Hořava–Lifshitz Gravity from Dynamical Newton–Cartan Geometry

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Introduction

- Why Newton–Cartan geometry?
- In relativistic field theory it can be very useful to couple to a background geometry to compute EM tensors, study anomalies, Ward identities, etc.
- Background field methods for systems with NR symmetries requires NC geometry (with torsion) [talk by Niels Obers].
- Recent examples: Son's approach to the effective field theory for the FQHE [Son, 2013], [Geracie, Son, Wu, Wu, 2014] and NR hydrodynamics [Jensen, 2014].
- Torsional NC geometries occur in Lifshitz holography [Christensen, JH, Obers, Rollier, 2013], [Kiritsis, JH, Obers, 2014].

Outline Talk

- Newton–Cartan geometry: fields, connections, torsion, curvatures, etc.
- ADM decomposition: dictionary with HL gravity [Horava, 2008/9]
- Effective actions
- The local $U(1)$ of HL gravity [Horava, Melby-Thompson, 2010].
- Summary/Outlook

Newton–Cartan Geometry

- GR is a diff invariant theory whose tangent space invariance group is the Poincaré group.
- Newton–Cartan gravity is a diffeomorphism invariant theory on a manifold whose tangent space invariance group which is the Bargmann algebra: H, P_a, G_a, J_{ab}, N with N central and

$$[H, G_a] = P_a\,, \qquad [P_a, G_b] = N\delta_{ab}$$

- NC geometry (with torsion) is the natural geometric framework for HL gravity.

From Poincaré to GR

- Local Poincaré: P_a, M_{ab} (gauging):

$$
\begin{aligned}
\mathcal{A}_\mu &= P_a e^a_\mu + \frac{1}{2} M_{ab} \omega_\mu{}^{ab} \\
\mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu \,, \mathcal{A}_\nu] = P_a R_{\mu\nu}{}^a(P) + \frac{1}{2} M_{ab} R_{\mu\nu}{}^{ab}(M) \\
\delta \mathcal{A}_\mu &= \partial_\mu \Lambda + [\mathcal{A}_\mu \,, \Lambda] \,, \qquad \Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma \,, \qquad \Sigma = \frac{1}{2} M_{ab} \lambda^{ab} \\
\bar{\delta} \mathcal{A}_\mu &= \delta \mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu \,, \Sigma]
\end{aligned}
$$

- ∇_μ defined via VP : $\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho - \omega_\mu{}^a{}_b e^b_\nu = 0$
- Lorentz invariant $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. Affine $\Gamma^\rho_{\mu\nu}$: $\nabla_\mu g_{\nu\rho} = 0$.
- $R_{\mu\nu}{}^a(P) = 2\Gamma^\rho_{[\mu\nu]} =$ torsion
- $R_{\mu\nu}{}^{ab}(M) =$ Riemann curvature 2-form

Gauging Bargmann I

- Gauging Bargmann [Andringa, Bergshoeff, Panda, de Roo, 2011] H, P_a, G_a, J_{ab}, N (a is a spatial index):

$$\mathcal{A}_\mu = H\tau_\mu + P_a e^a_\mu + G_a \Omega_\mu{}^a + \frac{1}{2} J_{ab} \Omega_\mu{}^{ab} + N m_\mu$$

$$\mathcal{F}_{\mu\nu} = H R_{\mu\nu}(H) + P_a R_{\mu\nu}{}^a(P) + G_a R_{\mu\nu}{}^a(G) + \frac{1}{2} J_{ab} R_{\mu\nu}{}^{ab}(J) + N R_{\mu\nu}(N)$$

$$\delta\mathcal{A}_\mu = \partial_\mu \Lambda + [\mathcal{A}_\mu\,, \Lambda]\,, \quad \Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma\,, \quad \Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab} + N\sigma$$

$$\bar{\delta}\mathcal{A}_\mu = \delta\mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu\,, \Sigma]$$

- Vielbein postulates (introduction of $\Gamma^\rho_{\mu\nu}$):

$$\mathcal{D}_\mu \tau_\nu = \partial_\mu \tau_\nu - \Gamma^\rho_{\mu\nu} \tau_\rho = 0$$

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho - \Omega_\mu{}^a \tau_\nu - \Omega_\mu{}^a{}_b e^b_\nu = 0$$

