Spin Matrix Theory

A quantum mechanical model of the AdS/CFT correspondence

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Based on JHEP 1411: 134 (arXiv:1409.4417) by TH and M. Orselli

Motivation:
How to see the emergence of gravity, space and time from a quantum theory?

The Holographic Principle

Gravity$_D$ is dual to QFT$_{D-1}$

Gravity$_D$ at high energies: Described by QFT$_{D-1}$ (without gravity)

QFT$_{D-1}$ strongly coupled: Described by gravity in D-dimensions

Solves in principle the problem of quantizing gravity: Space, time and gravity emerges in a non-local way from a quantum theory without gravity

Space, time and gravity are emergent rather than fundamental concepts
→ The quantum theory lives in a separate "space"

But how does this work in practice?

→ We need a concrete realization of the Holographic Principle to study this

Best understood holographic duality: The AdS_5/CFT_4 duality

AdS_5: 5-dim. gravity with negative cosmological constant (asympt. AdS_5)

CFT_4: $\mathcal{N}$=4 Super-Yang-Mills theory (SYM) with gauge group SU(N)

Small λ	Finite λ	Large λ	λ
CFT_4	?	5-dim. gravity	

To see emergence of gravity, space and time from quantum theory we need a unifying framework to describe what happens at finite λ

→ Such a unifying framework is not known in general, not even for this duality

Exception: The planar limit N = ∞

Small λ	Finite λ	Large λ	λ
planar CFT_4	Spin chain	Tree-level string theory	

But gravitational coupling ~ $\frac{1}{N^2}$ → No gravity in planar limit!

Can we find a unifying framework of AdS/CFT for finite, large N?

Is there a N x N matrix generalization of the spin chain?

Not possible to find in general without some kind of organizing principle

- Would amount to describing CFT_4 at any finite λ and N
- Integrability symmetry presumably absent for $N < \infty$
- Spin chain has infinite-range interaction

→ We need to simplify the problem to make progress

Our discovery:

- There exists zero-temperature critical points of CFT_4
- Near these CFT_4 simplifies drastically → Described by a new type of QM theory we call Spin Matrix theory
- Spin Matrix theory: An N x N matrix generalization of any nearest-neighbor spin chain

What is Spin Matrix theory?

A new type of quantum mechanical theory, generalizes nearest-neighbor spin chains

Hilbert space:

Defined given

- Representation R_s of semi-simple (super-) Lie group G_s (called the "spin" group)
- The adjoint representation of U(N) → N x N matrices

Hilbert space built from harmonic oscillators: $(a_s^\dagger)^i{}_j$ — $s \in R_s$: Spin index; (i,j): Matrix indices

$$\left[(a^s)^j{}_i\,,\,(a^\dagger_{s'})^k{}_l\right] = \delta^s_{s'}\delta^j_l\delta^k_i$$ with vacuum $$(a^s)^j{}_i|0\rangle = 0$$

Singlet condition: $\Phi^i{}_j|0\rangle = 0$ with $\Phi^i{}_j = \sum_{s\in R_s}\sum_{k=1}^N \Big((a_s^\dagger)^i{}_k (a^s)^k{}_j - (a_s^\dagger)^k{}_j (a^s)^i{}_k \Big)$

Spin Matrix theory Hilbert space spanned by the multi-trace states

$$\mathrm{Tr}(a^\dagger_{s_1} a^\dagger_{s_2}\cdots a^\dagger_{s_k})\,\mathrm{Tr}(a^\dagger_{s_{k+1}}\cdots)\cdots\mathrm{Tr}(\cdots a^\dagger_{s_L})|0\rangle$$

Note that such states are not linearly independent for finite N

Hamiltonian (interacting part):

- Annihilate 2, create 2
- Commute with G_s generators
- Spin and matrix parts factorize

→ Fixes interacting part of Hamiltonian to be

$$H_{\text{int}} = \frac{1}{N} U_{sr}^{s'r'} \sum_{\sigma \in S(4)} T_\sigma \, (a^\dagger_{s'})^{i_{\sigma(1)}}{}_{i_3} (a^\dagger_{r'})^{i_{\sigma(2)}}{}_{i_4} (a^s)^{i_{\sigma(3)}}{}_{i_1} (a^r)^{i_{\sigma(4)}}{}_{i_2}$$

