The Membrane Paradigm at Large D

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References

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Builds on earlier work by Emparan, Tanabe, Suzuki (EST) and collaborators

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Introduction

- In this talk we study the classical vacuum Einstein Equations.

$$R_{\mu\nu} = 0$$

- Deceptively simple equations capture a great deal of complicated dynamics, in particular involving black holes.
- E.g. consider the collision of two black holes. Analytically intractable. Phenonenon seems too complicated to ever admit an exact solution. Progress in numerics, but also very difficult.
- Natural instinct of a theorist: search for a parameter and do perturbation theory. However Einstein's equations do not have a parameter. Can we invent one?

Introduction

- Similar issue with analysis of (quantum) $SU(3)$ Yang Mills theory in four dimensions. To make progress t' Hooft invented a parameter by generalizing to the study of $SU(N)$. New effective parameter $\frac{1}{N}$.
- In the early 1980s Witten studied the problem of quantum bound states with the $\frac{1}{r}$ potential in a large number of dimensions. Emergent semiclassical picture of (e.g.) the Helium atom, with definite inter nuclear separation and 'bond angle'. While Witten's motivations were pedegogical, some chemists today find this approximation in the study of complicated real world molecules.
- This talk. Buiding on earlier work by Emparan, Suzuki, Tanabe and collaborators, we adopt a similar strategy for the analysis of black hole dynamics in classical gravity. Our parameter is $\frac{1}{D}$. D is the number of spacetime dimensions.

Introduction: the membrane region

- Schwarschild Black hole in D spacetime dimenstions.

$$ds^2 = -\left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right) dt^2 + \frac{dr^2}{\left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right)} + r^2 d\Omega_{D-2}^2 \tag{1}$$

- EST made the following important observation. If r is held fixed at any value greater than r_0 as $D \to \infty$ then metric reduces to flat space.
- On the other hand set $r = r_0(1 + \frac{R}{D-3})$ and keep R fixed as D is sent to ∞ then

$$\lim_{D\to\infty} \left(\frac{r_0}{r}\right)^{D-3} = \lim_{D\to\infty} (1 + \frac{R}{D-3})^{-(D-3)} = e^{-R}.$$

Thus 'tail' of the black hole extends only over the distance $\frac{r_0}{D}$. We refer to this thin layer as the 'membrane' region.

Introduction: Light Qasinormal Modes

- EST proceeded to compute the quasinormal modes of the Schwarzschild Black hole in an expansion in $\frac{1}{D}$. As usual there are an infinite number of qasinormal modes at every angular momentum. At any angular momentum EST find that all but a finite number of quasinormal modes have frequency of order $\frac{D}{r_0}$, the inverse thickness of the membrane.
- At each angular momentum, however, there are also a small finite number light modes (more below), whose frequency is $\mathcal{O}(1/r_0)$. The frequencies of these modes have imaginary parts comparable to real parts and so are dissipative. Moreover they are localized entirely in the membrane region.

Introduction: Effective low energy theory?

- When a few light modes are separated from a tower of heavy modes, it is usually possible to find a consistent autonomous nonlinear effective theory of the light modes obtained by ‘integrating out’ the heavy modes.
- As the light quasinormal modes are localized in the membrane region in the current context, the effective theory should capture the dynamics of the membrane.
- In this talk we will explain how this works and determine the equations of motion of the effective theory (to leading order) by a direct analysis of Einstein’s equations. The method we employ is reminicent of the Fluid Gravity correspondence. However we work in flat space; our approximations are justified by the expansion in $\frac{1}{D}$ rather than the long wavelength limit; we do not require derivatives to be small in units of the Schwarschild radius.

Einstein's equations with $SO(d+1)$ Isometry

- The large D limit is a simplification only when observables are kept fixed as D is taken to infinity.
- We divide up the D dimensions into two sets of $p+2$ and $d+1$ respectively ($D = p+d+3$). We then study only those spacetimes that enjoy an $SO(d)$ invariance.
- In other words we require the metric to take the form

$$ds^2_{full} = g_{\mu\nu} dx^\mu dx^\nu + e^\phi d\Omega_d^2 \tag{2}$$

E.g. flat space

$$ds^2 = dw_a dw^a + ds^2 + s^2 d\Omega_d^2$$

- In this talk we take the limit $d \to \infty$ with p held fixed.

