Entanglement Entropy and Boundary Terms: Two Short Stories

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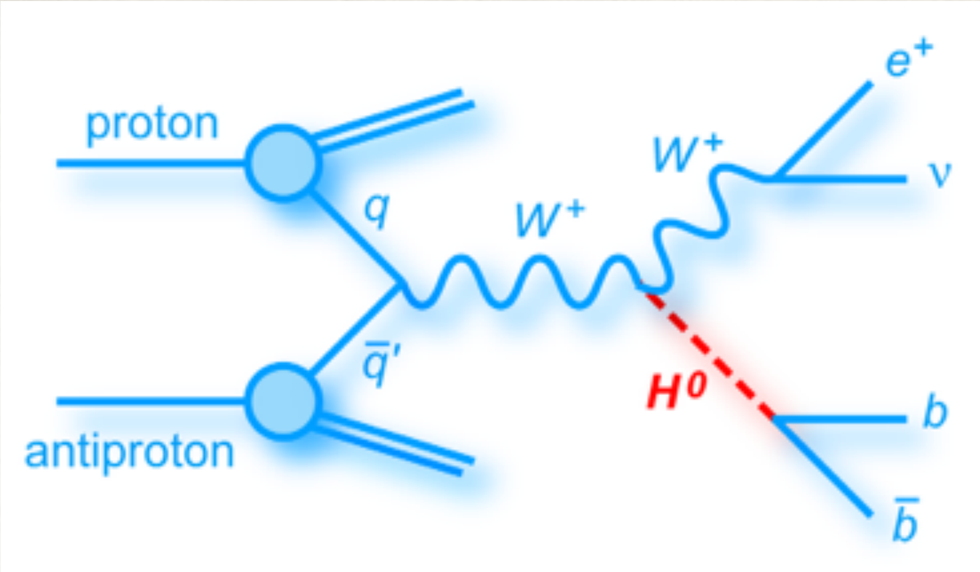
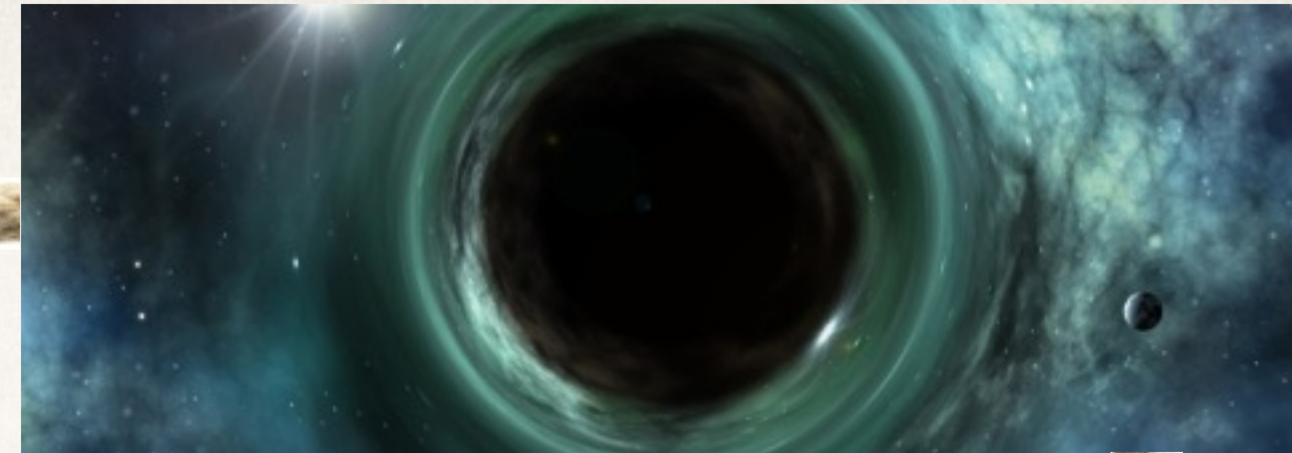
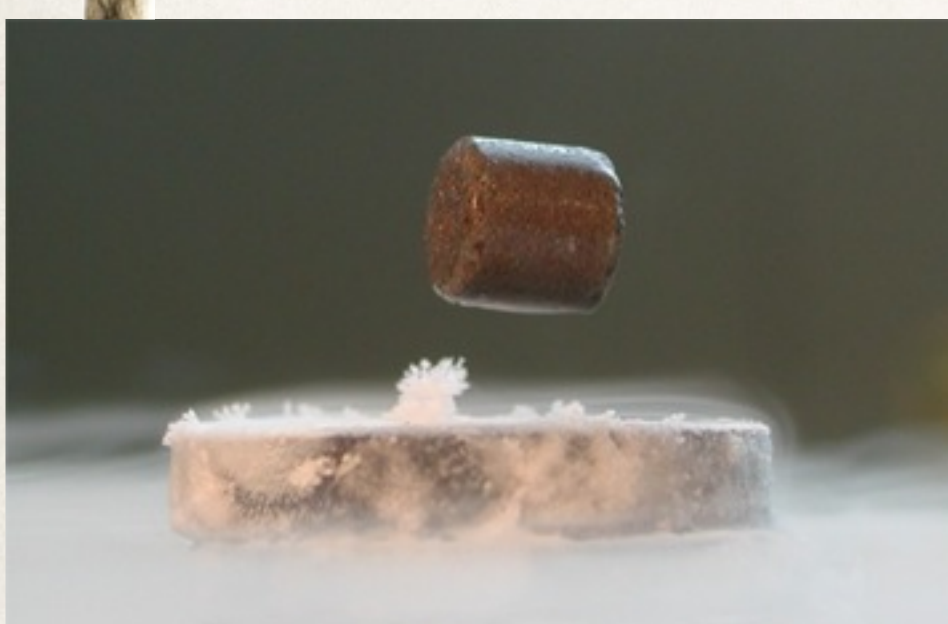
March 28, 2015

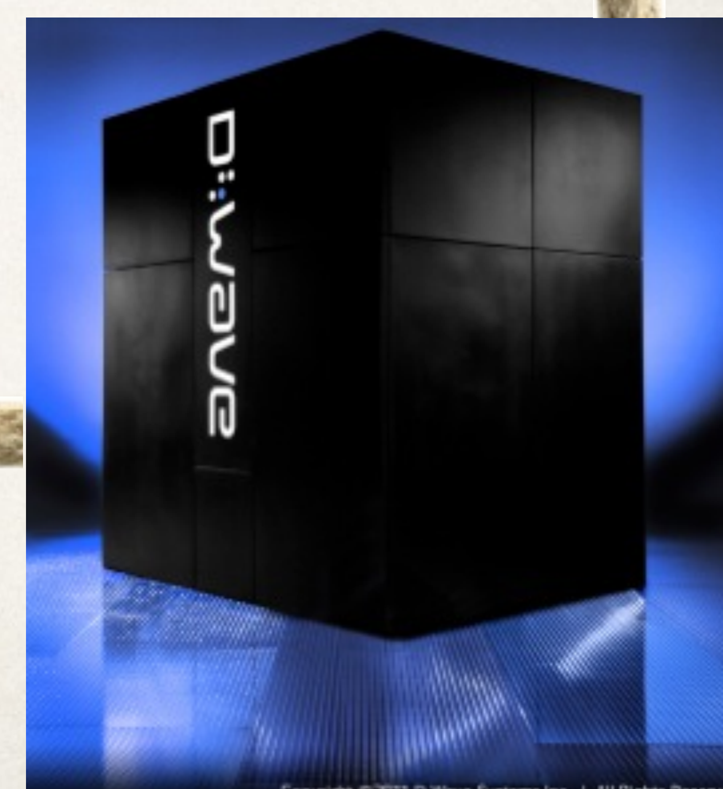




For a nonlocal, nonobservable, ultraviolet cut-off dependent quantity, entanglement entropy has become surprisingly important in theoretical physics today.

A Unifying Theme





Why is It Important?

- Quantum information, communication and computation — measure of entanglement in quantum systems
- Condensed matter physics — order parameter for exotic phase transitions (Osborne-Nielsen 2002, Vidal et al. 2003)
- Quantum field theory (QFT) — measure of renormalization group flow (a and c theorems) (Casini-Huerta 2006, 2012)
- Gravity — relations to black hole entropy (Bombelli et al. 1986, Srednicki 1993); Bekenstein bound (Casini 2008)
- String theory — Ryu-Takayanagi (2006) formula and AdS/CFT ties QFT and gravity aspects together.

Entanglement Entropy

- Consider a state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ in a factorizable Hilbert space. (*A* and *B* spatial.)

For the EPR pair

$$\rho_A = \frac{1}{2}\left(|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow|\right)$$

$$S_E = \log 2$$

- Form density matrix: $\rho = |\psi\rangle\langle\psi|$

- Perform the partial trace: $\rho_A = \mathrm{tr}_B\, \rho$

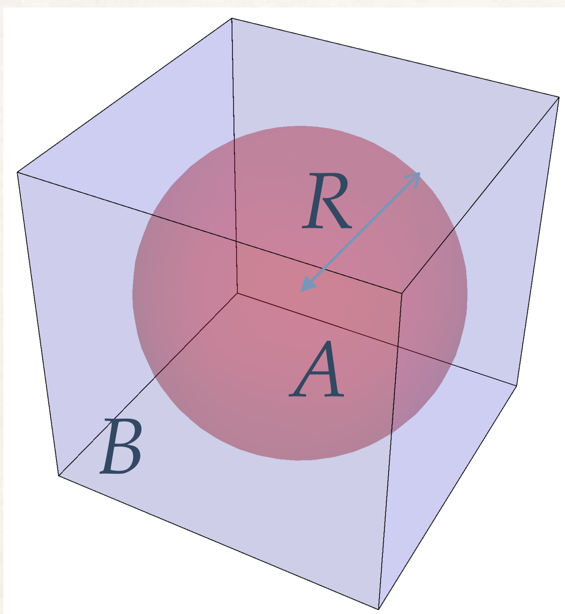
- Compute the von Neumann entropy of ρ_A

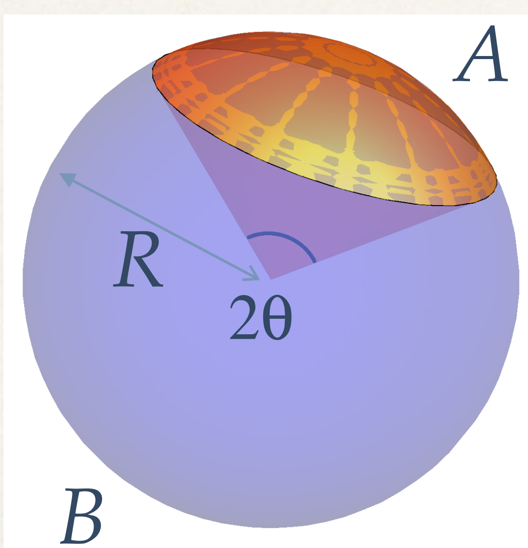
$$S_E \equiv -\,\mathrm{tr}(\rho_A \log \rho_A)$$

The Challenges in QFT

- The assumption that the Hilbert space can be factorized wrt to A and B is often problematic.
- The infinite number of degrees of freedom means EE is badly divergent.
- That the density matrix grows exponentially with the size of the Hilbert space means EE is difficult to compute.