Spin part: $U : R_s \otimes R_s \to R_s \otimes R_s$

Choice for matrix part: $\sum_{\sigma \in S(4)} T_\sigma \sigma = (14) + (23) - (12) - (34)$

$$R_s \otimes R_s = \sum_J V_J$$

$$U_{sr}^{s'r'} = \sum_J C_J (P_J)_{sr}^{s'r'}$$

Motivation: Limits of CFT_4 + Spin chain generalization

Spin Matrix theory partition function:

$$Z(\beta, \mu_p) = \mathrm{Tr}(e^{-\beta(L + g H_{\text{int}} - \sum_p \mu_p K_p})$$

(Choice: Chem. pot. for L equal to one)

"Length" operator:

$$L = \sum_s \mathrm{Tr}(a_s^\dagger a^s)$$

K_p: Cartan gen's of G_s

g: Coupling of interaction

Spin Matrix theory a generalization of nearest-neighbor spin chains:

For N = ∞ limit multi-trace states linearly independent
→ Interpret single-trace states as spin chains

$$|s_1 s_2 \cdots s_L\rangle = \mathrm{Tr}(a^\dagger_{s_1} a^\dagger_{s_2} \cdots a^\dagger_{s_L})|0\rangle$$

Then the interacting part of Hamiltonian gives a nearest-neighbor spin chain:

$$H_{\mathrm{int}}|s_1 s_2 \cdots s_L\rangle = 2\sum_{k=1}^{L} U^{mn}_{s_k s_{k+1}} |s_1 \cdots s_{k-1} m n s_{k+2} \cdots s_L\rangle$$

Zero-temperature critical points of CFT_4 ($\mathcal{N}$=4 SYM):

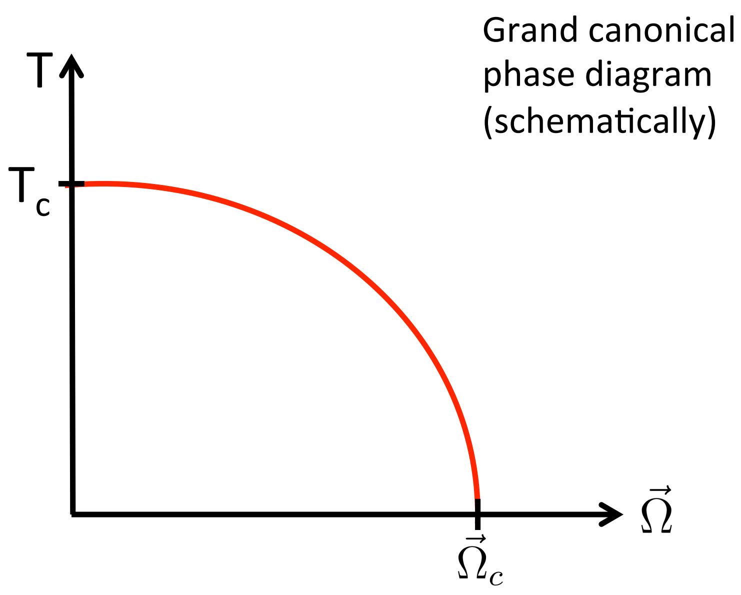
CFT_4 ($\mathcal{N}$=4 SYM) on R x S^3 in the grand canonical ensemble

→ Temp. T and chemical potentials turned on $\vec{\Omega} = (\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3)$

For zero chemical potentials, any coupling λ and large N there is a temperature T_c where one has a phase transition from confining behavior of the partition function $\log Z \sim \mathcal{O}(1)$ to deconfining behavior $\log Z \sim \mathcal{O}(N^2)$

Turn on chemical potentials

→ Curve/surface of phase transition temp's



Zero-temperature critical points: Points with zero temperature and critical values of the chemical potentials that lie arbitrarily close to phase transition points

There are nine zero-temperature critical points that are protected by supersymmetry (listed on a later slide)

Spin Matrix theory from CFT_4 near zero-temp. critical points:

Partition function of CFT_4 ($\mathcal{N}$=4 SYM) on R x S^3 in the grand canonical ensemble

$$Z(\beta,\vec{\Omega}) = \mathrm{Tr}\left(e^{-\beta D+\beta\vec{\Omega}\cdot\vec{J}}\right)$$

Dilatation operator: $D = D_0 + \lambda D_2 + \lambda^2 D_4 + \cdots$

Charges: $\vec{J} = (S_1, S_2, R_1, R_2, R_3)$

Chemical pot's: $\vec{\Omega} = (\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3)$

