion intermediate state in both channels:



# Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \ldots$$

two-pion intermediate state in first channel: 

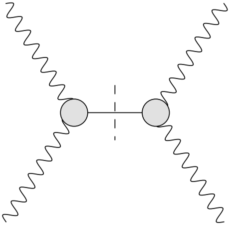
# Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$

neglected: higher intermediate states

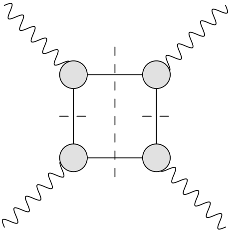
## Pion pole



- input the doubly-virtual and singly-virtual pion transition form factors $\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$
- dispersive analysis of transition form factor:

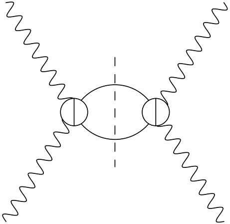
$\rightarrow$ Hoferichter et al., EPJC **74** (2014) 3180

# Box contributions



- simultaneous two-pion cuts in two channels
- analytic properties correspond to sQED loop
- $q^2$-dependence given by multiplication with pion vector form factor $F_\pi^V(q^2)$ for each off-shell photon ($\Rightarrow$ 'FsQED')

# Rescattering contribution

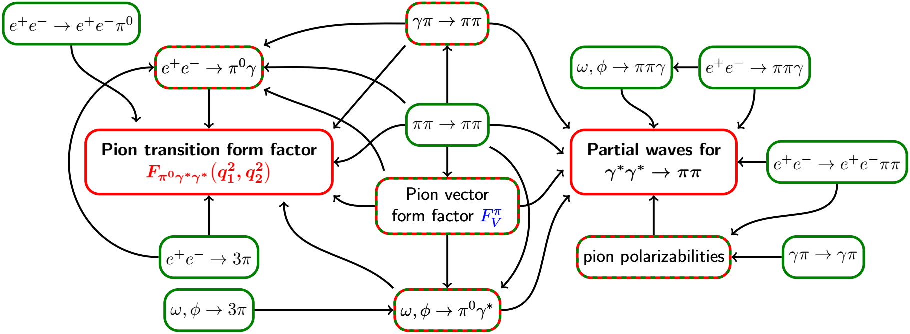


- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel
- expansion into partial waves
- unitarity relates it to the helicity amplitudes of the subprocess $\gamma^*\gamma^{(*)} \to \pi\pi$

# Summary

- our dispersive approach to HLbL scattering is based on fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- we take into account the lowest intermediate states: $\pi^0$-pole and $\pi\pi$-cuts
- relation to experimentally accessible (or again with data dispersively reconstructed) quantities
- a step towards a model-independent calculation of $a_\mu$
- numerical evaluation is work in progress

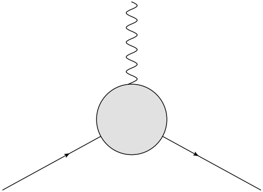
# A roadmap for HLbL




→ Flowchart by M. Hoferichter

# Backup

# Interaction of a muon with an external electromagnetic field



$$= (-ie)\langle \mu^-(p_2, s_2)|j^\mu_{\rm em}(0)|\mu^-(p_1, s_1)\rangle$$
$$= (-ie)\bar{u}(p_2)\Gamma^\mu(p_1, p_2)u(p_1)$$

$\Gamma^\mu(p_1, p_2)$: vertex function

# Form factors of the vertex function

Lorentz decomposition:

$$\Gamma^\mu(p_1, p_2) = \gamma^\mu F_E(k^2) - i\frac{\sigma^{\mu\nu}k_\nu}{2m}F_M(k^2) - \frac{\sigma^{\mu\nu}k_\nu}{2m}\gamma_5 F_D(k^2) + \left(\gamma^\mu + \frac{2mk^\mu}{k^2}\right)\gamma_5 F_A(k^2)$$

Form factors depend only on $k^2 = (p_1 - p_2)^2$.

# Form factors of the vertex function

Lorentz decomposition:

$$\Gamma^\mu(p_1, p_2) = \gamma^\mu F_E(k^2) - i\frac{\sigma^{\mu\nu}k_\nu}{2m}F_M(k^2) - \frac{\sigma^{\mu\nu}k_\nu}{2m}\gamma_5 F_D(k^2) + \left(\gamma^\mu + \frac{2mk^\mu}{k^2}\right)\gamma_5 F_A(k^2)$$

Form factors depend only on $k^2 = (p_1 - p_2)^2$.