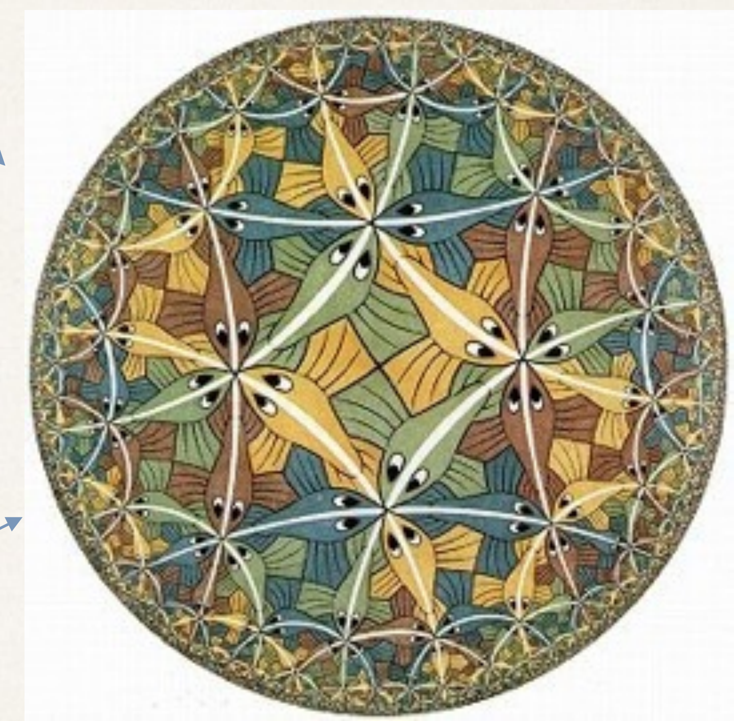
Trick for Calculating EE of CFTs





conformal transformation

Has $T = 1/2\pi R$



For caps on spheres and balls in flat space, "A" gets mapped to all of hyperbolic space.

Map to Hyperbolic Space

- Density matrix on hyperbolic space is thermal

$$\rho_H = \frac{e^{-H_M}}{\operatorname{tr} e^{-H_M}}$$

H_M called the modular Hamiltonian

- $\rho_A = U^{-1} \rho_H U$ for some unitary operator U.

- EE invariant under U implies thermal entropy of hyperbolic space is EE. (see e.g. Casini-Huerta-Myers 2011)

Two applications of this map

For conformal field theories (CFTs)

- Thermal corrections to EE (work with M. Spillane, J. Nian, R. Vaz, and J. Cardy).
- Universal contributions to EE at zero temperature (work with K.-W. Huang and K. Jensen).

Moral: The importance of boundary terms.

Thermal Corrections?

The initial density matrix is not that of a pure state!

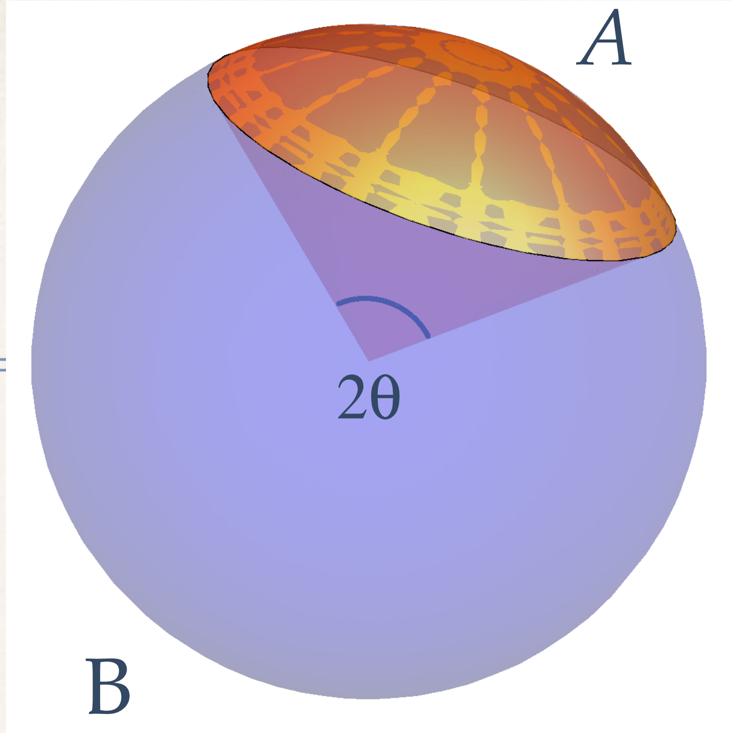
$$\rho(T) = \frac{e^{-H/T}}{\mathrm{tr}(e^{-H/T})}$$

Entanglement entropy measures some combination of thermal entropy and quantum entanglement.

Why bother with thermal effects?

- Nice to be able to remove them.
- Lessons to be learned from EE in non-traditional contexts.
- Connection to black hole physics.

A Universal Result



In the $RT \ll 1$ limit, for a cap A of opening angle 2θ on the S^3,

$$S_E(A,T) - S_E(B,T) = 2\pi g m R \cot(\theta)\, e^{-m/T} + o(e^{-m/T})$$

(Herzog 2014)

m is the mass gap, ~ $1/R$
g is the degeneracy of the 1st excited state

- Turns out to be true for any CFT in any dimension!
- Subleading in a large N expansion.
- The exp($-m/T$) Boltzmann suppression should be true of any gapped QFT (Herzog-Spillane 2012).

Where does it come from?

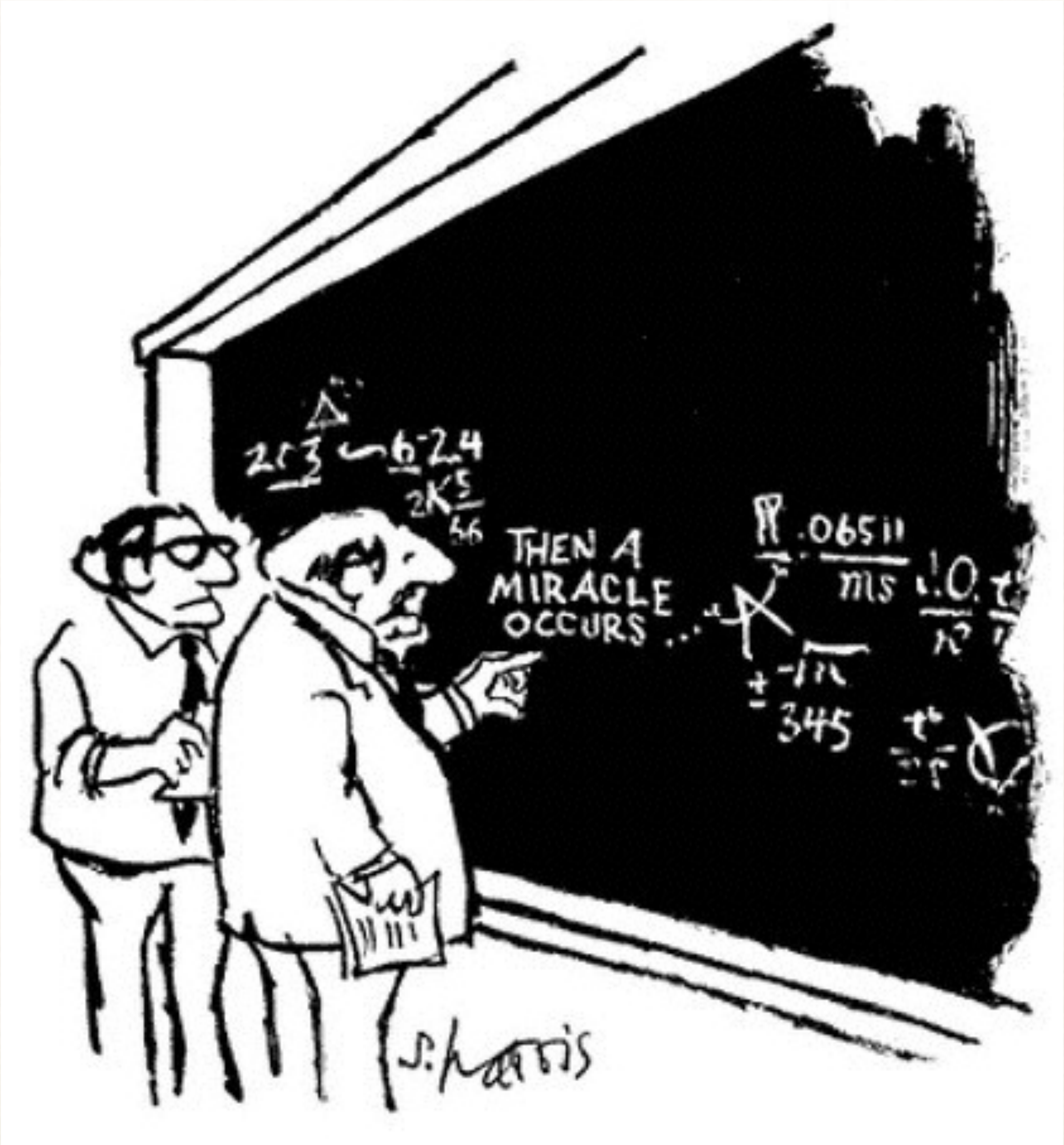
Start with a thermal density matrix $\rho(T) = \frac{e^{-H/T}}{\mathrm{tr}(e^{-H/T})}$

(That ρ is mixed means we're not really measuring quantum entanglement.)