The large d limit

- Einstein's equations become

$$\begin{aligned} e^{-\phi}(d-1) - \frac{d}{4}(\partial\phi)^2 - \frac{1}{2}\nabla^2\phi = 0 \\ R_{\mu\nu} = \frac{d}{2}\nabla_\mu\nabla_\nu\phi + \frac{d}{4}\nabla_\mu\phi\nabla_\nu\phi \end{aligned} \quad (3)$$

- Note factors of d behind $\partial\phi$. Intuitive reason: ϕ controls the size of a d sphere. Wiggles of ϕ much more expensive than those of $g_{\mu\nu}$. Sensible large d limit requires $g_{\mu\nu}$ and ϕ to be treated asymmetrically. ϕ varies on length scale unity. $g_{\mu\nu}$ varies on length scale $\frac{1}{d}$.
- Solutions of interest are nontrivial over length scales of order unity. However metric varies over length scale $\frac{1}{d}$. In order to write metric we think of spacetime collection of approximately d^{p+2} patches, each of size $\frac{1}{d}$. Solve Einsten's equations in each region and smoothly match.

Coordinates in a patch

- Consider a particular patch centred around x_0^μ. We use rescaled patch coordinates

$$x^\mu = x_0^\mu + \alpha_a^\mu \frac{y^a}{d} \tag{4}$$

- Also use rescaled metric and 'dilaton' gradient

$$\begin{aligned} G_{ab} &= d^2 g_{ab} \\ g_{\mu\nu} &= d^2 \alpha_\mu^a \alpha_\nu^b g_{ab} = \alpha_\mu^a \alpha_\nu^b G_{ab} \\ \chi_a &= \nabla_a \phi \times d = \alpha_a^\mu \nabla_\mu \phi \end{aligned} \tag{5}$$

Equations in the patch

- Adapted to patch coordinates the equations of motion become

$$\nabla_a \chi^a = e^{-\phi} \frac{d-1}{d} - \frac{1}{4}\chi^2$$
$$R_{ab} = \frac{1}{2}\nabla_a \chi_b + \frac{1}{4d}\chi_a \chi_b \qquad (6)$$

Here R_{ab} is the curvature with G_{ab} regarded as the metric. Similarly for covariant derivatives

- At leading order in $\frac{1}{d}$ the equations of motion reduce to

$$-4G^{ab}\Gamma^c_{ab}\chi_c = 4e^{-\phi} - \chi^2$$
$$2R_{ab} = -\Gamma^c_{ab}\chi_c \qquad (7)$$

ϕ and χ are constants in this equation. Note d has disappeared so we have a good large d limit.

Black Brane Solutions

- We have found a class of exact solutions to the equations of motion preserve translational invariance in $p+2$ of the $p+3$ directions.
- Extremely simple solutions given by

$$G_F + e^{-R} O^2 \tag{8}$$

- Here R is the 'non translationally invariant' coordinate, G_F is a constant metric and O a constant oneform s.t.

$$e^{\phi}\chi.\chi = 4, \quad (2dR-\chi).dR = 0, \quad (2dR-\chi).O = 0, \quad O.O = 0 \tag{9}$$

(all dot products evaluated using G_F).

Standard Coordinates for Black Branes

- Given any black brane solution of the form described above, it is possible to choose coordinates to put it in the form

$$e^{\phi} = x_0^2, \quad \chi = 2dR + \frac{2}{x_0} dX$$
$$ds^2 = 2dRdV - a\left(1 - e^{-R}\right) dV^2 + dY^i dY^i + \frac{dX^2}{1 - ax_0^2} \quad (10)$$

The parameter x_0 determines the constant value of ϕ. The parameter a is a modulus of the solutions: it has its origin in the scale symmetry of the vacuum Einstein equations. Can check that we obtain the black brane metric upon zooming into a patch of the Schwarzschild black hole centred about its origin. a determined by r_0.

