Towards a data-driven analysis of hadronic light-by-light scattering

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G. Colangelo, MH, M. Procura, P. Stoffer, JHEP 09 (2014) 091, arXiv:1309.6877

G. Colangelo, MH, B. Kubis, M. Procura, P. Stoffer, PLB 738 (2014) 6

MH, B. Kubis, S. Leupold, F. Niecknig, S. Schneider, arXiv:1410.4691

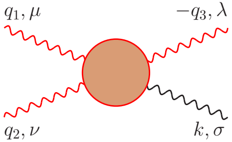
Outline

HLbL scattering

- Large uncertainty and **model dependence**
- 5 kinematic variables, (at least) 29 Lorentz structures
- **Dispersive point of view**
  - Analytic structure: poles and cuts

    ↪ **residues** and **imaginary parts** ⇒ by definition **on-shell** quantities

    ↪ **form factors** and **scattering amplitudes** from experiment
  - Expansion: mass of intermediate states, partial waves
  - Pseudoscalar poles most important, next $\pi\pi$ cuts

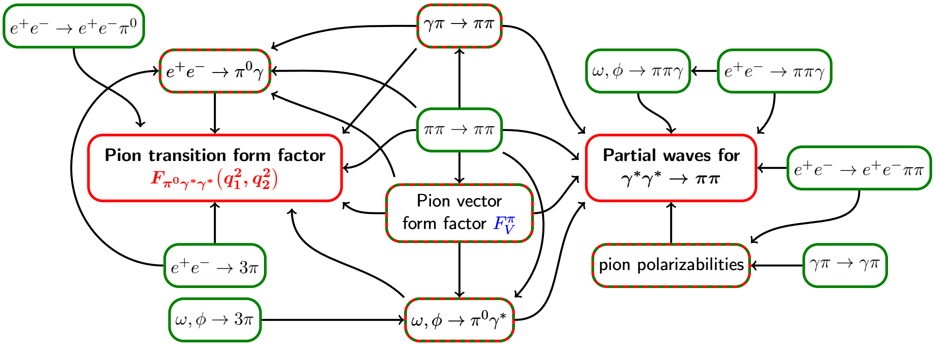


- Decompose the tensor according to

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

↪ accounts for **one-** and **two-pion** intermediate states

Towards a data-driven analysis of HLbL



- Reconstruction of $\gamma^*\gamma^* \to \pi\pi, \pi^0$: combine experiment and theory constraints
- Beyond: η, η', $K\bar{K}$, multi-pion channels (resonances), pQCD constraints, …

Master formulae for a_μ

Knecht, Nyffeler 2001

Master formula for pion-pole contribution

$$a_\mu^{\pi^0\text{-pole}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s\big((p+q_1)^2-m^2\big)\big((p-q_2)^2-m^2\big)}$$
$$\times \left\{ \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi^0\gamma^*\gamma^*}(s,0)}{s-M_\pi^2} T_1(q_1,q_2;p) + \frac{F_{\pi^0\gamma^*\gamma^*}(s,q_1^2)F_{\pi^0\gamma^*\gamma^*}(q_2^2,0)}{q_2^2-M_\pi^2} T_2(q_1,q_2;p) \right\}$$