$\vec{\Omega}\cdot\vec{J} = \omega_1 S_1 + \omega_2 S_2 + \Omega_1 R_1 + \Omega_2 R_2 + \Omega_3 R_3$

Consider a zero-temp. critical point given by T=0 and $\vec{\Omega} = \vec{\Omega}_c$

Limit towards critical point: $(T,\vec{\Omega}) \to (0,\vec{\Omega}_c)$ and $\lambda \to 0$ with $\beta\lambda$ and $\beta(\vec{\Omega}-\vec{\Omega}_c)$ fixed

$$\beta D - \beta\vec{\Omega}\cdot\vec{J} = \beta(D-D_0) + \beta(D_0 - \vec{\Omega}_c\cdot\vec{J}) - \beta(\vec{\Omega}-\vec{\Omega}_c)\cdot\vec{J}$$

$\beta\lambda D_2$ gives $\tilde{\beta} g H_{\mathrm{int}}$

Higher loops → 0

Only states with $D_0 = \vec{\Omega}_c\cdot\vec{J}$ survives

Gives terms of the form $\tilde{\beta}(L - \sum_p \mu_p K_p)$

→ Spin Matrix theory describes the physics near the zero-temp. critical point

Compared to CFT_4: 1) Reduced set of states. 2) Only one-loop interaction survives

Limit towards critical point: $(T, \vec{\Omega}) \to (0, \vec{\Omega}_c)$ and $\lambda \to 0$ with $\beta\lambda$ and $\beta(\vec{\Omega} - \vec{\Omega}_c)$ fixed

This limit is a low energy, non-relativistic limit

Magnon dispersion relation: $$D - D_0 = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} - 1$$

QFT → QM

Zero-temperature critical points of CFT$_4$ (N=4 SYM):

Critical point $(T, \omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3)$	Spin group G_s	Cartan diagram for algebra	Representation R_s
$(0,0,0,1,1,0)$	$SU(2)$	○	$[1]$
$(0,\frac{2}{3},0,1,\frac{2}{3},\frac{2}{3})$	$SU(1\|1)$	⊗	$[1]$
$(0,\frac{1}{2},0,1,1,\frac{1}{2})$	$SU(1\|2)$	○—⊗	$[1,0]$
$(0,0,0,1,1,1)$	$SU(2\|3)$	○—⊗—○—○	$[0,0,0,1]$
$(0,1,0,1,0,0)$	$SU(1,1)$	○	$[-1]$
$(0,1,0,1,\frac{1}{2},\frac{1}{2})$	$SU(1,1\|1)$	⊗—⊗	$[0,1]$
$(0,1,0,1,1,0)$	$SU(1,1\|2)$	⊗—○—⊗	$[0,1,0]$
$(0,1,1,1,0,0)$	$SU(1,2\|2)$	○—⊗—○—⊗	$[0,0,0,1]$
$(0,1,1,1,1,1)$	$SU(1,2\|3)$	○—⊗—○—○—⊗	$[0,0,0,1,0]$

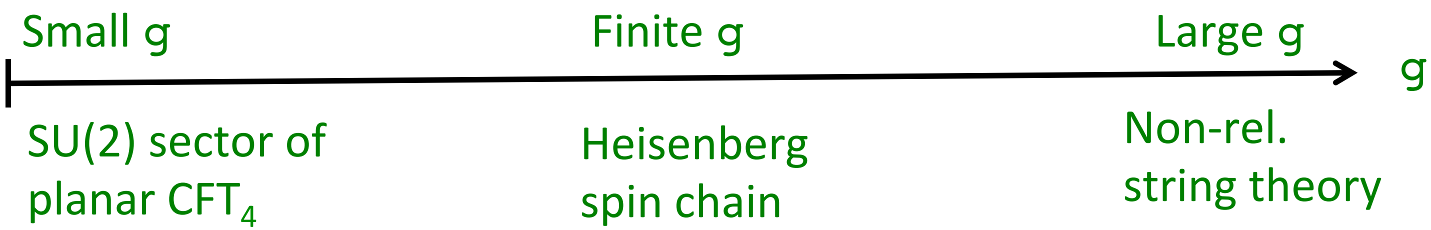
Table 1: Critical points of $\mathcal{N}=4$ SYM that can be described by Spin Matrix theory. Listed are the spin groups, the Cartan diagram for the corresponding algebra and the representations (in terms of Dynkin labels) that defines the Spin Matrix Theory for a given critical point.