Gauging Bargmann II

- Inverse vielbeins: v^μ and e^μ_a via

$$v^\mu \tau_\mu = -1\,, \qquad v^\mu e^a_\mu = 0\,, \qquad e^\mu_a \tau_\mu = 0\,, \qquad e^\mu_a e^b_\mu = \delta^b_a\,.$$

- Metric: $h^{\mu\nu} = \delta^{ab} e^\mu_a e^\nu_b$ and τ_μ
- $\Gamma^\rho_{\mu\nu}$ is affine and inert under G, J, N.
- $\Omega_\mu{}^{ab} = \Omega_\mu{}^{[ab]}$ so that $\nabla_\mu h^{\nu\rho} = 0$. Also $\nabla_\mu \tau_\nu = 0$.
- Torsion: $2\Gamma^\rho_{[\mu\nu]} = -v^\rho R_{\mu\nu}(H) + e^\rho_a R_{\mu\nu}{}^a(P)$
- Curvature: $[\nabla_\mu\,, \nabla_\nu] X_\sigma = R_{\mu\nu\sigma}{}^\rho X_\rho - 2\Gamma^\rho_{[\mu\nu]} \nabla_\rho X_\sigma$
- where via VPs: $R_{\mu\nu\sigma}{}^\rho = e^\rho_a \tau_\sigma R_{\mu\nu}{}^a(G) - e_{\sigma a} e^\rho_b R_{\mu\nu}{}^{ab}(J)$

Affine Connection I

- The most general metric compatible $\Gamma^\rho_{\mu\nu}$:

$$\Gamma^\rho_{\mu\nu} = -v^\rho\partial_\mu\tau_\nu + \frac{1}{2}h^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) + \frac{1}{2}h^{\rho\sigma}\left(\tau_\mu K_{\sigma\nu} + \tau_\nu K_{\sigma\mu} + L_{\sigma\mu\nu}\right)$$

  where $h_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu$ (not G invariant) and $K_{\mu\nu} = -K_{\nu\mu}$, $L_{\sigma\mu\nu} = -L_{\nu\mu\sigma}$ are arbitrary.

- Transformations of τ_μ, e^a_μ and m_μ:

$$\bar\delta\tau_\mu = \mathcal{L}_\xi\tau_\mu\,, \quad \bar\delta e^a_\mu = \mathcal{L}_\xi e^a_\mu + \lambda^a\tau_\mu + \lambda^a{}_b e^b_\mu\,, \quad \bar\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu\sigma + \lambda_a e^a_\mu$$

- Demanding local Galilean invariance ($\lambda_\mu = \lambda_a e^a_\mu$):

$$\delta_G K_{\sigma\mu} = \partial_\sigma\lambda_\mu - \partial_\mu\lambda_\sigma$$

$$\delta_G L_{\sigma\mu\nu} = \lambda_\sigma\left(\partial_\mu\tau_\nu - \partial_\nu\tau_\mu\right) - \lambda_\mu\left(\partial_\nu\tau_\sigma - \partial_\sigma\tau_\nu\right) - \lambda_\nu\left(\partial_\mu\tau_\sigma - \partial_\sigma\tau_\mu\right)$$

Affine Connection II

- Local Galilean invariance (Milne boosts in [Jensen, 2014]) realized by taking:

$$K_{\sigma\mu} = \partial_\sigma m_\mu - \partial_\mu m_\sigma$$

$$L_{\sigma\mu\nu} = m_\sigma\left(\partial_\mu\tau_\nu - \partial_\nu\tau_\mu\right) - m_\mu\left(\partial_\nu\tau_\sigma - \partial_\sigma\tau_\nu\right) - m_\nu\left(\partial_\mu\tau_\sigma - \partial_\sigma\tau_\mu\right)$$

- The connection becomes:

$$\Gamma^\rho_{\mu\nu} = -\hat{v}^\rho\partial_\mu\tau_\nu + \frac{1}{2}h^{\rho\sigma}\left(\partial_\mu\bar{h}_{\nu\sigma} + \partial_\nu\bar{h}_{\mu\sigma} - \partial_\sigma\bar{h}_{\mu\nu}\right)$$

  where $\hat{v}^\mu = v^\mu - h^{\mu\nu}m_\nu$ and $\bar{h}_{\mu\nu} = h_{\mu\nu} - \tau_\mu m_\nu - \tau_\nu m_\mu$ are G and J invariant.