- Crucial ingredient: **pion transition form factor**

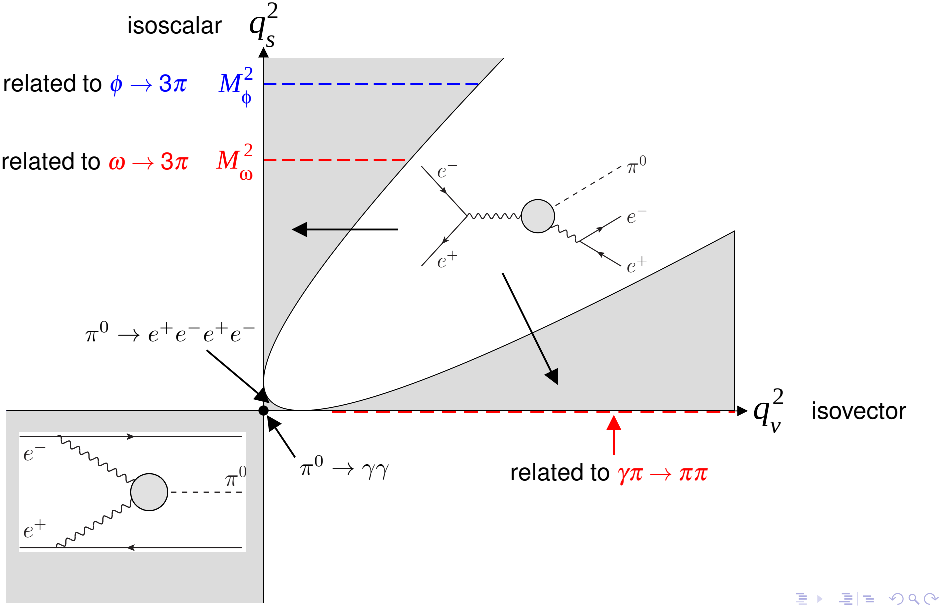
Colangelo, MH, Procura, Stoffer 2014

Master formula for $\pi\pi$ intermediate states

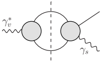
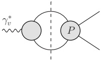
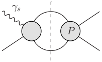
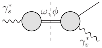
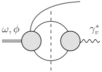
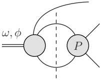
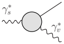
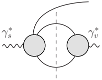
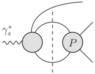
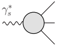
$$a_\mu^{\pi\pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_i T_i(q_1,q_2;p) I_i(s,q_1^2,q_2^2)}{q_1^2 q_2^2 s\big((p+q_1)^2-m^2\big)\big((p-q_2)^2-m^2\big)}$$

- $I_i(s,q_1^2,q_2^2)$: dispersive integrals over $\gamma^*\gamma^* \to \pi\pi$ **helicity partial waves**

Pion transition form factor: physical regions



Pion transition form factor: unitarity relations

process	unitarity relations	SC 1	SC 2
			$F_{\pi^0\gamma\gamma}$
		$F_{3\pi}$	$\sigma(\gamma\pi\to\pi\pi)$
			$\Gamma_{\pi^0\gamma}$
		$\Gamma_{3\pi}$	$\frac{d^2\Gamma}{dsdt}(\omega,\phi\to 3\pi)$
			$\sigma(e^+e^-\to\pi^0\gamma)$
		$\sigma(e^+e^-\to 3\pi)$	$\sigma(\gamma\pi\to\pi\pi)$ $\frac{d^2\Gamma}{dsdt}(\omega,\phi\to 3\pi)$
		$F_{3\pi}$	$\sigma(e^+e^-\to 3\pi)$

$\gamma\pi\to\pi\pi$

$\omega\to 3\pi$, $\phi\to 3\pi$

$\gamma^*\to 3\pi$

resummation of

$\pi\pi$ rescattering

Phenomenological analysis of the singly-virtual form factor

- Dispersive representation for
  - $\omega, \phi \to 3\pi$ Niecknig, Kubis, Schneider 2012

    $\hookrightarrow$ normalization fixed by **decay width**
  - $\gamma\pi \to \pi\pi$ MH, Kubis, Sakkas 2012

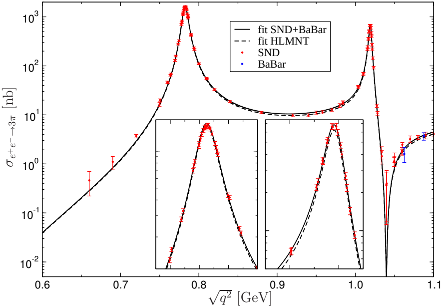
    $\hookrightarrow$ normalization fixed by **chiral anomaly** $F_{3\pi} = e/(4\pi^2 F_\pi^3)$
- General virtualities: fit to $e^+e^- \to 3\pi$