Make a small *T* perturbative expansion

Need to calculate $\langle \psi(x)\psi(y) \log \rho_A(0) \rangle$

where $\psi(x)$ creates the first excited state.


THEN A
MIRACLE
OCCURS
S. Harris

A Special Trick for CFTs

For CFTs and "A" a cap on a sphere,

$$H_M = -\log \rho_A(0)$$

also called the modular Hamiltonian, is known.

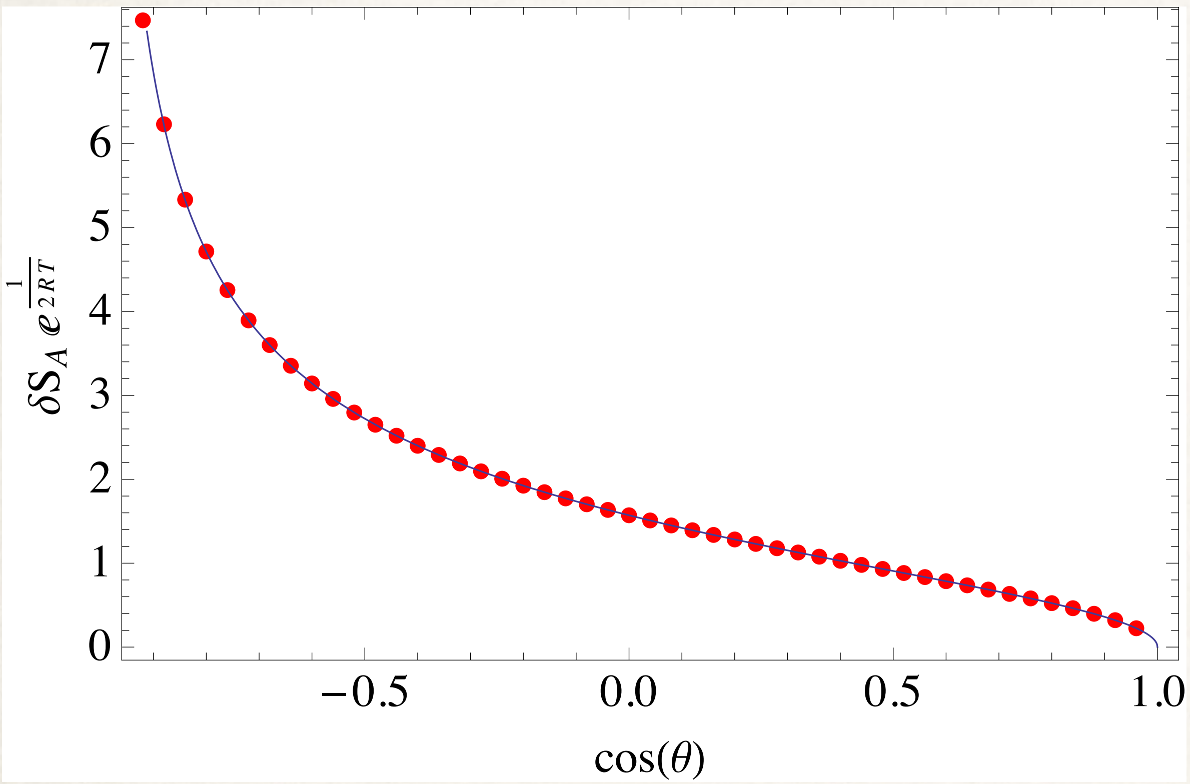
(see e.g. Casini-Huerta-Myers 2011)

H_M is proportional to the stress-energy tensor $T_{\mu\nu}$.

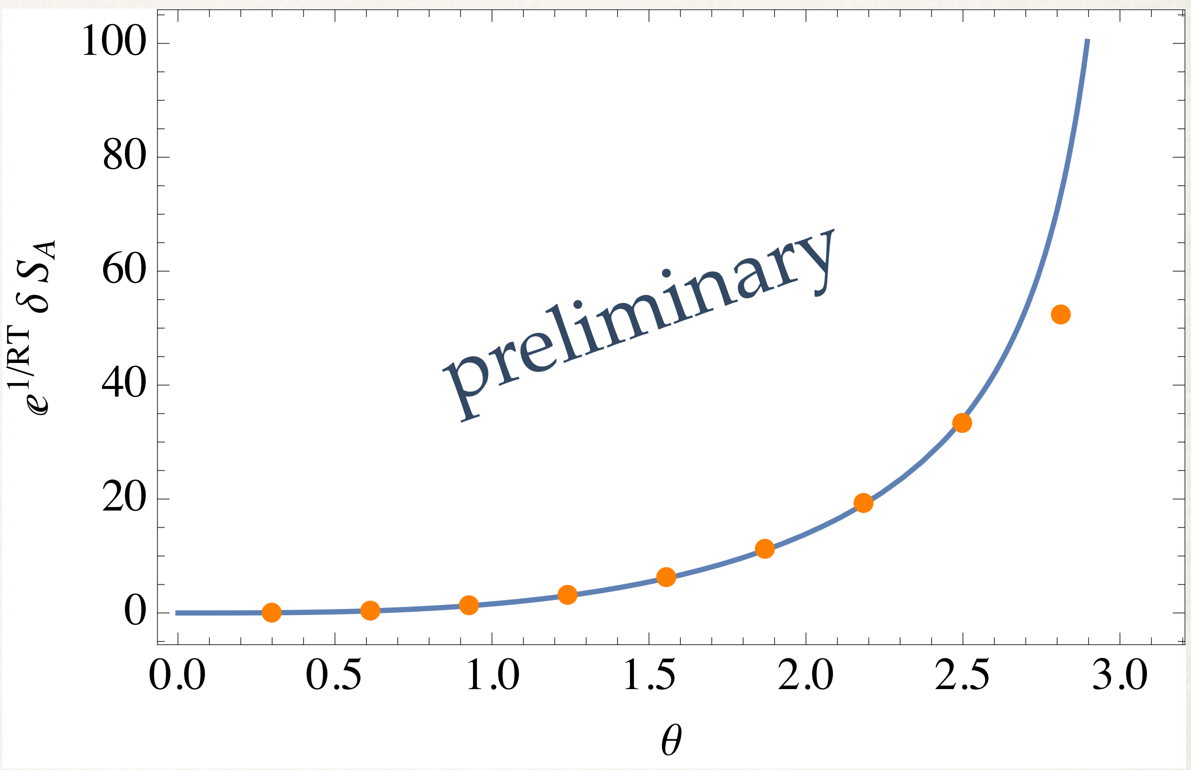
$$\langle \psi(x)\psi(y) \log \rho_A(0) \rangle \to \langle \psi(x)\psi(y) T_{\mu\nu}(0) \rangle$$

Three point functions involving the stress tensor in CFTs are constrained by symmetry to take relatively simple forms.

Numerical Check



free (conformally coupled) scalar in 3d (Herzog 2014)



free fermion in 3d

(Herzog, Nian, Spillane, Vaz to appear)

points: modernized version of Srednicki's (1993) method.

