Holographic topological entanglement entropy and ground state degeneracy

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April 16, 2015

Introduction: topological order

There are interesting systems in 2+1 dimensions whose phases are not distinguished by local order parameters.

The ground state might have topological order: long range correlations in the fields. These properties are reflected in the degeneracy of the ground state as a function of genus.

One way to distinguish phases/measure topological order it to compute topological entanglement entropy (TEE). (Levin, Wen; Kitaev, Preskill) Can we have this in holography?

Outline

Topological entanglement entropy and ground state degeneracy
- Entanglement entropy and topological entanglement entropy
- TEE and ground state degeneracy in Chern-Simons theory

Topological entanglement entropy in (EH) holography
- TEE in compactified N=4

TEE in holographic models with Gauss-Bonnet term
- Holographic Gauss-Bonnet gravity
- Holographic entanglement entropy in soft-wall models

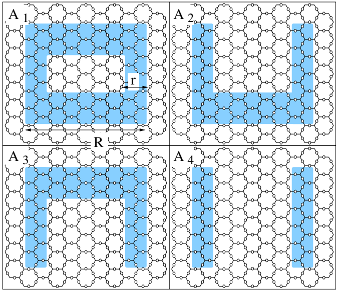
Entanglement entropy

Divide the system into subsystems A and B. Consider state descibed by a density matrix ρ. Let $\rho_A = tr_B \rho$. Entanglement entropy = Van Neumann entropy $S_A = -tr_A \rho_A \log \rho_A$. Example:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|1\rangle_A |1\rangle_B + |2\rangle_A |2\rangle_B \right)$$

For this state entanglement entropy $S_A = \log 2$. Measures entanglement between A and B.

Topological entanglement entropy



Consider $(S_1 - S_2) - (S_3 - S_4) = -2\gamma$. [Levin, Wen] The value of topological entropy γ can also be computed via the constant term in the entanglement entropy of a disk whose radius $R \to \infty$:

$$S = R/\epsilon - \gamma + \ldots$$

Chern-Simons

Chern-Simons is a good example of topological theory. Quantum dimensions

$$d_a = \lim_{q \to 1} \frac{\chi_a(q)}{\chi_0(q)} = \lim_{q \to 0} \frac{\sum_b \mathcal{S}_a^b \chi_b(q)}{\sum_b \mathcal{S}_0^b \chi_b(q)} \approx \frac{\mathcal{S}_a^0}{\mathcal{S}_0^0}$$

The topological entanglement entropy can be written as

$$\gamma = \log \mathcal{D} = -\log S_0^0 \qquad ; \qquad \mathcal{D} = \sqrt{\sum_a |d_a|^2}$$

$U(1)_k$ Chern-Simons

Quantum dimension and TEE can be easily computed:

$$\mathcal{D} = \sqrt{\sum_{a=1}^{k} |d_a|^2} = \sqrt{k}; \qquad \gamma = \frac{1}{2}\log(k)$$

The ground state degeneracy and the associated entropy are

$$\mathcal{Z}[\Sigma_g \times S^1; U(1)_k] = k^g \qquad ; \qquad S_g = g \log k$$

So we have a relation

$$2g\gamma = S_g$$

$SU(N)_k$ Chern-Simons

Level-rank duality

$$\frac{Z[S^3, SU(N)_k]}{Z[S^3, SU(k)_N]} = \sqrt{\frac{k}{N}}$$

One can use the expression for $Z[S^3, SU(k)_N]$ for finite k and $N \to \infty$ limit, $\log \mathcal{Z}[S^3, SU(k)_N] = -\frac{1}{2}(k^2 - 1)\log N$

$$\gamma = \frac{k^2}{2}\log N$$

$SU(N)_k$ Chern-Simons

For the ground state degeneracy we use

$$\frac{\mathcal{Z}[\Sigma_g \times S^1, SU(N)_k]}{\mathcal{Z}[\Sigma_g \times S^1, SU(k)_N]} = \left(\frac{N}{k}\right)^g$$

and

$\lim_{N\to\infty} \mathcal{Z}[\Sigma_g \times S^1, SU(k)_N] = (g-1)(N^2-1)\log k + \mathcal{O}(k^0)$. So

$$S_g = \log Z[\Sigma_g \times S^1, SU(N_k)] = g\log(N/k) + (g-1)(k^2-1)\log N + O(N^0)$$

In the limit $N \gg k \gg 1$ we have

$$2(g-1)\gamma = S_g$$

Entanglement entropy in holography

According to Ryu and Takayanagi, the value of entanglement entropy between a region and its complement is obtained by evaluating the volume of the minimal codimension-2 hypersurface $t = const$ which asymptotes to the border between the region and its complement at $z = 0$.