SU(2) Spin Matrix theory: Simplest non-trivial case

Spin ½ representation of SU(2) with Hamiltonian $H_{\text{int}} = -\frac{1}{8\pi^2 N} \text{Tr}\left([a_\uparrow^\dagger a_\downarrow^\dagger][a^\uparrow, a^\downarrow] \right)$

Partition function: $Z(\beta) = \text{Tr}\left(e^{-\beta(L + g H_{\text{int}})} \right)$

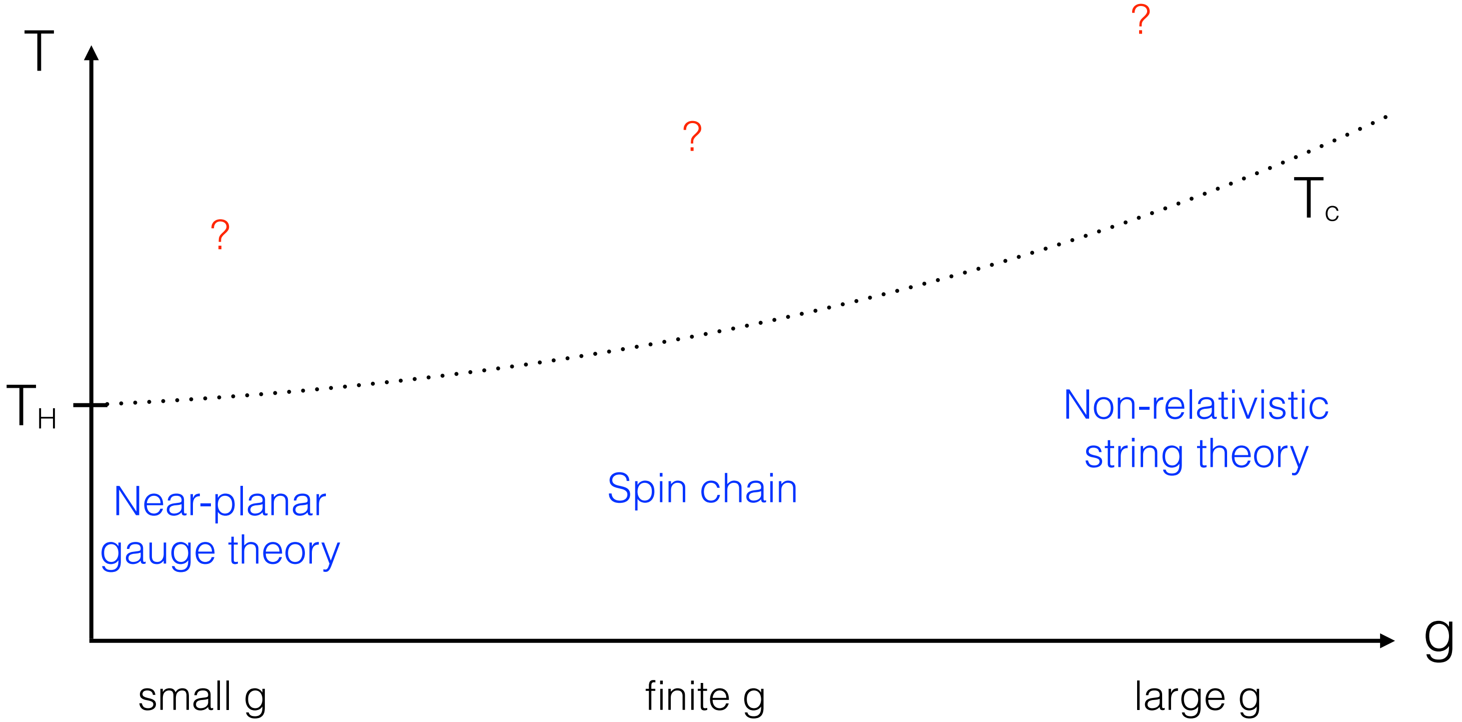
In the N = ∞ limit: $XXX_{1/2}$ Heisenberg spin chain:



For a given large N:

- Low T ⟷ N = ∞ + 1/N perturbations
- High T ⟷ Finite N effects (non-perturbative in 1/N)

Phase diagram for SU(2) Spin Matrix theory at low T:



At T=T_c we go from confining $\log Z \sim \mathcal{O}(1)$ to deconfining behavior $\log Z \sim \mathcal{O}(N^2)$

For N = ∞ this becomes the Hagedorn temperature

→ We can describe well what happens for low T
→ What about the high T phases?