The wiggly moving membrane

- We will now present the metric that describes a wiggly moving membrane, and that also solves the large d equations of motion everywhere outside its event horizon.
- Consider a scalar function B in flat space which vanishes on (for instance) a single compact surface. B divides the spacetime into an inside and outside. We require that B is negative inside, positive outside, and that the dot product of dB with $d\phi = ds$ is everywhere positive. The zeroes of B give the world volume of the membrane.
- In terms of B define an auxilliary function ψ by

$$\psi = 1 + \frac{dB.d\phi}{2dB.dB} B \tag{11}$$

ψ is less than one inside and greater than one outside the membrane.

Wiggly Moving Membrane

- We also need a null oneform field O in flat space. O is additionally required to obey the conditions

$$O.(\frac{d\Phi}{2} - d\psi)|_{B=0} = 0. \quad (12)$$

- The membrane metric is given by

$$ds^2 = ds^2_{flat} + \frac{O_\mu O_\nu dx^\mu dx^\nu}{\psi^{d+p}} \quad (13)$$

- Note that when $(d+p)(\psi-1)$ is of order unity, $\frac{1}{\psi^{d-1}} = e^{-(d+p)(\psi-1)}$ in the large d limit. Consider any patch centred around a point on the membrane. Use the coordinate $R = (d+p)(\psi-1)$. Then

$$ds^2 = ds^2_{flat} + e^{-R} O_\mu O_\nu dx^\mu dx^\nu \quad (14)$$

(13) is a black brane at leading order in large d.

Note also that

- Our metric reduces to flat space for $\psi - 1 \gg \frac{1}{d}$. In particular it solves Einstein's equations upto terms of order e^{-d} when $\psi - 1$ is held fixed as d is taken to infinity.
- In the membrane region it solves Einstein's equations at leading order in large d, because it reduces to the membrane.
- The metric does not solve Einstein's equations for $\psi - 1$ of order unity and negative. However it is easily verified that the event horizon of our metric deviates from the zeroes of B only at order $\frac{1}{d}$. For predictability of the outside we only need to solve Einstein's equations everywhere outside the event horizon. Our metric achieves this at leading order in $\frac{1}{d}$.

Data for the leading order solution

- It follows that the membrane metric above is a good starting point for a construction of solutions of Einstein's equations in a power series expansion in $\frac{1}{d}$.
- Our membrane metrics are naively parametereized by a scalar and null oneform function over all of $p+3$ dimensional spacetime.
- Recall, however, that membrane metrics that differ from each other only at order $\frac{1}{d}$ are not inequivalent starting points for perturbation theory. It is easily verified that inequivalent membrane metrics are parameterized only by the zeroes of B (shape of the membrane) and the values of O where B vanishes, i.e. on the membrane.

The velocity field in O

- The oneform O is everywhere null. The most general null vector can be parameterized as

$$O = e^h(1, u_\mu) \tag{15}$$

where the first component refers to the s direction and

$$u^2 = -1, \quad u.k = k^s - \frac{k^2}{k^s} \tag{16}$$

where k is the unit normalized outward pointing normal vector of the membrane surface.

- The scalar function h is not really data for the membrane, as it can be absorbed into a shift of order $\frac{1}{d}$ in the position of the membrane. Total data: $p + 1$ functions of $p + 2$ variables.

The first $\frac{1}{d}$ correction to the membrane metric

- Method: Choose a point x_o^μ on the membrane. Use scaled coordinates and expand the metric. Choose coordinates s.t. you get the black brane in standard coordinates at leading order. Continue the expansion to first subleading order. Obtain black brane plus an explicitly determined first order correction, written in terms of the extrinsic curvature of the membrane and first derivatives of the velocity field.
- Allow membrane metric to be corrected at first order in $\frac{1}{d}$. Demand equations of motion are solved at first subleading order in $\frac{1}{d}$. Yields ordinary differential equations for functions (in R) of the correction metric. Turns out equations are all analytically solvable.
- We find that the solution generically has physically unacceptable singularities at finite values of R. The singularities are avoided, and we have a completely regular solution, if and only if the membrane location and velocities obey the following equations of motion.