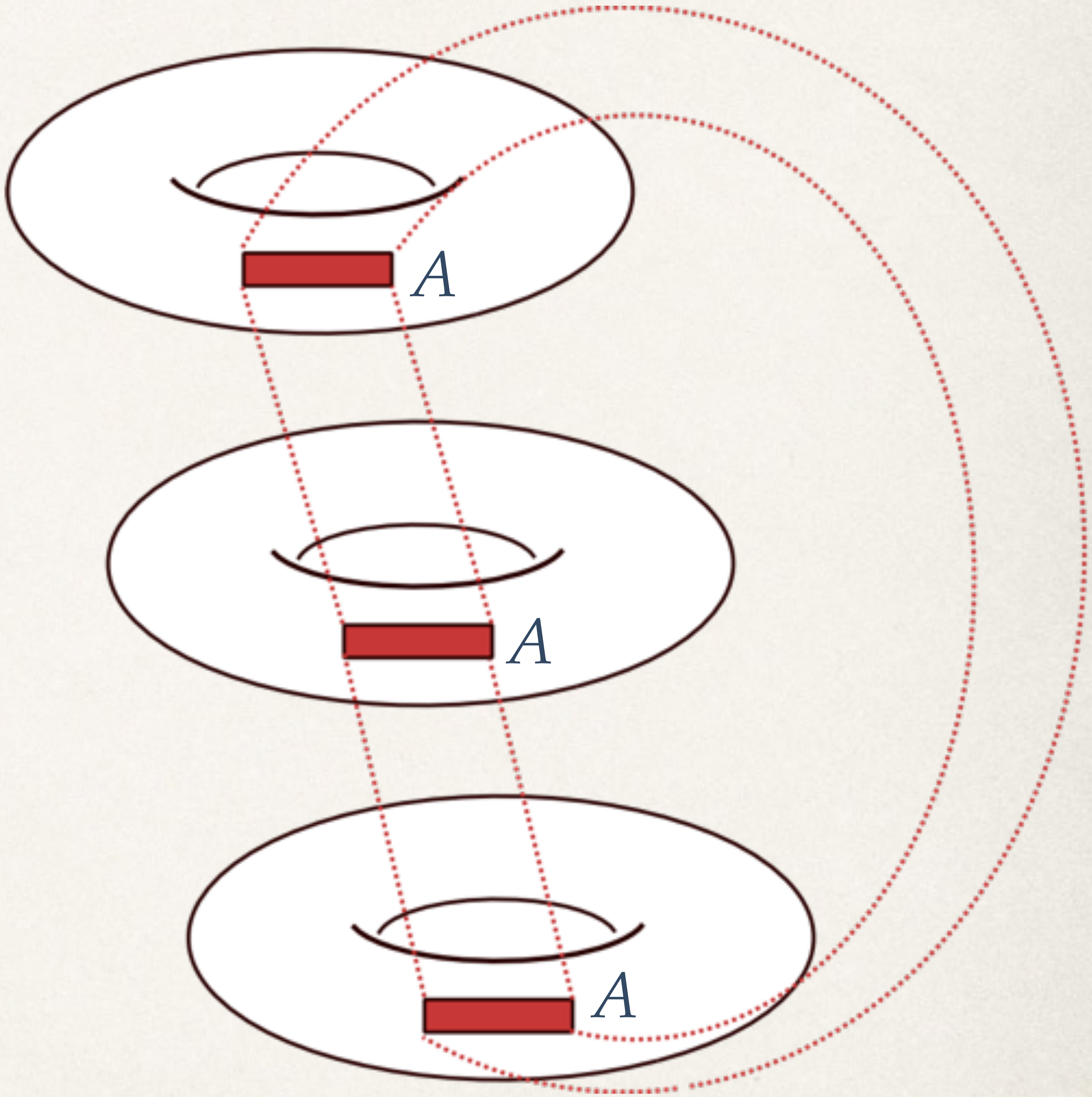
line: analytic prediction

$$\delta S_A = S_E(A,T) - S_E(A,0)$$

Analytic Checks via the Replica Method

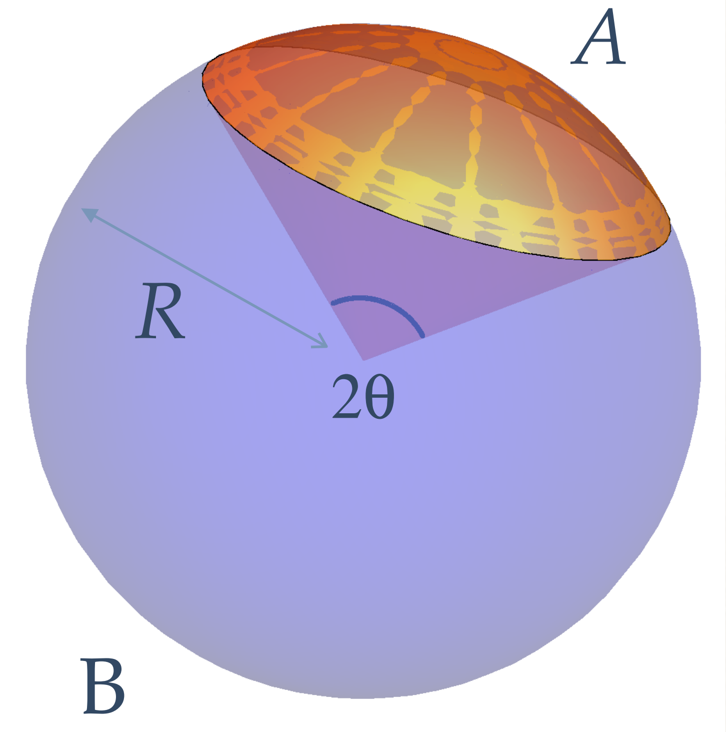
- Free scalar and fermion can also be checked analytically using the method of images (Herzog, Nian 2014; Herzog, Nian, Spillane, Vaz to appear).

- Results in 2d can be checked independently using a conformal transformation (Cardy-Herzog 2014).



These results also yield Rényi entropies.

Related Result Not Quite Right



From the modular Hamiltonian method

$$S_E(A,T) - S_E(A,0) = gmR\, I_d(\theta)\, e^{-m/T} + \ldots$$

where

$$I_d(\theta) = 2\pi \frac{\mathrm{Vol}(S^{d-2})}{\mathrm{Vol}(S^{d-1})} \int_0^{\theta_0} \frac{\cos\theta - \cos\theta_0}{\sin\theta_0} \sin^{d-2}\theta \, \mathrm{d}\theta$$

But for a scalar field, it turns out the other methods match $I_{d-2}(\theta)$.

WHAT'S GOING ON!?!

A Resolution

$$\Delta H_M = 2\pi\xi \int_{\partial H^{d-1}} \mathrm{d}^{d-2}x \sqrt{\gamma}\phi^2$$

- There exists a boundary term that can correct H_M.
- When $\xi = (d\text{-}2)/4(d\text{-}1)$ (the conformal coupling)

$$I_{d-2}(\theta) \to I_d(\theta)$$

- Suggests whenever CFT has operators of dimension d-2, H_M may get corrected by boundary terms.

Casini, Mazitelli, Teste (2014)

This particular case

The conformally coupled scalar

$$S = -\frac{1}{2}\int_M [(\partial\phi)^2 + \xi R\phi^2] - \xi \int_{\partial M} K\phi^2$$

trace of extrinsic curvature

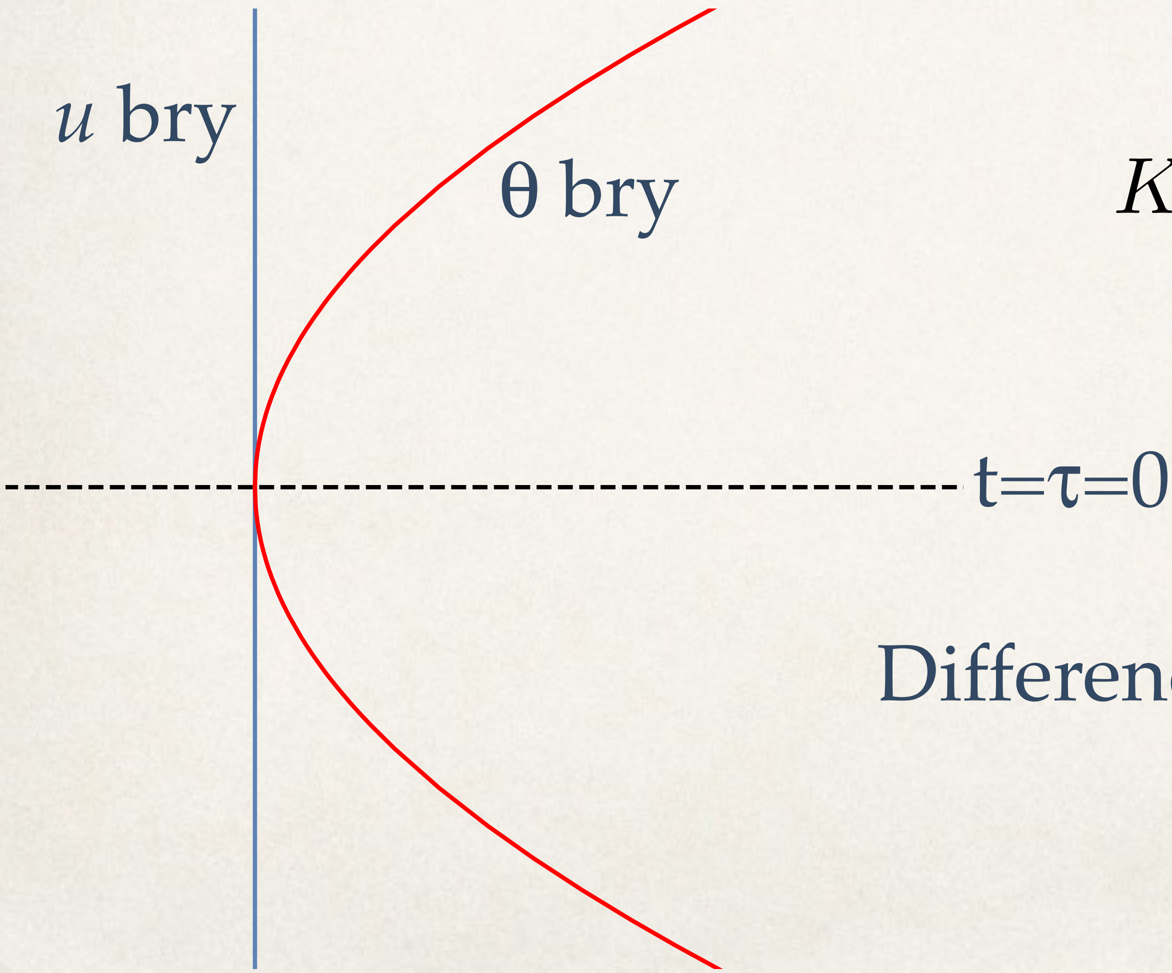
- To define the stress tensor.
- To preserve Weyl scaling symmetry.

Boundary term in action translates into a boundary term in H_M.

In more detail

Let u be the radius of hyperbolic space.
Let θ be the polar angle on the sphere.

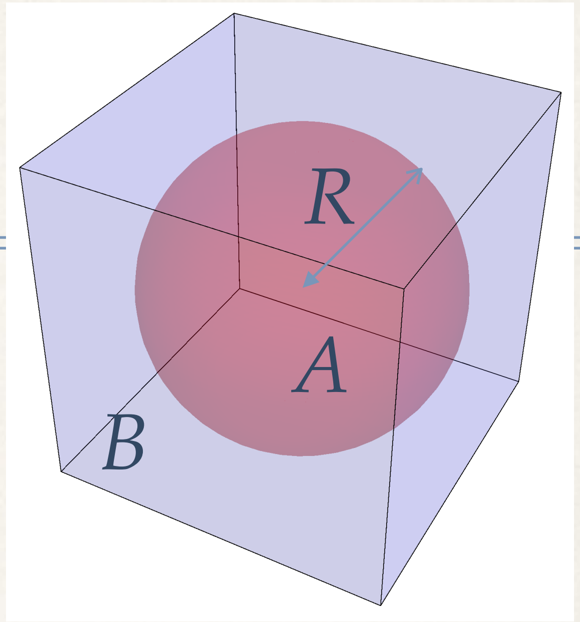
Constant u boundary different from pull back of constant θ boundary.



$$K_{(\theta)}|_{\tau=0} = \frac{d-1}{R} \quad , \quad K_{(u)}|_{\tau=0} = \frac{d-2}{R}$$

Difference in K reproduces the shift in H_M.

Universal contributions to EE at zero T



There is a "universal" contribution to EE that is proportional to "a" anomaly coefficient in $\langle T^\mu_\mu \rangle$.

$$\langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^{d/2}} \left[\sum c_j I_j - (-)^{d/2} a_d E_d + \nabla_\mu J^\mu \right]$$

Weyl curvature invariants

Euler density

UV cutoff

$$S_E(A,0) = \ldots + 2(-1)^{d/2} \left(\frac{d}{2}\right)!\, a_d \log \frac{\epsilon}{R} + \ldots$$

Euler character of sphere.