- $\Gamma^\rho_{\mu\nu}$ is still not unique: can replace $\bar{h}_{\mu\nu}$ by $\bar{h}_{\mu\nu} + \alpha\tau_\mu\tau_\nu\tilde{\Phi}$ where $\tilde{\Phi} = -v^\mu m_\mu + \frac{1}{2}h^{\mu\nu}m_\mu m_\nu$ is G, J invariant. But the action will be independent of α.

Torsion

- $\delta_N \Gamma^\rho_{\mu\nu} \neq 0$ will be fixed later.
- The affine connection has torsion:
  $2\hat{\Gamma}^\rho_{[\mu\nu]} = -\hat{v}^\rho \left(\partial_\mu \tau_\nu - \partial_\nu \tau_\mu\right)$.
- We distinguish three cases :
  - No torsion: $\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = 0$ (NC geometry)
  - Twistless torsion: $\tau_{[\mu}\partial_\nu \tau_{\rho]} = 0$ (TTNC geometry)
  - No constraint on τ_μ (TNC geometry)
- Here: TTNC (which includes NC) geometry. Torsion measured by one vector a_μ because:

$$\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = a_\mu \tau_\nu - a_\nu \tau_\mu \,, \qquad a_\mu = \hat{v}^\rho \left(\partial_\rho \tau_\mu - \partial_\mu \tau_\rho\right)$$

ADM Decomposition I

- Local Galilean invariant vielbeins: τ_μ, $\hat{e}^a_\mu = e^a_\mu - \tau_\mu e^{\nu a} m_\nu$ and inverses: $\hat{v}^\mu$ and e^μ_a.
- Lorentzian metric: $g_{\mu\nu} = -\tau_\mu \tau_\nu + \hat{h}_{\mu\nu}$ where $\hat{h}_{\mu\nu} = \delta_{ab} \hat{e}^a_\mu \hat{e}^b_\nu = \bar{h}_{\mu\nu} + 2\tau_\mu \tau_\nu \tilde{\Phi}$
- $\hat{v}^\mu = g^{\mu\nu} \tau_\nu$ and $e^\mu_a = g^{\mu\nu} \hat{e}_{\nu a}$
- ADM: $ds^2 = -N^2 dt^2 + \gamma_{ij} \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right)$
- TTNC $\tau_\mu = \psi \partial_\mu \tau$ (τ is Khronon field of [Blas, Pujolas, Sibiryakov, 2010])
- Fix foliation $\tau = t$ this implies

$$\tau_t = N \,, \qquad \hat{h}_{ti} = \gamma_{ij} N^j \,, \qquad \hat{h}_{ij} = \gamma_{ij} \,, \qquad m_i = -N^{-1} \gamma_{ij} N^j$$

ADM Decomposition II

- Since $\tau_t = N$ it follows that
  - NC: $\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = 0$ is equivalent to $N = N(t)$: projectable HL gravity
  - TTNC: $N = N(t, x)$: non-projectable HL gravity, extra field (torsion) $a_i = N^{-1}\partial_i N$
- ADM decomposition becomes dynamical and is described by τ_μ (lapse), m_μ (shift) and $\hat{h}_{\mu\nu}$ (spatial metric on cst time slices).
- Actually $m_t = -\frac{1}{2N}\gamma_{ij}N^i N^j + N\tilde{\Phi}$ is an additional field (denoted by A in [Horava, Melby-Thompson, 2010]=HMT)
- Bargmann $U(1)$: $\delta_N m_\mu = \partial_\mu \sigma$ is the $U(1)$ discussed in HMT including χ appearing as $m_\mu - \partial_\mu \chi$.

Effective Actions I

- Extrinsic curvature: $\nabla_\mu \hat{v}^\rho = -h^{\rho\sigma} K_{\mu\sigma}$ where $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_{\hat{v}}\hat{h}_{\mu\nu}$
- Integration measure $e = \det\left(\tau_\mu, e^a_\mu\right)$ is G, J, N invariant.
- Add terms (built out of tangent space invariants) to the action that are relevant or marginal (up to dilatation weight $d+z$)

invariant	τ_μ	$\hat{h}_{\mu\nu}$	$\hat{v}^\mu$	$h^{\mu\nu}$	e	$\tilde{\Phi}$	χ
dil. weight	$-z$	-2	z	2	$-(z+d)$	$2(z-1)$	$z-2$

- We work in 2+1 dimensions with $1 < z \leq 2$. Weight of each term is determined by number of $h^{\mu\nu}$ and $\hat{v}^\mu$.