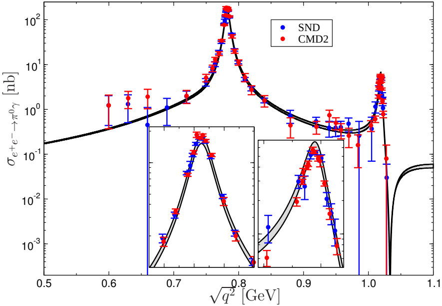
$$a(q^2) = \alpha + \beta q^2 + \frac{q^4}{\pi}\int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im}\,\mathscr{A}(s')}{s'^2(s'-q^2)}$$

$$\mathscr{A}(q^2) = \frac{c_\omega}{M_\omega^2 - q^2 - i\sqrt{q^2}\Gamma_\omega(q^2)} + \frac{c_\phi}{M_\phi^2 - q^2 - i\sqrt{q^2}\Gamma_\phi(q^2)}$$

- α fixed by $F_{3\pi}$, $\Gamma_{\omega/\phi}(q^2)$ include 3π, $K\bar{K}$, $\pi^0\gamma$ channels
- Good analytic properties, free parameters: β, c_ω, c_ϕ
- Valid up to 1.1 GeV, also fit including ω', ω'' to estimate uncertainties

Predicting $\sigma(e^+e^- \to \pi^0\gamma)$ from $\sigma(e^+e^- \to 3\pi)$





- Perfect fit to SND + BaBar up to 1.1 GeV
- Large χ^2 for world data, but small shift of central curve
- Result for the form factor checked in $e^+e^- \to \pi^0\gamma$

Extraction of slope and space-like continuation

- Another dispersion relation

$$F_{\pi^0\gamma^*\gamma}(q^2,0) = F_{\pi\gamma\gamma} + \frac{q^2}{\pi}\int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \frac{\operatorname{Im} F_{\pi^0\gamma^*\gamma}(s',0)}{s'(s'-q^2)}$$

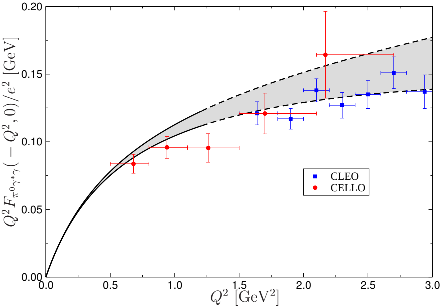
- Sum rules for $F_{\pi\gamma\gamma}$ and **slope parameters**

$$a_\pi = \frac{M_{\pi^0}^2}{F_{\pi\gamma\gamma}} \frac{1}{\pi}\int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \frac{\operatorname{Im} F_{\pi^0\gamma^*\gamma}(s',0)}{s'^2}$$

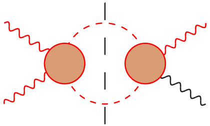
$$= (30.7 \pm 0.6) \times 10^{-3}$$

$$b_\pi = (1.10 \pm 0.02) \times 10^{-3}$$

- Main uncertainties
  - Low energies: variation in $\pi\pi$ shift and cutoffs
  - High energies: treatment of asymptotic region, $\omega', \omega'', \ldots$
- Soon to be tested at BESIII

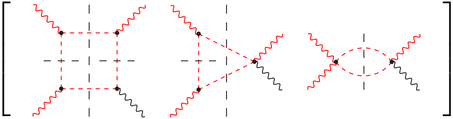


$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



- Separate terms with **simultaneous cuts**

$$\Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} = F_\pi^V(q_1^2)\,F_\pi^V(q_2^2)\,F_\pi^V(q_3^2) \times \Big[\quad \Big]$$



- Multiplication of sQED diagrams with F_π^V gives correct q^2-dependence
  $\hookrightarrow$ **not an approximation**
- Remaining $\pi\pi$ contribution included in $\bar{\Pi}_{\mu\nu\lambda\sigma}$ has cuts only in one channel
  $\hookrightarrow$ partial-wave expansion, dispersion relations for this part