Membrane Equations of Motion

- $$(2-C)^2 K_{ss} + (1-C)^2 K_{uu} - 2(1-C)(2-C)K_{su}$$
  $$= -\frac{n.u(1-C)}{s}$$
  $$\mathcal{P}_a^b \Big((1-C)\left(u.\nabla - (u.n)n.\nabla - (2-C)\right) u_b +$$
  $$\left(\partial_s - n^s n.\nabla\right) u_b \Big) = 0 \qquad (17)$$

  Here $\mathcal{P}$ is the projector orthogonal to the three dimensional subspace spanned by n, u, ds and

  $$C = -n^s(u.n)$$

- Total number of equations $1 + p$. As many equations as variables. Have an intitial value problem for the shape of the membrane and the velocity field.

First order corrected metric

- When the equation of motion above is obeyed, the first order corrected metric is simple, and is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \psi^{-(d+p)} (O_\mu dx^\mu)^2 + \frac{\psi^{-d}}{d} (O_\mu dx^\mu) \Big[K_1(x^\alpha)(O_\nu dx^\nu) + 2K_2(x^\alpha)(ds - d\psi) - K_V(x^\alpha)(P^\beta_\nu U_{s\beta}) dx^\nu \Big] + \mathcal{O}\left(\frac{1}{d}\right)^2 \tag{18}$$

Details of first order correction to metric

$$
\begin{aligned}
K_1(x^\alpha) = &- \left[\frac{-c^6 + 11c^4 - 23c^2 + 11}{\left(c^2-2\right)^2 \left(c^2-1\right)} \right] R \\
&- \left[\frac{\left(c^2-1\right)\left(3c^2-8\right)}{2\left(c^2-2\right)^2} \right] R^2 \\
&+ \left[\frac{c^2 R^2 + 2\left(-4c^4 + 7c^2 + 1\right) R}{2c^2 \left(2-c^2\right)^2} \right] (u^\alpha u^\beta K_{\alpha\beta})(sk^\theta \partial_\theta s) \\
&- \left[\frac{c^2 R^2 + \left(-4c^4 + 6c^2 + 2\right) R}{\left(1-c^2\right)\left(2-c^2\right)} \right] (u^\beta K^\alpha_\beta \partial_\alpha s)(sk^\theta \partial_\theta s) \\
&+ \frac{R}{1-c^2} \left[2(\partial^\nu s)(\partial_\nu h) - (P^{\mu\nu} K_{\mu\nu})(sk^\theta \partial_\theta s) \right]
\end{aligned}
\tag{19}
$$

First order correction: details

$$K_2(x^\alpha) = \exp(h)\left[\frac{(3c^4 - 6c^2 + 1) + \left(c^6 - 3c^4 + 2c^2\right) R}{c^2\left(c^2 - 2\right)^2\left(c^2 - 1\right)}\right]$$

$$+ \exp(h)\left[\frac{(3c^4 - 6c^2 + 1) + \left(c^2 - 2\right) c^2 R}{c^4\left(2 - c^2\right)^2}\right](u^\alpha u^\beta K_{\alpha\beta})(sk^\theta \partial_\theta s)$$

$$- \exp(h)\left[\frac{-\left(2 - c^2\right) R + 3c^2 - 5}{c^2\left(1 - c^2\right)\left(2 - c^2\right)}\right](u^\beta K^\alpha_\beta \partial_\alpha s)(sk^\theta \partial_\theta s)$$

$$- \frac{s\exp(h)}{c^2(1 - c^2)}\Big[2(2 - c^2)(\partial^\theta s)(\partial_\theta h) - 2(1 - c^2)(u^\theta \partial_\theta h)$$

$$- (\mathcal{P}^{\mu\nu} K_{\mu\nu})(k^\theta \partial_\theta s) + (1 - c^2)(\mathcal{P}^{\mu\nu} U_{\mu\nu})\Big]$$