Compactified N=4

Consider N D3 branes on a circle of radius R_3 at large t'Hooft coupling $\lambda = g_{YM}^2 N$. The metric of the gravity dual

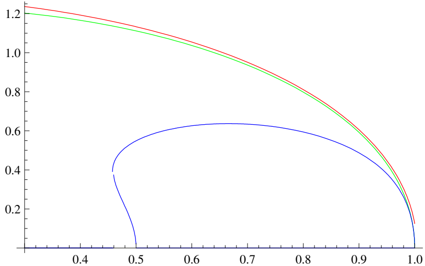
$$ds^2 = L^2 \left[\frac{dz^2}{z^2 h(z)} + \frac{dx_\mu dx^\mu}{z^2} + h(z)\frac{dx_3^2}{z^2} + d\Omega_5^2 \right]$$

where $\mu = 0, 1, 2$, $h(z) = 1 - (z/2R_3)^4$, $L^4 \sim \lambda$. Finite R_3 gives

rise to strongly coupled 2+1 dimensional theory with confinement and a mass gap (the scale of the correlation length is set by R_3)

D3 branes: conformal case

Consider, as a warm-up exersise, conformal case: $R_3 \to \infty$, $h(z) = 1$.



$z(r)$. Red curve $[z'(r=0)=0]$ corresponds to the cylinder $x_1^2 + x_2^2 = R^2$ on the boundary at $z=0$. Blue curve gives rise to two concentric cylinders at $z=0$.

D3 branes: conformal case

Action with parameterization $z(r)$:

$$S = \frac{4N_c^2 l}{15\pi} \int dr \frac{r}{z^3} \sqrt{1 + (z')^2}$$

Equation of motion:

$$\frac{d}{dr} \left(\frac{rz'}{z^3 \sqrt{1 + (z')^2}} \right) = -\frac{3r\sqrt{1 + (z')^2}}{z^4}$$

Near the boundary

$$z \simeq 2\sqrt{R}\sqrt{R - r}$$

D3 branes: conformal case

To compute entanglement entropy for the cylinder $x_1^2 + x_2^2 \leq R^2$ we need to subtract the divergent part. Solve equations of motion, and evaluate the action. The integral is cut off at $z = \epsilon$.

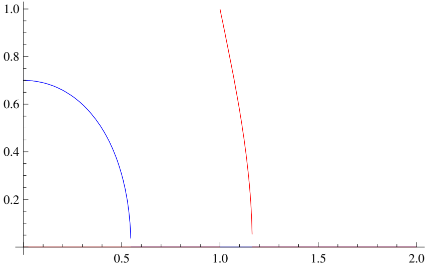
$$S = \frac{2N_c^2}{15\pi}\left(\frac{lR}{\epsilon^2} + \frac{l}{4R}\log\frac{\epsilon}{R}\right) - \frac{4N_c^2 l}{15\pi R}\tilde{\gamma}$$

where l is the length of the cylinder. This gives $\tilde{\gamma} = 0.305$. This is

similar to topological entanglement entropy, but is computed in the theory with infinite correlation length.

D3 branes on a circle

Consider finite correlation length R_3. rescale coordinates $z \to z/2R_3$



The space ends at $z = 1$. There are two solutions which asymptote to the circle at $z = 0$: disk topology (blue) and cylinder topology (red), where Kaluza-Klein circle shrinks to zero size.

D3 branes on a circle

Action

$$S = \frac{4N_c^2}{15} \int_0^{\frac{R}{z_0}} dr \frac{r}{z^3} \sqrt{1 - z^4 + (z')^2}$$

Equation of motion

$$\frac{d}{dr}\left(\frac{rz'}{z^3\sqrt{1-z^4+(z')^2}}\right) = \frac{r(z^4 - 3[1+(z')^2])}{z^4\sqrt{1-z^4+(z')^2}}$$

Same small z behavior as in the conformal case

D3 branes on a circle

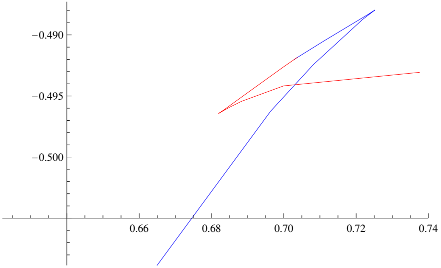
There are also other solutions, which asymptote to the annulus at $z = 0$. For $R \ll R_3$ the structure is similar to the conformal case. To compute entanglement entropy, we need to extract the

UV-divergent part:

$$S = \frac{4N_c^2}{15}\left(\frac{RR_3}{\epsilon^2} + \frac{R_3}{4R}\log\frac{\epsilon}{R}\right) + \frac{4N_c^2}{15}\tilde{S}$$

Note that topological entropy is encoded in $\tilde{S}$. In the conformal case hypersurface of disk topology gave non-vanishing $\tilde{S}$

D3 branes on a circle



$\tilde{S}(R)$. Disk (cylinder) topology– blue (red) curve. For small (large) R disk-type (cylinder-type) solutions dominate the computation of entanglement entropy. For large R, $\tilde{S} = -R/4R_3 + \mathcal{O}(R^{-1})$ which implies vanishng topological entropy!