$\pi\pi$ intermediate states: reconstruction theorem

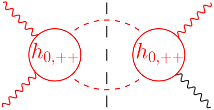
$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_i \Big(A^{\mu\nu\lambda\sigma}_{i,s}\Pi_i(s) + A^{\mu\nu\lambda\sigma}_{i,t}\Pi_i(t) + A^{\mu\nu\lambda\sigma}_{i,u}\Pi_i(u) \Big)$$

- Π_i functions with a **right-hand cut** only

  $\hookrightarrow$ similar to **reconstruction theorem** of $\pi\pi$ scattering Stern, Sazdjian, Fuchs 1993
- Keep discontinuity of lowest partial waves
- Dispersion integrals for Π_i required to have correct **soft-photon zeros**

  $\hookrightarrow$ forces subtraction constants to zero
- Gives relations such as

$$\Pi_1(s) = \frac{s-q_3^2}{\pi} \int\limits_{4M_\pi^2}^{\infty} \frac{ds'}{s'-q_3^2} K_1(s',s) \operatorname{Im} \bar{h}^0_{++,++}(s';q_1^2,q_2^2;q_3^2,0)$$



$$\operatorname{Im} h^0_{++,++}(s;q_1^2,q_2^2;q_3^2,0) = \frac{\sigma(s)}{16\pi} h^*_{0,++}(s;q_1^2,q_2^2) h_{0,++}(s;q_3^2,0)$$

$$\sigma(s) = \sqrt{1-\frac{4M_\pi^2}{s}} \qquad K_1(s',s) = \frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda(s',q_1^2,q_2^2)} \qquad \lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz+yz)$$

$\pi\pi$ intermediate states: non-diagonal terms

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_i \Big(A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u) \Big)$$

- Need to choose $A_i^{\mu\nu\lambda\sigma}$ so that Π_i are **free of kinematic singularities**
- General procedure for finding such a basis Bardeen, Tung 1968, Tarrach 1975
- Results in **non-diagonal terms**

$$\Pi_1(s) = \frac{s-q_3^2}{\pi} \int\limits_{4M_\pi^2}^{\infty} \frac{ds'}{s'-q_3^2} \bigg(K_1(s',s) \operatorname{Im} \bar{h}^0_{++,++}(s') + \frac{2\xi_1\xi_2}{\lambda(s',q_1^2,q_2^2)} \operatorname{Im} \bar{h}^0_{00,++}(s') \bigg)$$

- Solved for S-wave, D-wave calculation in progress

Digression: $\gamma^*\gamma^* \to \pi\pi$

- Similar analysis for $\gamma^*\gamma^* \to \pi\pi$: Bardeen–Tung–Tarrach basis

  $\hookrightarrow$ partial-wave dispersion relations (**Roy–Steiner equations**)

- Find similar non-diagonal kernels
- Check within 1-loop ChPT

$$\frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} dt' \left\{ \left(\frac{1}{t'-t} - \frac{t'-q_1^2-q_2^2}{\lambda(t',q_1^2,q_2^2)} \right) \mathrm{Im}\, h_1(t';q_1^2,q_2^2) + \frac{2q_1^2q_2^2}{\lambda(t',q_1^2,q_2^2)} \mathrm{Im}\, h_2(t';q_1^2,q_2^2) \right\}$$

$$= 1 + 2\left(M_\pi^2 + \frac{tq_1^2q_2^2}{\lambda(t,q_1^2,q_2^2)} \right) C_0(t,q_1^2,q_2^2) + \frac{t(q_1^2+q_2^2)-(q_1^2-q_2^2)^2}{\lambda(t,q_1^2,q_2^2)} \bar{J}(t)$$

$$- \frac{q_1^2(t+q_2^2-q_1^2)}{\lambda(t,q_1^2,q_2^2)} \bar{J}(q_1^2) - \frac{q_2^2(t+q_1^2-q_2^2)}{\lambda(t,q_1^2,q_2^2)} \bar{J}(q_2^2)$$