- The possibilities are: $\hat{v}^\mu$ (z), $h^{\mu\nu}$ (2), $\hat{v}^\mu\hat{v}^\nu$ ($2z$), $h^{\mu\nu}\hat{v}^\rho$ ($2+z$), $h^{\mu\nu}h^{\rho\sigma}$ (4). We make scalars out of them by contracting with ∇_μ and a_μ (note that $\hat{v}^\mu a_\mu = 0$).
- In 2+1 dimensions there is one curvature invariant: $\mathcal{R} = h^{\mu\nu}R_{\rho\mu\nu}{}^\rho$ (2) which is the Ricci curvature of γ_{ij}.
- Do not allow terms that break time reversal invariance.
- Two kinetic terms (the HL λ parameter):

$$c_1\nabla_\nu\hat{v}^\mu\nabla_\mu\hat{v}^\nu + c_2\nabla_\mu\hat{v}^\mu\nabla_\nu\hat{v}^\nu = C\left(h^{\mu\rho}h^{\nu\sigma}K_{\mu\nu}K_{\rho\sigma} - \lambda\left(h^{\mu\nu}K_{\mu\nu}\right)^2\right)$$

- The potential term matches [Blas, Pujolas, Sibiryakov, 2010], [Zhu, Shu, Wu, Wang, 2010]

$$\begin{aligned}\mathcal{V} &= c_3 h^{\mu\nu}a_\mu a_\nu + c_4\mathcal{R} + \delta_{z,2}\left[c_5\left(h^{\mu\nu}a_\mu a_\nu\right)^2 + c_6 h^{\mu\rho}a_\mu a_\rho\nabla_\nu\left(h^{\nu\sigma}a_\sigma\right)\right.\\ &\left.+c_7\nabla_\nu\left(h^{\mu\rho}a_\rho\right)\nabla_\mu\left(h^{\nu\sigma}a_\sigma\right) + c_8\mathcal{R}^2 + c_9\mathcal{R}\nabla_\mu\left(h^{\mu\nu}a_\nu\right) + c_{10}\mathcal{R}h^{\mu\nu}a_\mu a_\nu\right]\end{aligned}$$

Local $U(1)$

- TTNC identity (note $\lambda = 1$):

$$\delta_N \left(\nabla_\nu \hat{v}^\mu \nabla_\mu \hat{v}^\nu - \nabla_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu\right) = -\mathcal{R}\hat{v}^\mu \partial_\mu \sigma + \text{torsion terms}\,,$$

- The additional field $\tilde{\Phi}$ transforms as: $\delta_N \tilde{\Phi} = -\hat{v}^\mu \partial_\mu \sigma$.

$$S = \int d^3 x e \left[C\left(h^{\mu\rho}h^{\nu\sigma}K_{\mu\nu}K_{\rho\sigma} - \left(h^{\mu\nu}K_{\mu\nu}\right)^2 - \tilde{\Phi}\mathcal{R}\right) - \mathcal{V}\right]$$

  Is the $U(1)$ invariant HMT action for projectable HL gravity in 3D [Horava, Melby-Thompson, 2010].

- The non-projectable HMT action can only be made $U(1)$ invariant by adding a Stückelberg scalar [HMT]. By replacing m_μ by $m_\mu - \partial_\mu \chi$ we reproduce precisely all the terms of [Zhu, Shu, Wu, Wang, 2010].

Summary/Outlook

- Dynamical (TT)NC geometry is exactly the same as (non-)projectable HL gravity.
- What does this teach us about the ground state? Flat NC space-time has different symmetries than Minkowski space-time (see talk by Niels Obers).
- χ is an essential part of the TNC geometry. Under special circumstances it can drop out (e.g some HL actions or Schrödinger scalar model).
- Black holes?, phase space formulation, etc.
- Gauging other NR symmetry groups: Schrödinger space-times [Andrade, Keeler, Peach, Ross, 2014], [Armas, Blau, JH, in progress] or warped AdS_3 [Hofman, Rollier, 2014].