$$K_V(x^\alpha) = \frac{2\exp(h)}{(1 - c^2)}(1 + R)$$

$$c^2 = -(k^\theta \partial_\theta s)(u^\mu k_\mu)$$

‘Duality’ between black holes and membranes

- The ‘membrane’ equations of motion presented above are the main result of this talk. Let us recap their significance.
- Given any solution to the membrane equations, we have constructed a corresponding solution to large d gravity to first subleading order in $\frac{1}{d}$. The gravity solution reduces to flat space outside the membrane region. We expect that every large d solution of gravity that reduces to flat space outside the world volume of a compact world tube is dual to some membrane solution by our construction.
- A solution of gravity that vanishes outside the membrane clearly describes the intrinsic dynamics *of* the black hole in flat space. It follows the intrinsic dynamics of black holes in large dimensions is governed by our membrane equations (well defined initial value problem). It is tempting to use the name ‘membrane paradigm’ for this phenomenon.

Checks: Schwarschild black hole

- I will now present several checks of our membrane equations. To start with we demonstrate that well known exact black hole solutions obey the membrane equations.
- Start with the Schwarschild Black hole of unit radius. The metric can be recast in the form

$$ds^2 = -dt^2 + ds^2 + dw_i dw^i + \left(dt + \frac{w_i dx^i + s ds}{\sqrt{r^2 + s^2}} \right)^2 \left(\frac{1}{\sqrt{s^2 + w_i w^i}} \right)^{D-3} \quad (21)$$

It follows that $\psi = \sqrt{s^2 + w_a w^a}$, and the membrane surface is time indpendent and given by $s^2 + w_i w^i = 1$. while

$$O = \frac{s}{\sqrt{s^2 + w_i w^i}} \left(\frac{\sqrt{w_i w^i + s^2}}{s} dt + \frac{w_i dw^i}{s} + ds \right) \quad (22)$$

Checks: Schwarschild black hole

- O is clearly null. The u_μ oneform field is

$$u = \left(\frac{\sqrt{w_i w^i + s^2}}{s} dt + \frac{w_i dw^i}{s} + ds \right)$$

  It is easily verified that u obeys the constraint which determines $u.k$.
- With the membrane surface and velocity field in hand the extrinsic curvatures and velocity derivatives are computed in a straightforward manner. We have verified that the membrane equations are indeed obeyed.
- This is, of course, not a very strong check as the Schwarschild black hole is a very symmetric solution. Stronger checks of our equations can may be obtained by analysing exact Kerr solutions.

- The Myers-Perry metric for rotating black holes can be cast into the form

$$ds^2 = -dt^2 + \sum_{i=1}^{\frac{p}{2}} (dx_i^2 + dy_i^2) + dz^2 +$$
$$\frac{m\rho}{\Pi F}\left(dt + \left(1 - \sum_{i=1}^{\frac{p}{2}} \frac{\mu_i^2 a_i^2}{\rho^2 + a_i^2}\right) d\rho - \sum_{i=1}^{\frac{p}{2}} \frac{a_i^2}{\rho^2 + a_i^2}(x_i dy_i - y_i dx_i)\right)^2 . \tag{23}$$

where

$$\Pi = \prod_{i=1}^{\frac{p}{2}} (\rho^2 + a_i^2), \quad F = 1 - \sum_{i=1}^{\frac{p}{2}} \frac{a_i^2 \mu_i^2}{\rho^2 + a_i^2}. \tag{24}$$

Check: Rotating black holes

And μ_i and ρ are defined by the following equations

$$\sum_{i=1}^{\frac{p}{2}} \frac{x_i^2 + y_i^2}{\rho^2 + a_i^2} + \frac{z^2}{\rho^2} = 1$$
$$\mu_i^2 = \frac{x_i^2 + y_i^2}{\rho^2 + a_i^2}. \tag{25}$$

It follows that $\psi = \rho$ and the membrane surface is given by

$$\sum_{i=1}^{\frac{p}{2}} \frac{x_i^2 + y_i^2}{\rho^2 + a_i^2} + \frac{z^2}{\rho^2} = 1$$