Free SU(2) Spin Matrix theory at high T:

For g=0: $Z(\beta)_{g=0} = \mathrm{Tr}\left(e^{-\beta L}\right)$

Low T → L < N : Spin chain interpretation

High T → L >> N: What is the high temperature limit of the g=0 partition function?

We computed explicit partition fct. for N=2,3,4,5 and N →∞

For high T: $Z(\beta)_{g=0} \simeq \frac{a_N}{(1-x)^{N^2+1}}$

N	a_N
2	2^{-3}
3	$2^{-4}\,3^{-4}$
4	$2^{-13}\,3^{-5}$
5	$193\;2^{-18}\,3^{-8}\,5^{-5}$
→∞	$(0.11)^{N^2}$

Conjecture:
Classical high T limit of free SU(2) Spin Matrix theory corresponds to $N^2 + 1$ decoupled harmonic oscillators

Semi-classical result for large N:
High T phase corresponds to N^2 distinguishable decoupled harmonic oscillators

Classical high T limit of SU(2) Spin Matrix theory:

High T limit → Classical limit of SU(2) Spin Matrix theory

Get classical description using coherent states

$$|\lambda\rangle = \mathcal{N}_\lambda \exp\Big(\sum_s \mathrm{Tr}(\lambda_s a_s^\dagger)\Big)|0\rangle \text{ with } \langle\lambda|\lambda\rangle = 1 \text{ and } \lambda_s = \frac{1}{\sqrt{2}}(X_s + iP_s)$$

Singlet condition $0 = \langle\lambda|\Phi^i{}_j|\lambda\rangle$ becomes Gauss constraint $\sum_s [X_s, P_s] = 0$

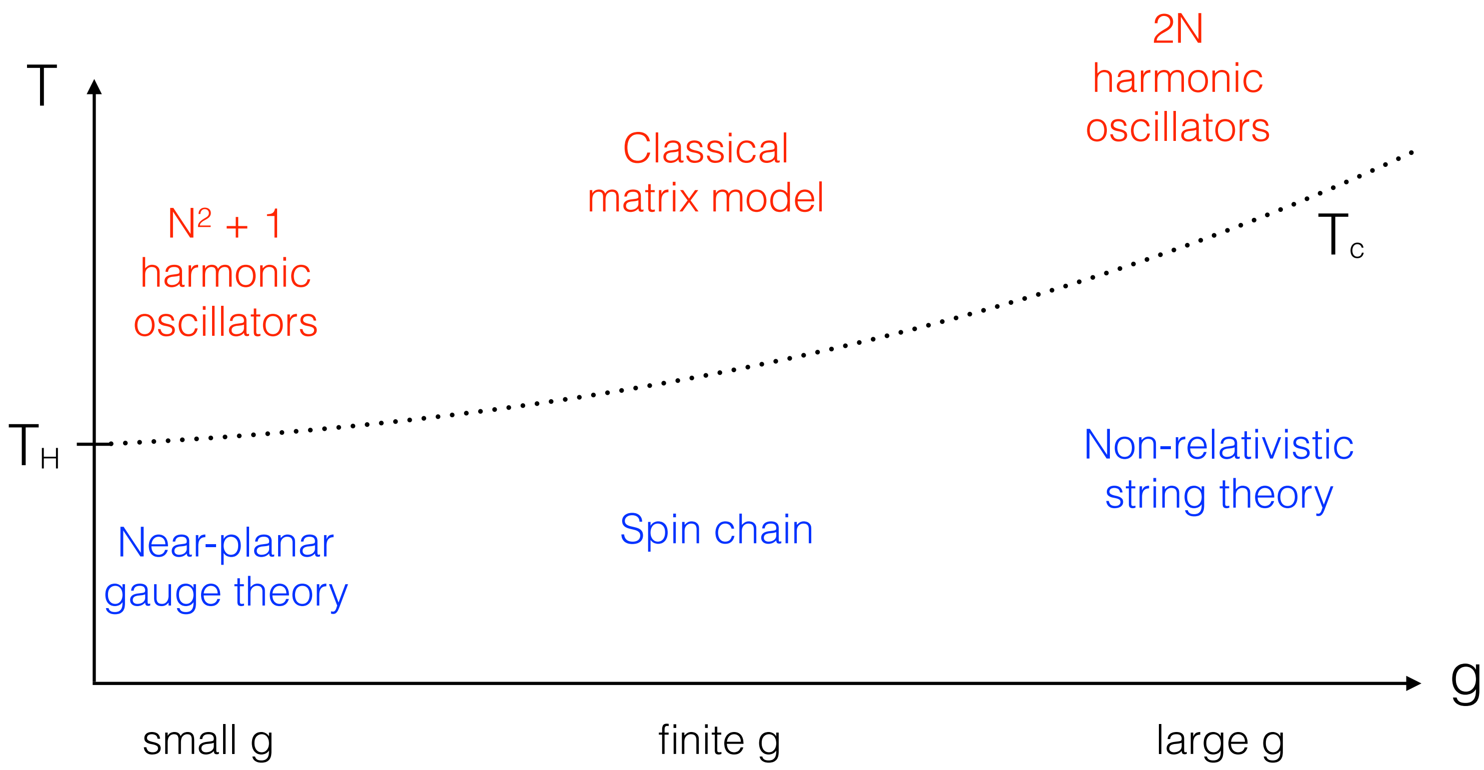
Classical Matrix model Hamiltonian for any coupling g:

$$H_{\mathrm{cl}}(X_s, P_s) = \langle\lambda|H|\lambda\rangle = \frac{1}{2}\sum_s \mathrm{Tr}(P_s^2 + X_s^2)$$
$$-\frac{g}{32\pi^2 N}\mathrm{Tr}\Big([X_\uparrow, X_\downarrow]^2 + [P_\uparrow, P_\downarrow]^2 + [X_\downarrow, P_\uparrow]^2 + [X_\uparrow, P_\downarrow]^2 + [X_\uparrow, P_\uparrow]^2 + [X_\downarrow, P_\downarrow]^2\Big)$$

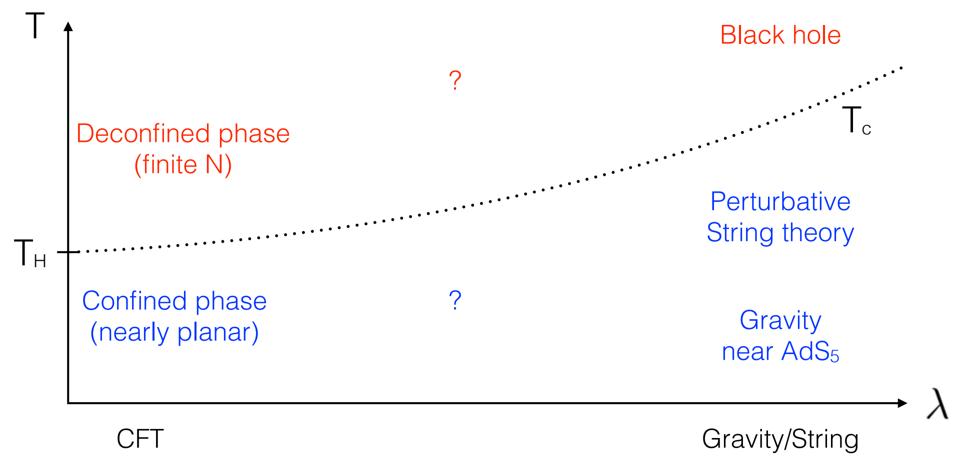
For g=0: $N^2+1 = 2N^2 - (N^2-1)$ → Counting works, but far from obvious that one gets N^2+1 decoupled harmonic oscillators for high T