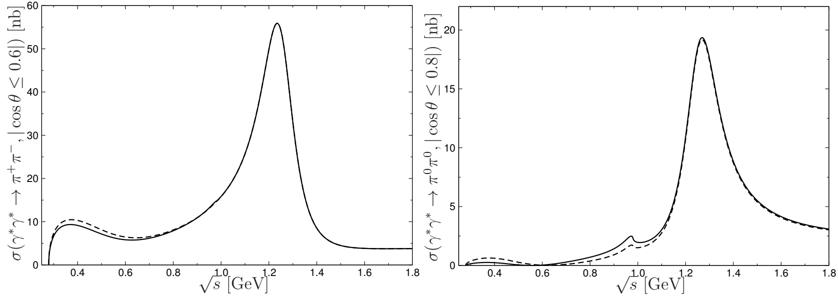
$$\mathrm{Im}\, h_1(t;q_1^2,q_2^2) = 2\left(M_\pi^2 + \frac{tq_1^2q_2^2}{\lambda(t,q_1^2,q_2^2)} \right) \mathrm{Im}\, C_0(t,q_1^2,q_2^2) + \frac{t(q_1^2+q_2^2)-(q_1^2-q_2^2)^2}{\lambda(t,q_1^2,q_2^2)} \mathrm{Im}\, \bar{J}(t)$$

$$\mathrm{Im}\, h_2(t;q_1^2,q_2^2) = -\frac{1}{\lambda(t,q_1^2,q_2^2)} \left[\left(t^2 - (q_1^2-q_2^2)^2 \right) \mathrm{Im}\, C_0(t,q_1^2,q_2^2) + 4t\,\mathrm{Im}\, \bar{J}(t) \right]$$

$\hookrightarrow$ non-diagonal kernels crucial for doubly-virtual case

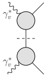
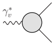
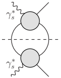
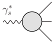
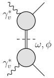
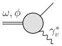
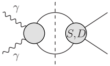
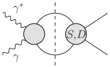
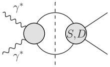
Digression: $\gamma^*\gamma^* \to \pi\pi$

- Reason for digression: simplified model for MC generators?
- **Rescattering** for large virtualities dominated by **non-diagonal terms**, but overall size moderate, e.g. for $Q_1^2 = Q_2^2 = 0.5\,\text{GeV}^2$



- Many caveats: definition of cross section, just pion pole for LHC, BW for $f_2(1270)$, dipole for q^2-dependence (Λ from Pauk, Vanderhaeghen 2014), subtractions, $f_0(980)$, ...

$\gamma^*\gamma^* \to \pi\pi$ partial waves: unitarity relations

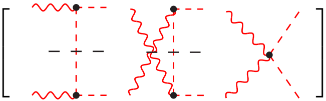
process	building blocks and SC	
		left-hand cut
		π
		2π
		$3\pi\,(\sim \omega, \phi)$
		unitarity relations
	$\alpha_1 \pm \beta_1,\, \alpha_2 \pm \beta_2$	on-shell
	$\alpha_1\,(q^2) \pm \beta_1\,(q^2)$, ChPT $e^+e^- \to \pi\pi\gamma$ $e^+e^- \to e^+e^-\pi\pi$	singly-virtual
	ChPT $(e^+e^- \to \pi\pi\gamma)$ $e^+e^- \to e^+e^-\pi\pi$	doubly-virtual

Simplified input for $\gamma^*\gamma^* \to \pi\pi$: pion pole

Omnès representation for S-wave

$$h^I_{0,++}(s) = N^I_{0,++}(s) + \frac{\Omega^I_0(s)}{\pi} \int_{4M_\pi^2}^{s_m} ds' \frac{\sin\delta^I_0(s')}{|\Omega^I_0(s')|} \left[\left(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda(s',q_1^2,q_2^2)} \right) N^I_{0,++}(s') + \frac{2\xi_1\xi_2}{\lambda(s',q_1^2,q_2^2)} N^I_{0,00}(s') \right]$$

- Starting point: **Roy–Steiner** equations for $\gamma^*\gamma^* \to \pi\pi$
- Omnès factors $\Omega^I_0(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{s_m} ds' \frac{\delta^I_0(s')}{s'(s'-s)} \right\}$
- LHC approximated by **pion pole** $N^I_{0,\lambda_1\lambda_2}$ only