(Solodukhin 2008; Casini-Huerta-Myers 2011)

Puzzle #2

$$S_E(A,0) = \ldots + 2(-1)^{d/2}\left(\frac{d}{2}\right)!\, a_d \log\frac{\epsilon}{R} + \ldots$$

- Casini-Huerta-Myers (2011) try and fail to get this log from the hyperbolic space map.
- They succeed using a sphere (Euclidean de Sitter) — no boundary; they succeed also using the RT formula.
- The result is consistent (predicted) by earlier work using the replica method and squashed cones Solodukhin (2008).

Can we succeed where CHM failed?

$$S_E(A,0) = \langle H_M \rangle + \log \operatorname{tr} e^{-H_M}$$

$$\frac{\text{energy density}}{\text{hyperbolic space } T}$$

log of partition function;
call it W

Guess: universal contributions encoded in an effective action that reproduces the "a" part of the trace anomaly.

Anomaly Action from Dim Reg

$$W[g_{\mu\nu}] = \frac{(-)^{d/2+1} a_d}{(4\pi)^{d/2}} \lim_{n\to d} \frac{1}{n-d} \int_M E_d + \dots$$

$$\text{under } g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu} \,, \sqrt{g} E_d \to e^{(n-d)\sigma} E_d + \text{total deriv}$$

But hyperbolic space has a boundary.

CS like term

Recall the definition of the Euler character

$$\chi(M) = \frac{1}{(\frac{d}{2})!(4\pi)^{d/2}} \left(\int_M E_d + \int_{\partial M} Q_d \right)$$

Suggests we include Q_d in the definition of $W[g_{\mu\nu}]$.

It works!

$$S_E(A,0) = \langle H_M \rangle + W$$

In computation of W, we find

$S_E(A,0)$ comes from $\displaystyle\int_{\partial M} Q_d$

$\langle H_M \rangle$ cancels against $\displaystyle\int_M E_d$

can check by comparing with $\displaystyle\langle T^{00} \rangle = 2\frac{\delta W[g_{\mu\nu}]}{\delta g_{00}}$

It works mostly…

$W[e^{2\sigma} g_{\mu\nu}, g_{\mu\nu}] \equiv W[e^{2\sigma} g_{\mu\nu}] - W[g_{\mu\nu}]$ to regulate 1/(*n*-*d*) divergence.

$$W[g_{\mu\nu}] = \frac{(-)^{d/2+1} a_d}{(4\pi)^{d/2}} \lim_{n\to d} \frac{1}{n-d} \left(\int_M E_d + \int_{\partial M} Q_d \right) + \ldots$$

$S^1 \times H^{d-1}$ is conformally flat

Define $W[\delta_{\mu\nu}]$ to be the O(*n*-*d*) term in $\int_{\partial M} Q_d$ in flat space.

essentially the Euler character of the sphere

Final Remarks

- For certain types of entanglement entropy, mapping to hyperbolic space is a useful tool.
- Hyperbolic space has a boundary, and the boundary has important effects.
    - Thermal corrections.
    - Log contribution to the zero T EE.

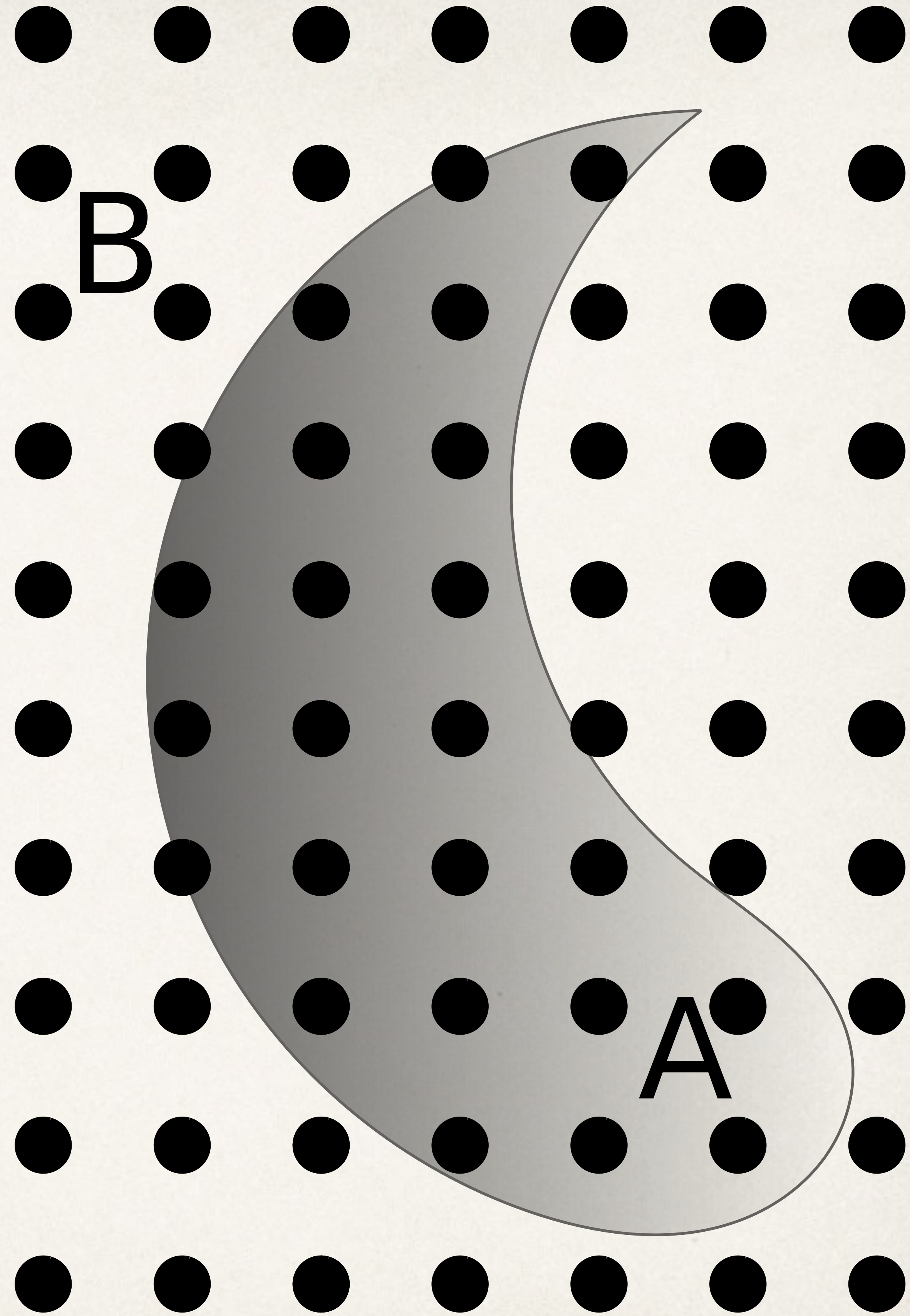
Thanks to my collaborators

- Michael Spillane (grad student)
- Tatsuma Nishioka (U. Tokyo)
- John Cardy (Oxford)
- Jun Nian (grad student)
- Ricardo Vaz (grad student)

(a chronological order)

Extra Slides

Further Restrictions



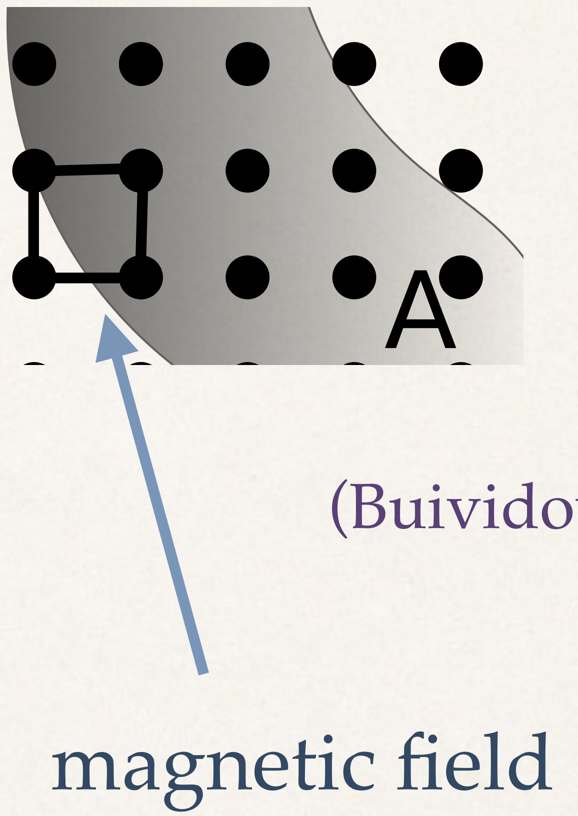
- For the gravity, QFT, and condensed matter applications, $\mathcal{H}$ is not finite.
- A and B are typically spatial regions.

These restrictions make it surprising I have anything to say to you today at all.

Challenge 1: Boundary terms

- For a lattice version of E&M, observables are loops. The Hilbert space does not factor well. Active area of research.

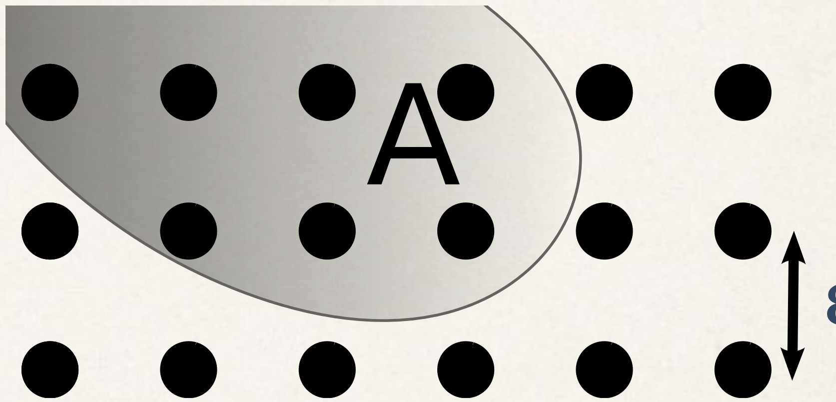
- We will see later that there are problems with boundary terms even for the simplest quantum field theory — a free scalar field!



(Buividovich-Polikarpov 2008)

Challenge 2: Ultraviolet Problems

EE is ultraviolet cut-off dependent!