electric charge or Dirac form factor, $F_E(0) = 1$

# Form factors of the vertex function

Lorentz decomposition:

$$\Gamma^\mu(p_1, p_2) = \gamma^\mu F_E(k^2) - i\frac{\sigma^{\mu\nu}k_\nu}{2m}F_M(k^2) - \frac{\sigma^{\mu\nu}k_\nu}{2m}\gamma_5 F_D(k^2) + \left(\gamma^\mu + \frac{2mk^\mu}{k^2}\right)\gamma_5 F_A(k^2)$$

Form factors depend only on $k^2 = (p_1 - p_2)^2$.

magnetic or Pauli form factor, $F_M(0) = a_\mu$

## Form factors of the vertex function

Lorentz decomposition:

$$\Gamma^\mu(p_1,p_2) = \gamma^\mu F_E(k^2) - i\frac{\sigma^{\mu\nu}k_\nu}{2m}F_M(k^2) - \frac{\sigma^{\mu\nu}k_\nu}{2m}\gamma_5 F_D(k^2) + \left(\gamma^\mu + \frac{2mk^\mu}{k^2}\right)\gamma_5 F_A(k^2)$$

Form factors depend only on $k^2 = (p_1 - p_2)^2$.

electric dipole form factor, $F_D(0)$ gives the $CP$-violating EDM

# Form factors of the vertex function

Lorentz decomposition:

$$\Gamma^\mu(p_1,p_2) = \gamma^\mu F_E(k^2) - i\frac{\sigma^{\mu\nu}k_\nu}{2m}F_M(k^2) - \frac{\sigma^{\mu\nu}k_\nu}{2m}\gamma_5 F_D(k^2) + \left(\gamma^\mu + \frac{2mk^\mu}{k^2}\right)\gamma_5 F_A(k^2)$$

Form factors depend only on $k^2 = (p_1 - p_2)^2$.

anapole form factor, $P$-violating

| | $10^{11} \cdot a_\mu$ | $10^{11} \cdot \Delta a_\mu$ | |
|---|---|---|---|
| BNL E821 | 116 592 091 | 63 | → PDG 2013 |
| QED total | 116 584 718.95 | 0.08 | → Kinoshita et al. 2012 |
| EW | 153.6 | 1.0 | |
| LO HVP | 6 949 | 43 | → Hagiwara et al. 2011 |
| NLO HVP | −98 | 1 | → Hagiwara et al. 2011 |
| NNLO HVP | 12.4 | 0.1 | → Kurz et al. 2014 |
| LO HLbL | 116 | 40 | → Jegerlehner, Nyffeler 2009 |
| NLO HLbL | 3 | 2 | → Colangelo et al. 2014 |
| Hadronic total | 6982 | 59 | |
| Theory total | 116 591 855 | 59 | |

| | $10^{11} \cdot a_\mu$ | $10^{11} \cdot \Delta a_\mu$ | |
|---|---|---|---|
| BNL E821 | 116 592 091 | 63 | → PDG 2013 |
| QED $\mathcal{O}(\alpha)$ | 116 140 973.32 | 0.08 | |
| QED $\mathcal{O}(\alpha^2)$ | 413 217.63 | 0.01 | |
| QED $\mathcal{O}(\alpha^3)$ | 30 141.90 | 0.00 | |
| QED $\mathcal{O}(\alpha^4)$ | 381.01 | 0.02 | |
| QED $\mathcal{O}(\alpha^5)$ | 5.09 | 0.01 | |
| QED total | 116 584 718.95 | 0.08 | → Kinoshita et al. 2012 |
| EW | 153.6 | 1.0 | |
| Hadronic total | 6982 | 59 | |
| Theory total | 116 591 855 | 59 | |

# Model calculations of HLbL

Table 13
Summary of the most recent results for the various contributions to $a_\mu^{\mathrm{LbL;had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

| Contribution | BPP | HKS | KN | MV | BP | PdRV | N/JN |
|---|---|---|---|---|---|---|---|
| $\pi^0, \eta, \eta'$ | 85±13 | 82.7±6.4 | 83±12 | 114±10 | – | 114±13 | 99±16 |
| $\pi, K$ loops | −19±13 | −4.5±8.1 | – | – | – | −19±19 | −19±13 |
| $\pi, K$ loops + other subleading in $N_c$ | – | – | – | 0±10 | – | – | – |
| axial vectors | 2.5±1.0 | 1.7±1.7 | – | 22± 5 | – | 15±10 | 22± 5 |
| scalars | −6.8±2.0 | – | – | – | – | −7± 7 | −7± 2 |
| quark loops | 21± 3 | 9.7±11.1 | – | – | – | 2.3 | 21± 3 |
| total | 83±32 | 89.6±15.4 | 80±40 | 136±25 | 110±40 | 105±26 | 116±39 |

→ Jegerlehner, Nyffeler 2009

- pseudoscalar pole contribution most important
- pion-loop second most important
- differences between models, large uncertainties