Holographic Gauss-Bonnet gravity

Consider holographic gravity with Gauss-Bonnet term:

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R + \frac{6}{L^2} + \frac{\lambda L^2}{2} E_4 \right].$$

where

$$E_4 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

Entropies with Gauss-Bonnet term

GB term does not affect equations of motion or entangling surface. But it does affect entropies:

$$S_{EE} = \frac{1}{4G}\left[\int_M d^2y\sqrt{\hat{h}}\left(1+\lambda L^2\hat{R}\right)\right]$$

while the Wald entropy is

$$S = \frac{1}{4G_N}\int_{\text{horizon}} d^2y\sqrt{h}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma} = \frac{1}{4G_N}\int_{\text{horizon}} d^2y\sqrt{h}(1+\lambda L^2\mathcal{R})$$

Holographic entropies from the GB term are topological

For the minimum entangling surface with the disc topology, $S_{EE}^{(1)}$ can be written as

$$S_{EE}^{(1)} = \frac{\lambda L^2}{4G}\int d^2y\sqrt{\hat{h}}\hat{R} = \frac{\pi\lambda L^2}{G}$$

On the other hand, Riemann surface of genus g can be obtained by identifying the hyperbolic space H^2 by a finite subgroup of $SL(2,Z)$. The Gauss-Bonnet term contributes a topological number

$$S_g^{(1)} = \frac{\lambda L^2}{4G}(4\pi\chi_g) = \frac{2\pi\lambda L^2}{G}(1-g)$$

Holographic Gauss-Bonnet

So, provided the entangling surface has a disk topology and there's no contribution from Einstein-Hilbert sector, we have

$$2(g-1)\gamma = S_g$$

similarly to $SU(N)_k$ Chern-Simons in the $N \gg k \gg 1$ regime.

Turns out, soft-wall confinement works well for TEE (disk topology, no Einstein-Hilbert contribution to γ). It is harder to deal with the ground state degeneracy.

TEE in holographic soft wall models

Soft-wall confinement geometry

$$ds^2 = \frac{e^{-(\mu z)^\nu}}{z^2}\left(-dt^2 + dz^2 + dx^2 + dy^2\right)$$

High energy glueball spectrum is $m_n \sim n^{2-2/\nu}$. $\nu = 1$ is the "maximally soft" regime, where there is continuum above a gap.

TEE in holographic soft wall models

Entangling surface parameterized by $z(r)$; the boundary condition is $z(r)\to 0$ as $r\to R$. It must satisfy

$$\left(\frac{2+z}{z}\right) r(1+(z')^2)+z'(1+(z')^2)+z''r=0$$

Cylinder topology implies $z'\to-\infty$ as $z\to\infty$, hence the leading terms become

$$(z')^3+z''r=0$$

and $z' =\simeq \frac{1}{\sqrt{R-r}}$ which is incompatible with $z\to\infty$ as $r\to R$.

The entangling surface necessarily has disk topology for $\nu<2$ and the area of the entangling surface does not contain $\mathcal{O}(R^0)$ term as $R\to\infty$. So the contribution to the TEE comes entirely from the Gauss-Bonnet term.

Entropy of hyperbolic horizons

Solutions with hyperbolic horizons

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 dH_{n-1}^2$$

where $f(r) = -1 - \frac{\mu}{r^{n-2}} + \frac{r^2}{L^2}$
have energy and entropy

$$E \simeq \frac{N^2}{\beta^4}\left(1 + \sqrt{1 + \frac{2\beta^2}{\pi^2 L^2}}\right)^2, \qquad S \simeq \frac{N^2}{\beta^3}\left(1 + \sqrt{1 + \frac{2\beta^2}{\pi^2 L^2}}\right)^3$$

Entropy of hyperbolic horizons

So there is finite entropy at $T = 0$. What about soft wall models? Assume the metric

$$ds^2_{\rm string} = \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + d\Sigma_2^2\right), \qquad ds^2 = e^{-2\varphi(z)} ds^2_{\rm string}$$

with $f(z) = 1 - \frac{z^2}{L^2} + Mz^3$. Then the area of the horizon is supressed

$$S \sim \frac{e^{-\sqrt{3}\mu L}}{3}$$

and in the limit of large mass gap the main contribution to the entropy comes from the Gauss-Bonnet term.

Summary

- Soft wall holographic models with GB term have nonvanishing topological entanglement entropy
- Relation between TEE and degeneracy of states is similar to that of CS theory
- Need better understanding of soft wall models with hyperbolic horizons

THE END

Thank you!