For a quantum field theory in the ground state

$$S_E \sim \frac{\text{Area}(\partial A)}{\varepsilon^{d-2}}$$

(Srednicki 1993)

Games involve extracting pieces which are argued to be universal and insensitive to ε.

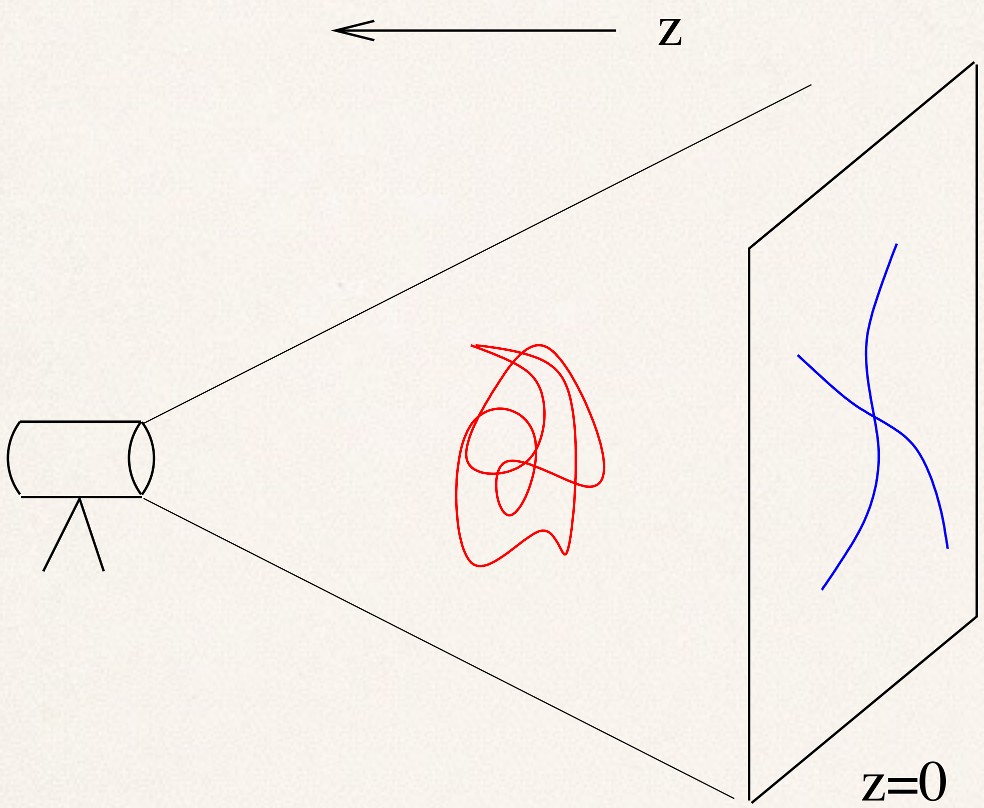
Challenge 3: Computability

- The standard tool for computing EE is the replica trick. Requires computing a partition function on an n-sheeted cover of space-time, branched over A, for all integer n, and then analytically continuing to compute a derivative at n=1.
- For free theories, a lattice regulated version of the density matrix can be computed numerically.
- For conformal field theories, various tricks, one of which we will see later.
- For quantum field theories with a dual classical gravity descriptions via the AdS/CFT correspondence, there is the Ryu-Takayanagi formula.
- Other numerical methods: Tensor networks, matrix product states.

AdS/CFT and Ryu-Takayanagi

A Statement of the Duality

Think of AdS as a half-space



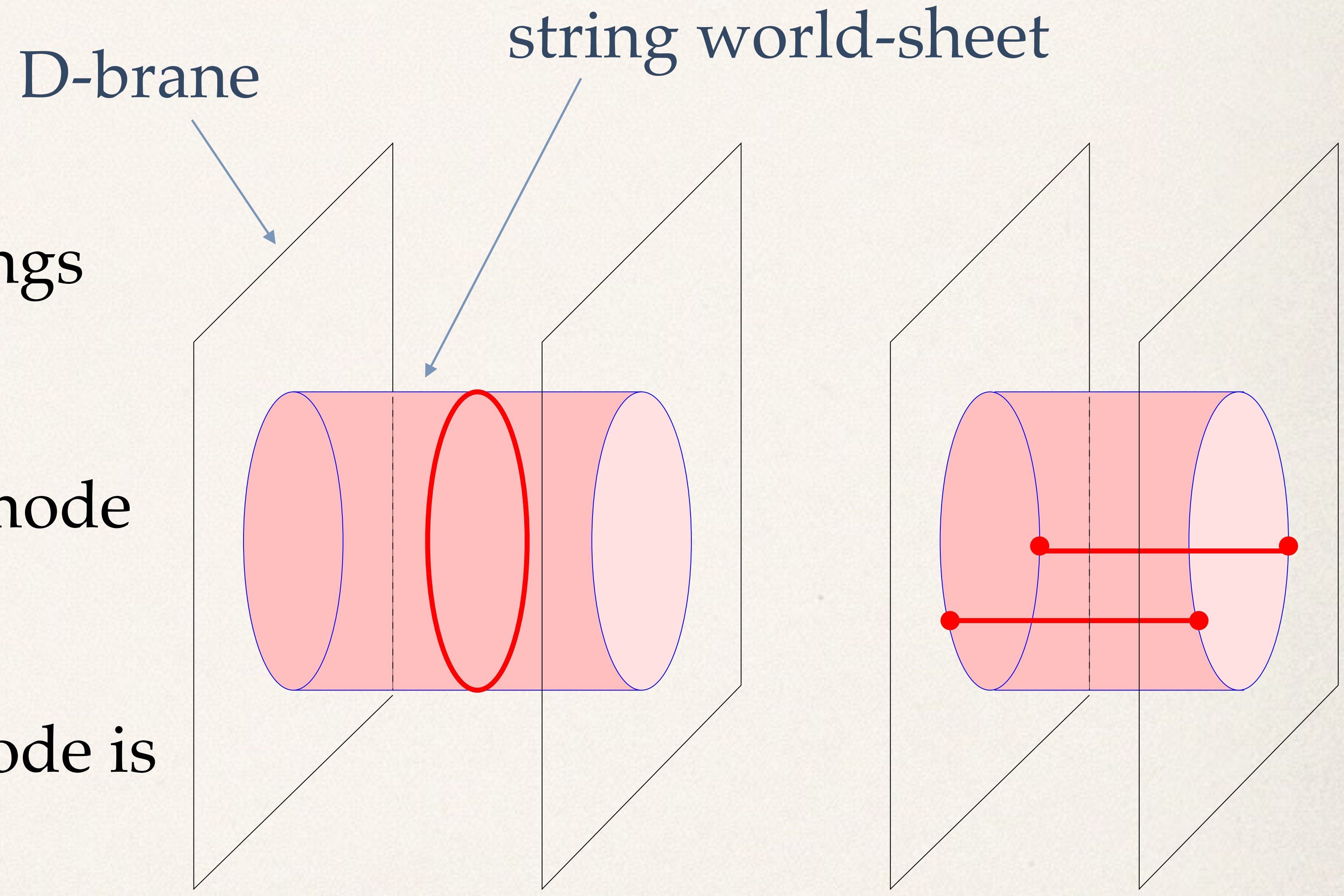
Bulk information is projected onto the boundary where the field theory lives.

Some QFTs have dual descriptions as quantum theories of gravity (string theory).

a) In a certain limit, the gravity becomes classical and we can use the correspondence to learn interesting things about QFT.
b) In another limit, we can use perturbative QFT to learn about quantum gravity.

What is AdS/CFT?
It depends on how you slice it.

- D-branes are surfaces strings end on.
- The lowest closed string mode is the graviton.
- The lowest open string mode is a gauge boson.