Moreover

$$O \propto \left(dt + \left(1 - \sum_{i=1}^{\frac{p}{2}} \frac{\mu_i^2 a_i^2}{\rho^2 + a_i^2} \right) d\rho - \sum_{i=1}^{\frac{p}{2}} \frac{a_i^2}{\rho^2 + a_i^2} (x_i dy_i - y_i dx_i) \right)$$

Check: Rotating black holes

- With these results in hand it is straightforward (though tedious) to explicitly check that O is null and that its dot product with the normal vector to the membrane is indeed constrained as required.
- It is also straightforward to compute the extrinsic curvatures and the velocity derivatives and check that our membrane equations are satisfied. We have performed the tedious algebra on mathematica; the final result is that both equations are beautifully obeyed.

Linearized spectrum about the Schwarschild black hole

- As a final check it is not difficult to linearize our membrane equations about the solution dual to the Schwarschild black holes.
- The linearized equations may then be solved. All modes are expanded in (scalar and vector) spherical harmonics on S^p and assumed to be harmonic in time. The dependence of modes on s is unknown. We obtain linear differential equations in s. These equations can be solved. The requirement that the solutions are regular imposes a qantization on the frequencies in time. For the scalar and vector modes respectively we find

$$w_s = -i(l-1) \pm \sqrt{l-1}, \quad w_V = i(l-1) \tag{26}$$

where l is a positive integer related to the angular momentum of the corresponding modes. Our results are in perfect agreement with the leading order spectrum of

Things to do

- This programme is very new, and I think there are a lot of interesting things to be done.
- In the $\frac{1}{d}$ expansion the wiggling membranes described in this talk do not radiate. However it is clear that there is radiation at order e^{-d}, which is nonperturbative from the viewpoint of the $\frac{1}{d}$ expansion. Because the radiation is so weak its backreaction on membrane dynamics can be ignored. However once we have solved for membrane dyanamics, we should be able to compute the radiation field this vibration produces of strength e^{-d}.
- Restated we should be able to determine the effective stress tensor of our membrane and its coupling to external gravitons. This sounds like an interesting thing to do

To be done

- We should be able to use the area form on the horizon and the Hawking area increase theorem to define an entropy current whose divergence is point wise non negative on every solution to the membrane equations of motion. This would demonstrate that our membrane equations are consistent with the second law of thermodyanmics, and perhaps help us establish unieqeness theorems for stationary solutions.
- It would also be interesting to study the interplay of the equation of conservation of the stress tensor from the previous subsection with the equations of motion derived in this talk.
- It would also be interesting to compare our equations with the equations governing the motion of an isometry preserving soap bubble in $p + d + 3$ dimensions.

Generalizations

- The obvious extension of our computation are to generalize the metric and equation of motion to one higher order in the $\frac{1}{d}$ expansion.
- It would also be interesting to redo our computations in the presence of a Maxwell field. In this situation the membrane will, presumably, have a new dynamical degree of freedom: the local charge density.

Application to four dimensions

- I started this talk asking for a parameter for general relativity. We have found a parameter but is it of any use at $D = 4$?
- Probably unlikely worth testing. The following strategy suggests itself. Take a tough problem in $D = 4$ (like the collision of two black holes). Solve the corresponding membrane equations. Then compute the resultant radiation field and boldly set $D = 4$. Compare with the results of a full simulation. If there are even qualitative similarities between the answers, our expansion might prove useful for physicists calliberating gravity wave detectors to measure black hole mergers.

Conclusions

- We have reduced the equations that govern intrinsic black hole dynamics in the limit of a large number of dimensions to a well defined initial value problem for wiggly membrane membrane. The degrees of freedom on this membrane are its shape and a velocity field.
- Our construction should be generalized in many ways: to understand radiation, stress energy, entropy, charge and higher orders, ...
- In my opinion the construction presented in this paper diserves the name 'the membrane paradigm of black hole physics': we see it emerges at large D.
- It will be interesting to see how well our large D solution compares with results from numerical simulations in $d = 4$.