$F^V_\pi(q_1^2) F^V_\pi(q_2^2) \times$ 

- **Finite matching point**: $h^I_{0,++}(s) = 0$ above s_m
- Take $\sqrt{s_m} = 0.98\,\text{GeV}$

  $\hookrightarrow$ no $f_0(980)$ or coupling to $K\bar{K} \Rightarrow$ "σ-contribution"

Preliminary numbers: $\pi\pi$ rescattering for S-waves

- $a_\mu^{\pi\pi}$ in units of 10^{-11}

phase shift δ_1^1	$I=0$ DR 1	$I=0$ DR 2	$I=2$ DR 1	$I=2$ DR 2
CCL	-7.13 ± 0.03	-6.75 ± 0.06	1.82 ± 0.01	1.68 ± 0.01
CCL + ρ',ρ''	-7.79 ± 0.03	-7.38 ± 0.06	2.00 ± 0.01	1.84 ± 0.01

- Adding the FsQED contribution

phase shift δ_1^1	FsQED	sum DR 1	sum DR 2
CCL	-13.77 ± 0.01	-19.08 ± 0.03	-18.84 ± 0.06
CCL + ρ',ρ''	-14.65 ± 0.01	-20.44 ± 0.03	-20.19 ± 0.06

- Comparing to the literature Jegerlehner, Nyffeler 2009

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	–	–	–	$2.3\pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

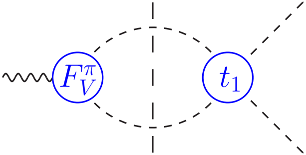
Summary

- **Dispersive framework** for the calculation of the HLbL contribution to a_μ
- Includes **one- and two-pion** intermediate states
- **Master formula** for $\pi\pi$ in terms of $\gamma^*\gamma^* \to \pi\pi$ **partial waves**
- Next steps
  - Doubly-virtual pion transition form factor
  - Refinement of $\gamma^*\gamma^* \to \pi\pi$ input
  - Comprehensive treatment of *D*-waves
  - Error analysis: which input quantity has the biggest impact on a_μ?

Pion vector form factor

- **Dispersive approach**: resum $\pi\pi$ rescattering F_V^π as example
- **Unitarity** for **pion vector form factor**

$$\operatorname{Im} F_V^\pi(s) = \theta\big(s - 4M_\pi^2\big) F_V^\pi(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

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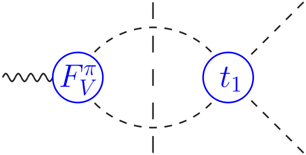

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- Solution in terms of **Omnès function** Omnès 1958

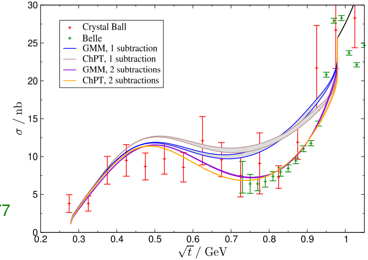
$$F_V^\pi(s) = P(s)\Omega_1(s) \qquad \Omega_1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1(s')}{s'(s'-s)}\right\}$$

- Asymptotics + normalization $\Rightarrow$ $P(s) = 1$

$\gamma^*\gamma^* \to \pi\pi$ partial waves

Roy(–Steiner) equations = Dispersion relations + partial-wave expansion + crossing symmetry + unitarity + gauge invariance

- **On-shell case** $\gamma\gamma \to \pi\pi$ Moussallam 2010, MH, Phillips, Schat 2011 $\hookrightarrow$ precision determination of $\sigma \to \gamma\gamma$ coupling
- **Singly-virtual** $\gamma^*\gamma \to \pi\pi$ Moussallam 2013
- **Doubly-virtual** $\gamma^*\gamma^* \to \pi\pi$: **anomalous thresholds**
  Colangelo, MH, Procura, Stoffer arXiv:1309.6877