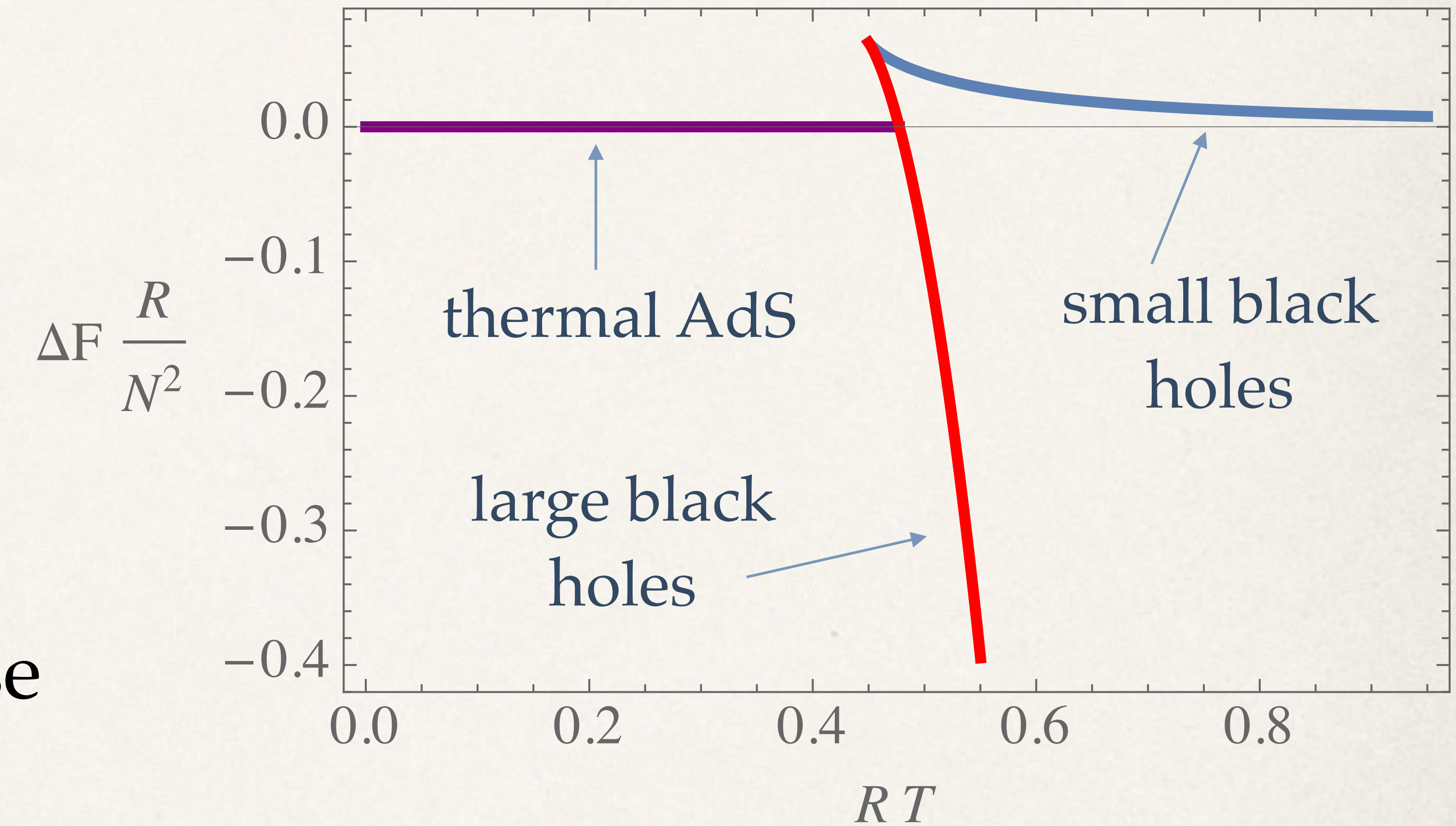
The Original AdS/CFT Correspondence

- Maximally supersymmetric SU(N) Yang-Mills theory (MSYM) — an example of a conformal field theory (CFT) — is dual to type IIB string theory in a $AdS_5 \times S^5$ background.
  - A theory like QCD. N colors instead of three. Supersymmetry means the gluons have scalar and fermionic partners that transform in the adjoint representation of SU(N).
  - The correspondence becomes useful (string theory becomes classical gravity) in the large N, large $\lambda = g_{\mathrm{YM}}^2 N$ limit.

Maldacena 1997

MSYM at Nonzero Temperature

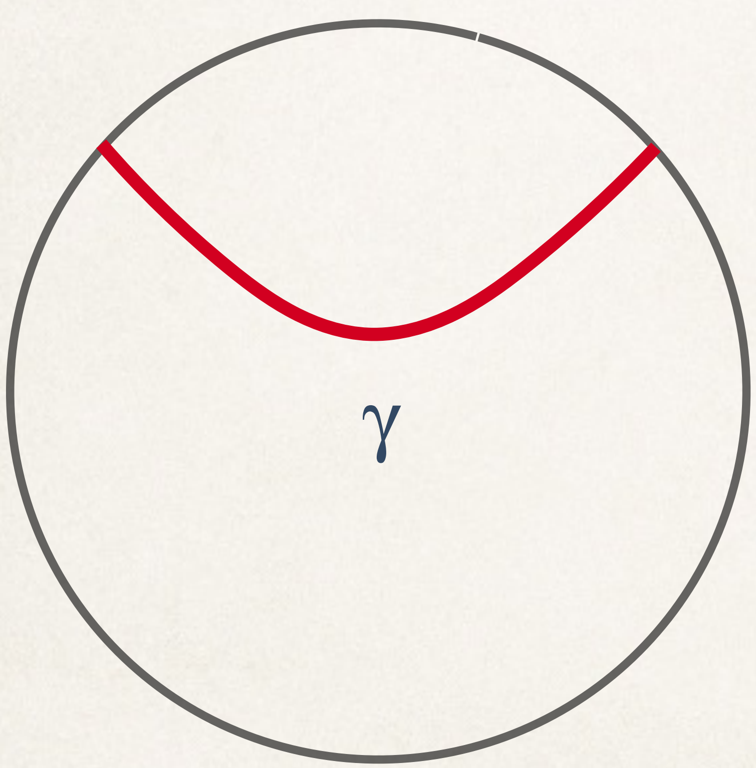
- Put MSYM on a three sphere with radius *R*.
- QFT tells us fields get a mass of order $1/R$.
- Gravity tells us there is a phase transition (Hawking-Page 1983) at $RT \sim 1$ between a solution with a black hole (high *T*) and a solution without (low *T*).



Witten pointed out relevance for MSYM (1998) —deconfinement phase transition

Calculating Entanglement Entropy in AdS/CFT (T=0)

A



B

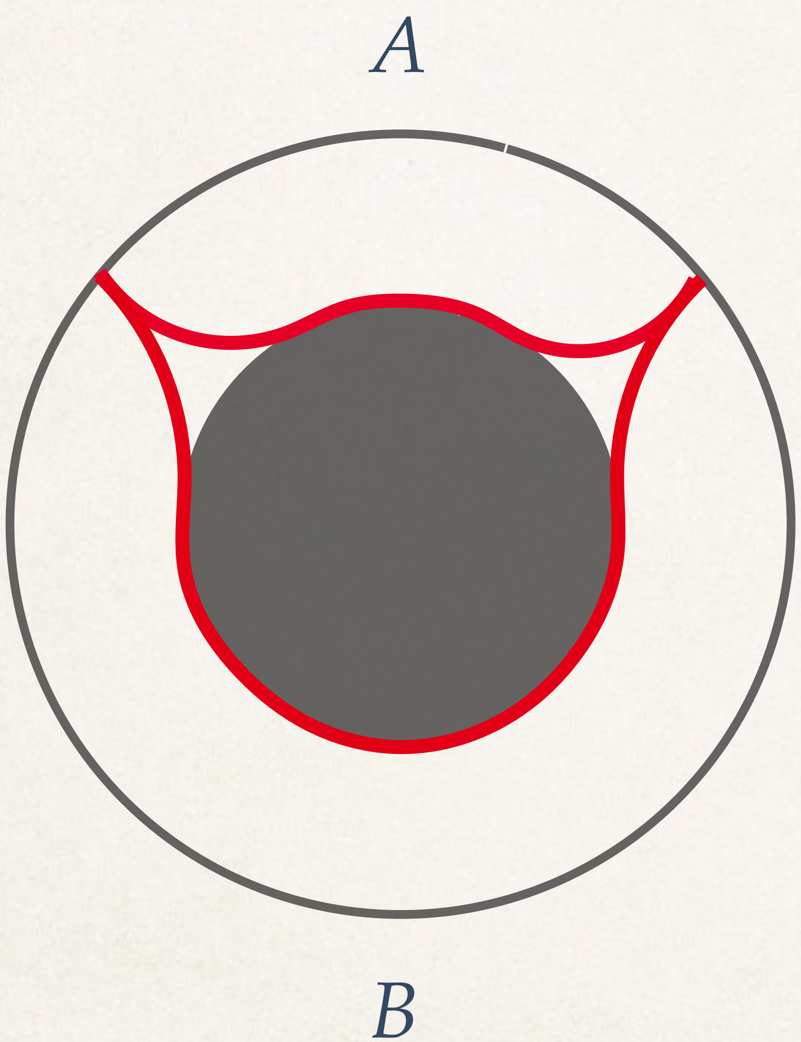
Take minimal surface γ in bulk such that $\partial A = \partial\gamma$.

$$S_E(A) = \frac{\text{area}(\gamma)}{4G_N}$$

Note: $S_E(A) = S_E(B)$

Ryu-Takayanagi (2006); Fursaev (2006);
Lewkowycz-Maldacena (2013)

Calculating EE at $T > 0$.



In presence of a black hole, instructed to consider different γ.

$$S_E(A) \neq S_E(B)$$

Note: EE serves as an order parameter for the phase transition.

Three comments

- Finite volume implies phase transition a large *N* effect.
- While it can be proven that $S_E(A) = S_E(B)$ at T=0, for T>0 the two are generically different.
- RT is only the leading order result: $\frac{1}{G_N} \sim N^2$

Where are we going?

- Given boundary term issues in construction of H_M are there more general lessons to be drawn? Probably yes. (Lee et al. 2014; Casini et al. 2014)
- Can these corrections can be computed in AdS/CFT? Yes in d=2 (Barrella et. al. 2013), but unknown in d>2.
- Can we go beyond $RT \ll 1$? Yes for fermions in d=2 (Herzog-Nishioka 2013), but unknown in general.

The Three Challenges

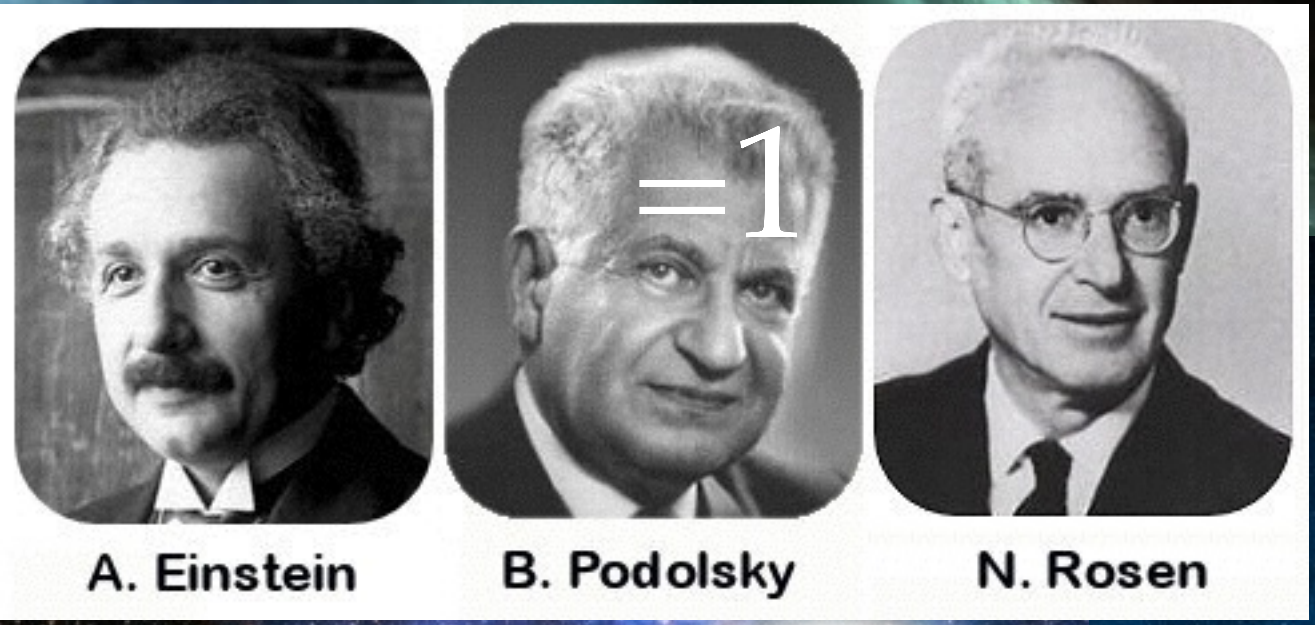
- Challenge 1: Boundary terms and factorizability issues can play a role even in the simplest field theories.