For large g: X_s and P_s matrices diagonal → 2N harmonic oscillators

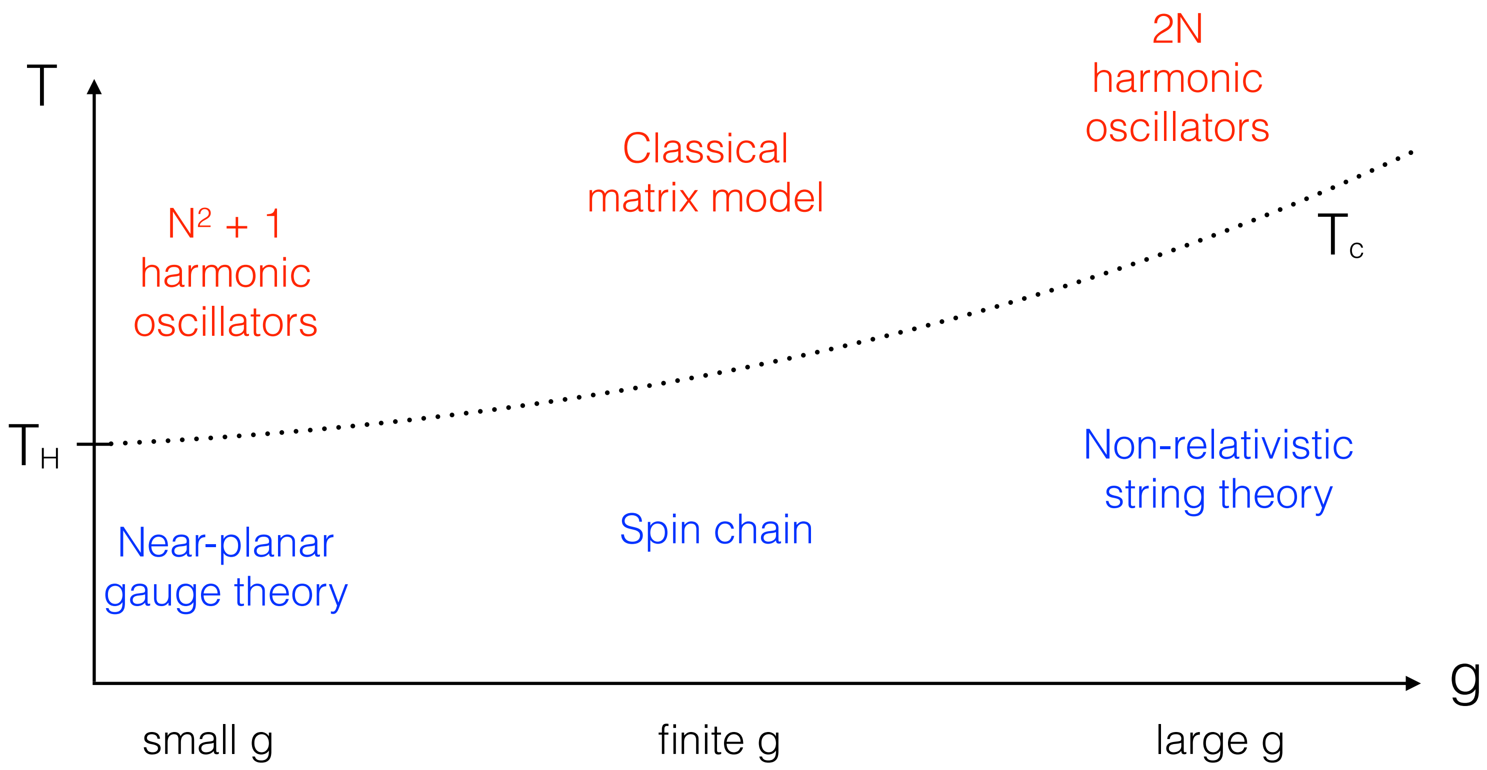
Full phase diagram for SU(2) Spin Matrix theory:



Quantum mechanical model
for AdS/CFT correspondence
(similar phase diagram)



Full phase diagram for SU(2) Spin Matrix theory:



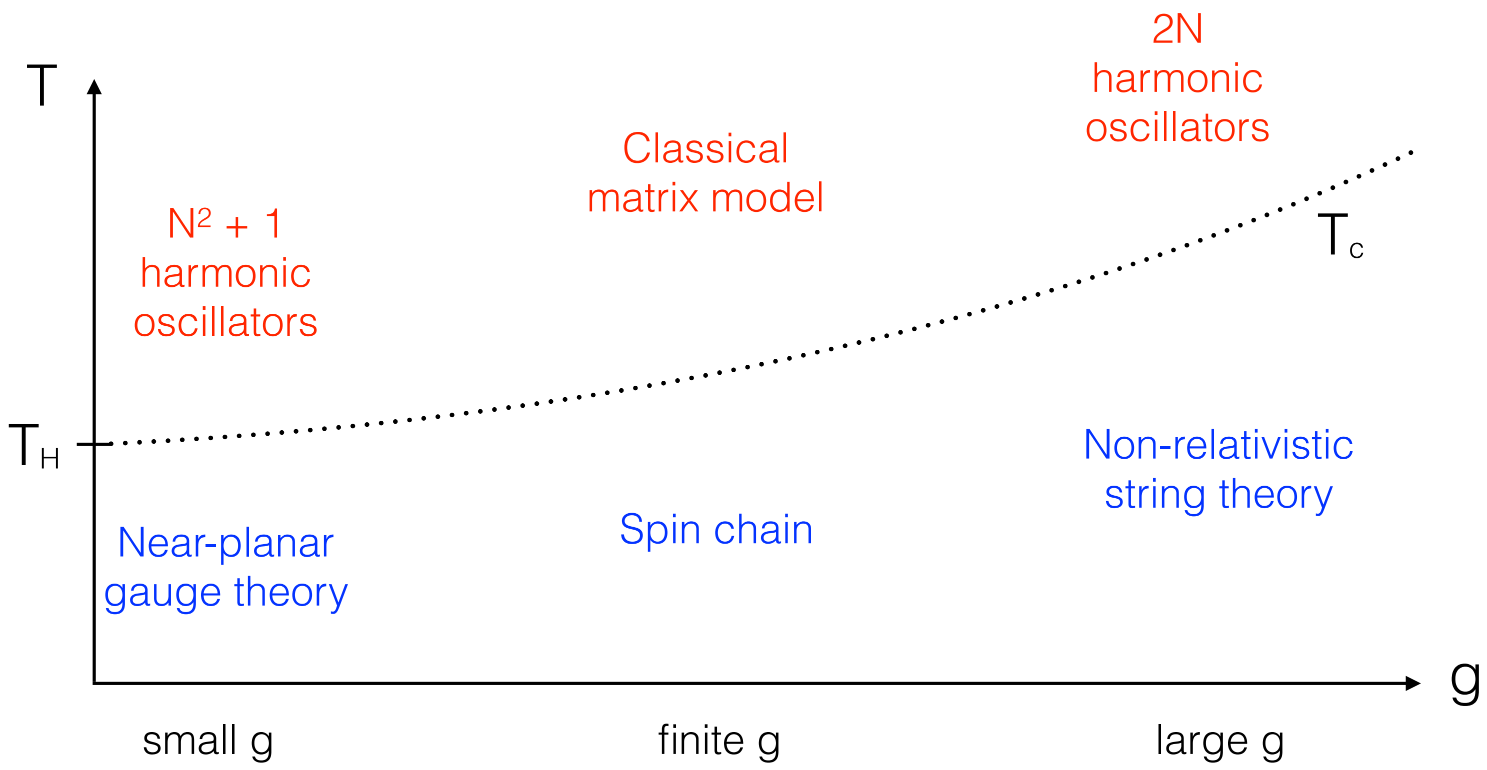
Spin Matrix theory provides a unifying framework for small g ("gauge theory") and large g (non-relativistic string theory) phases, enabling one to study finite g:

→ For low T: Spin chain gas (with splitting/joining)

→ For high T: Classical matrix model

We can interpolate between small and large coupling both in the (near-)planar regime and the finite N regime

Full phase diagram for SU(2) Spin Matrix theory:



We have shown that non-relativistic string theory can be obtained from Zero-temp. critical point limit of string theory on $AdS_5 \times S^5$

2N harmonic oscillators should be matched to highly excited gas of ¼ BPS Giant Gravitons

Can we extend this to near-BPS as well?

Conclusion and discussion:

Spin Matrix theory:

- A new type of Quantum Mechanical theory
- Describes $\mathcal{N}$=4 SYM near zero-temp. critical points
- Is a N x N matrix generalization of nearest-neighbor spin chains
- Is simple enough to be able to study the strong coupling limit

Our proposal: Spin Matrix theory is a unifying framework for certain holographic dualities

AdS/CFT correspondence:

SU(1,2|3) Spin Matrix theory potentially the most interesting since it is closest to the full AdS/CFT correspondence

$g \rightarrow \infty$ limit of SU(1,2|3) Spin Matrix theory = SUSY sector of string theory on $AdS_5 \times S^5$

1/16 BPS SUSY black hole: Should be described by SU(1,2|3) Spin Matrix theory

Can we also describe the near-BPS regime from SU(1,2|3) Spin Matrix theory at large g?

Holography:

Spin Matrix theory with non-compact spin group

→ N = ∞ , g >> 1: Non-relativistic string theory with non-compact target space

→ N < ∞, g >> 1: String coupling non-zero
Gravity, space and time could emerge as limit of Spin Matrix theory
What is the emergent gravitational theory?

If we manage to solve this:
Explicit example of emergence of gravity, space and time from a quantum theory!

Would mean that Spin Matrix theory provides new concrete realizations of the Holographic Principle
Moreover, Spin Matrix theory gives a unifying framework such that one can interpolate between the quantum side, and the gravity side