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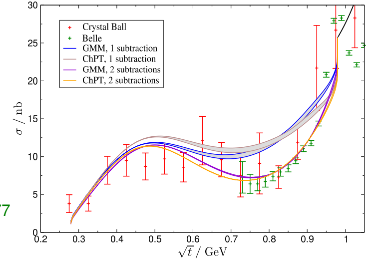
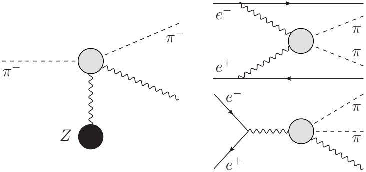
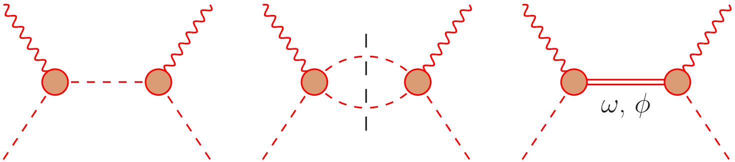
- Constraints
  - **Low energies**: pion polarizabilities, ChPT
  - **Primakoff**: $\gamma\pi \to \gamma\pi$ (COMPASS), $\gamma\gamma \to \pi\pi$ (JLab)
  - **Scattering**: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - **(Transition) Form factors**: F_V^π, $\omega, \phi \to \pi^0\gamma^*$

$\hookrightarrow$ discuss these constraints in the following





Left-hand cut



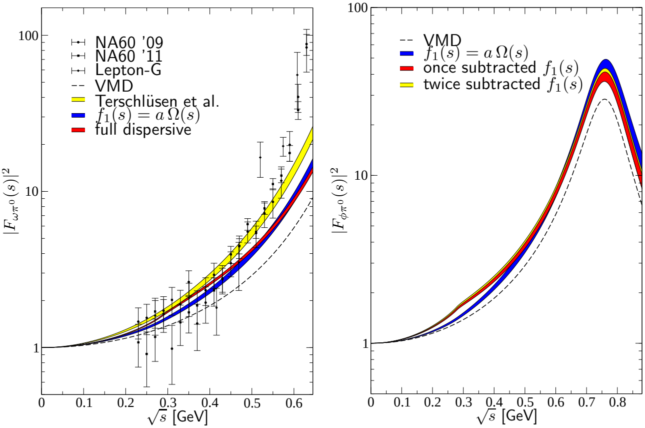
- **Pion pole**: coupling determined by F_V^π as before
- **Multi-pion intermediate states**: approximate in terms of **resonances**
  - $2\pi \sim \rho$: can even be done **exactly** using $\gamma^* \to 3\pi$ amplitude

    $\hookrightarrow$ see pion transition form factor
  - $3\pi \sim \omega, \phi$: narrow-width approximation

    $\hookrightarrow$ **transition form factors** for $\omega, \phi \to \pi^0\gamma^*$
  - Higher intermediate states also potentially relevant: **axials**, **tensors**

    $\hookrightarrow$ **sum rules** to constrain their transition form factors Pauk, Vanderhaeghen 2014

$\omega, \phi \to \pi^0 \gamma^*$ transition form factor



Schneider, Kubis, Niecknig 2012

- Puzzle of steep rise in $F_{\omega\pi^0}$

  $\hookrightarrow$ measurement of $F_{\phi\pi^0}$ would be extremely valuable

- Clarification important for pion transition form factor, but also $\gamma^*\gamma^* \to \pi\pi$

Subtraction functions

Omnès representation for S-wave

$$h_{0,++}(s) = \Delta_{0,++}(s) + \Omega_0(s)\Bigg[\frac{1}{2}(s-s_+)a_+(q_1^2,q_2^2) + \frac{1}{2}(s-s_-)a_-(q_1^2,q_2^2) + q_1^2 q_2^2 b(q_1^2,q_2^2)$$

$$+\frac{s(s-s_+)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_+)(s'-s)|\Omega_0(s')|} + \frac{s(s-s_-)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_-)(s'-s)|\Omega_0(s')|}$$

$$+\frac{2q_1^2 q_2^2 s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,00}(s')}{s'(s'-s_+)(s'-s_-)|\Omega_0(s')|}\Bigg] \qquad s_\pm = q_1^2 + q_2^2 \pm 2\sqrt{q_1^2 q_2^2}$$