- Challenge 2: By looking at certain EE differences, the result reduced to a local, observable — a three point function — and was UV cutoff independent.

- Challenge 3: A thermal correction turned out to be easily computable for CFTs and universal.

Big Questions

- Can EE help us understand black holes?
- Can EE help us map out the space of QFTs?
- How does AdS/CFT relate these two questions?
- Can EE give us deeper insight into why AdS/CFT might be correct?



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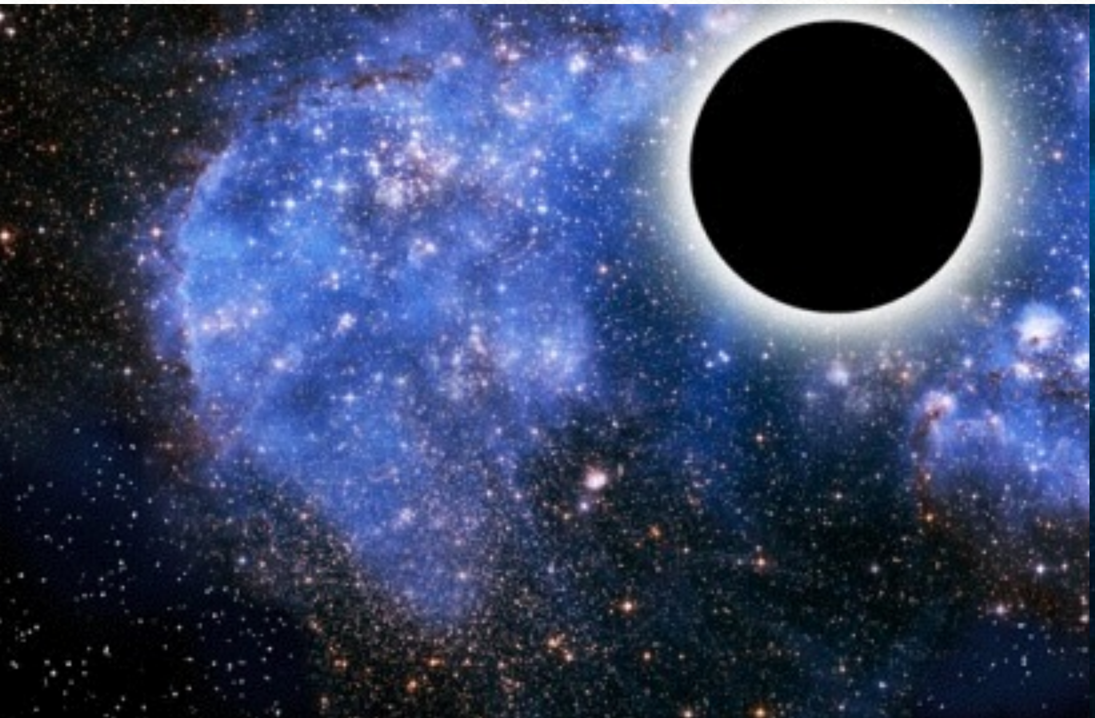
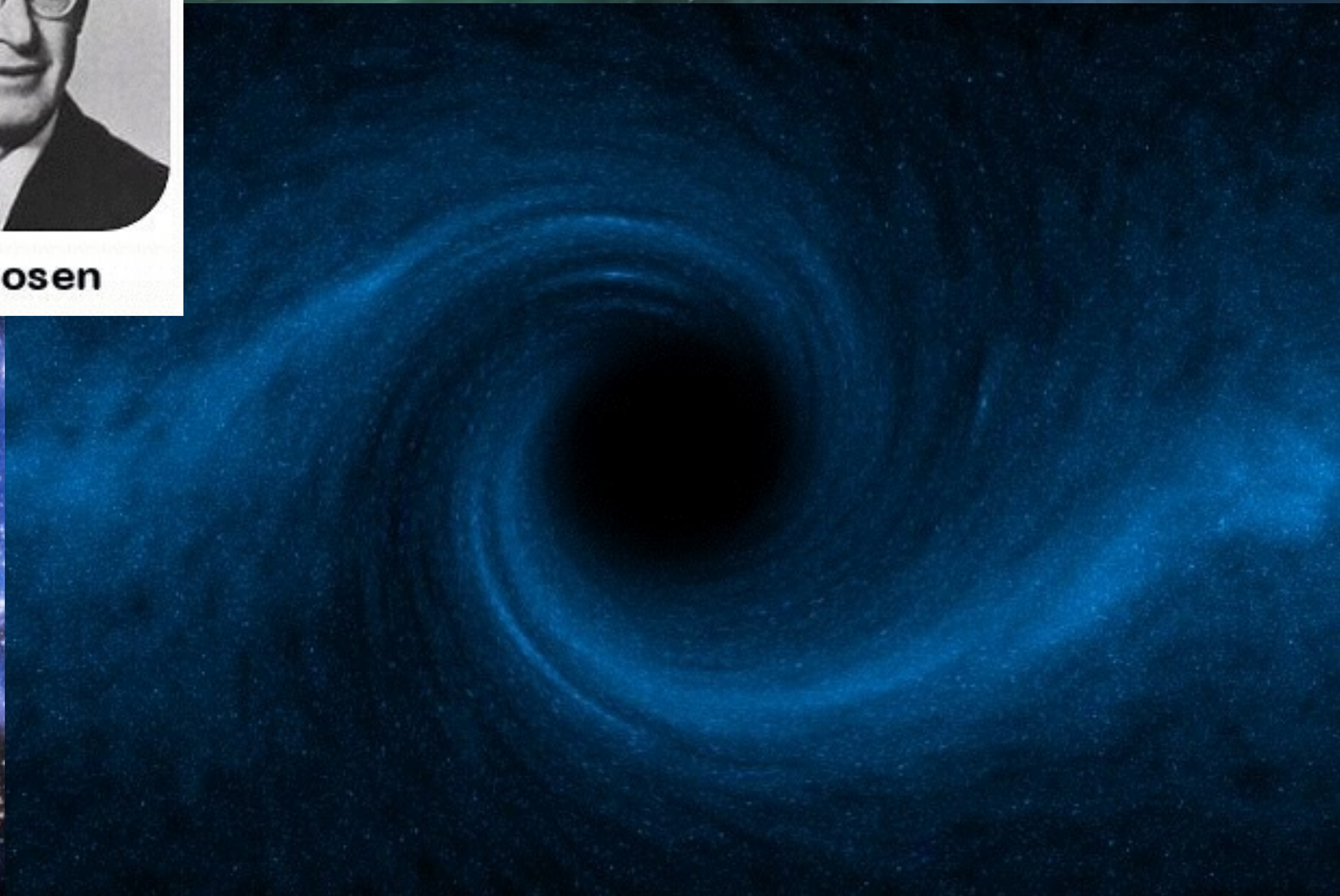
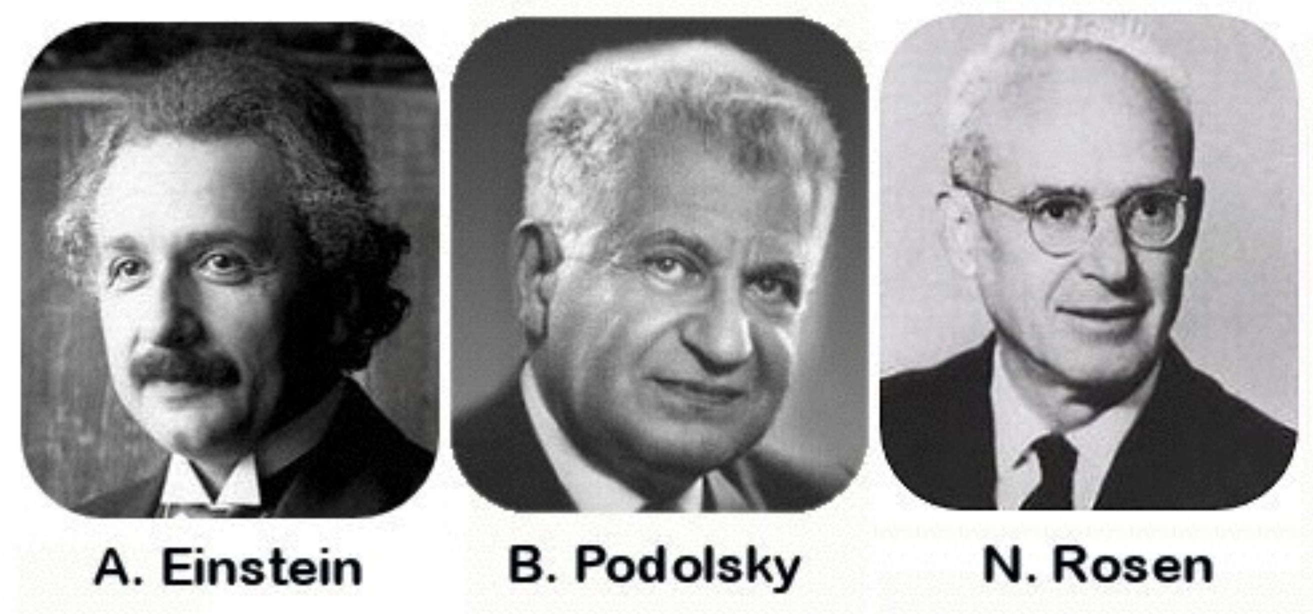




Entanglement



- We say two quantum systems are entangled when a measurement on one system affects the state of the other system.

- The classic entangled example, the EPR pair:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right)$$

- $|\psi\rangle = |\uparrow\rangle_A \otimes |\downarrow\rangle_B$ is not entangled.

- For larger vector spaces, how do you tell?