- Inhomogeneities $\Delta_{0,++}(s), \Delta_{0,00}(s)$ include left-hand cut
- **Subtraction functions**
  - $b(q_1^2,q_2^2)$ and $a_+(q_1^2,q_2^2) - a_-(q_1^2,q_2^2)$ multiply $q_1^2 q_2^2$ and $\sqrt{q_1^2 q_2^2}$
    $\hookrightarrow$ inherently doubly-virtual observables $\Rightarrow$ need ChPT (or lattice)
  - However: $a(q_1^2,q_2^2) = (a_+(q_1^2,q_2^2) + a_-(q_1^2,q_2^2))/2$ fixed by singly-virtual measurements
    $\hookrightarrow$ compare with chiral prediction, uncertainty estimates for the other functions

Subtraction functions: chiral constraints

- 1-loop result for arbitrary q_i^2, e.g.

$$a^{\pi^0}(q_1^2,q_2^2) = -\frac{M_\pi^2}{8\pi^2 F_\pi^2 (q_1^2-q_2^2)^2}\bigg\{q_1^2+q_2^2+2\Big(M_\pi^2(q_1^2+q_2^2)+q_1^2 q_2^2\Big)C_0(q_1^2,q_2^2)$$
$$+q_1^2\bigg(1+\frac{6q_2^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_1^2)+q_2^2\bigg(1-\frac{6q_1^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_2^2)\bigg\}$$

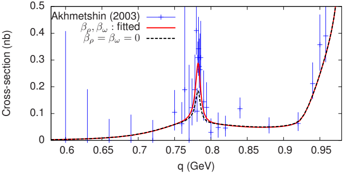
- Special case: $q_1^2=q_2^2=0$

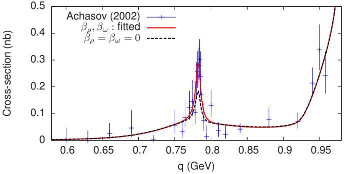
$$a^{\pi^\pm}(0,0) = \frac{\bar{l}_6-\bar{l}_5}{48\pi^2 F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^\pm} \qquad b^{\pi^\pm}(0,0)=0$$

$$a^{\pi^0}(0,0) = -\frac{1}{96\pi^2 F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^0} \qquad b^{\pi^0}(0,0) = -\frac{1}{1440\pi^2 F_\pi^2 M_\pi^2}+\cdots$$

$\hookrightarrow$ resum higher chiral orders into **pion polarizabilities**

Subtraction functions: dispersive representation





Moussallam 2013

- Singly-virtual case: phenomenological representation with chiral constraints

  $\hookrightarrow$ parameters fixed from $e^+e^- \to \pi^0\pi^0\gamma$ (CMD2 and SND) Moussallam 2013

- **Dispersive representation**: imaginary part from $2\pi, 3\pi, \ldots$

  $\hookrightarrow$ analytic continuation from time-like to space-like kinematics

- Example: $I = 2 \Rightarrow$ isovector photons $\Rightarrow 2\pi \sim \rho$

$$a^2(q_1^2, q_2^2) = \alpha_0\Big[\alpha^2 + \alpha\Big(q_1^2\mathscr{F}^\rho(q_1^2) + q_2^2\mathscr{F}^\rho(q_2^2)\Big) + q_1^2 q_2^2\mathscr{F}^\rho(q_1^2)\mathscr{F}^\rho(q_2^2)\Big]$$

$$\mathscr{F}^\rho(q^2) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds \frac{q_{\pi\pi}^3(s)\big(F_\pi^V(s)\big)^*\Omega_1(s)}{s^{3/2}(s-q^2)} \qquad q_{\pi\pi}(s) = \sqrt{\frac{s}{4} - M_\pi^2}$$

$\hookrightarrow$ α_0 and α can be determined from $a^2(q^2, 0)$ alone!