Holographic thermalization and AdS (in)stability

Ben Craps, work with Oleg Evnin and Joris Vanhoof

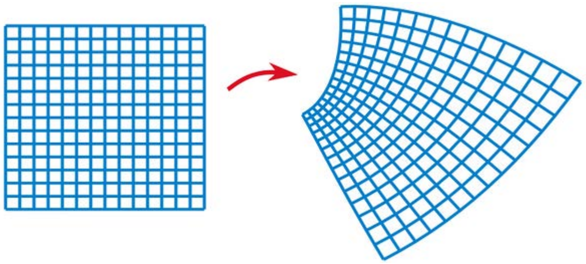
JHEP 1410 (2014) 48
JHEP 1501 (2015) 108

Gauge/Gravity Duality 2015, GGI, 16 April 2015

Holography relates thermalization to black hole formation

AdS

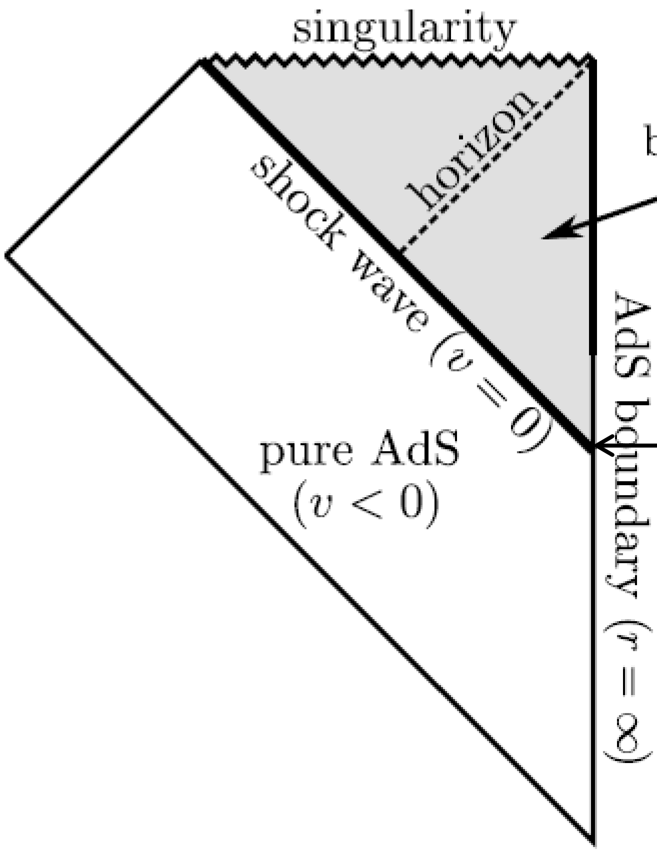
CFT



- Anti-de Sitter spacetime
- Black hole
- Black hole formation

- Conformal Field Theory
- Thermal state
- Thermalization

Weak-field BH formation in planar AdS$_{d+1}$



massless bulk scalar ϕ

homogeneous source

$\phi_0(t) = \epsilon\tilde{\phi}_0(t)$

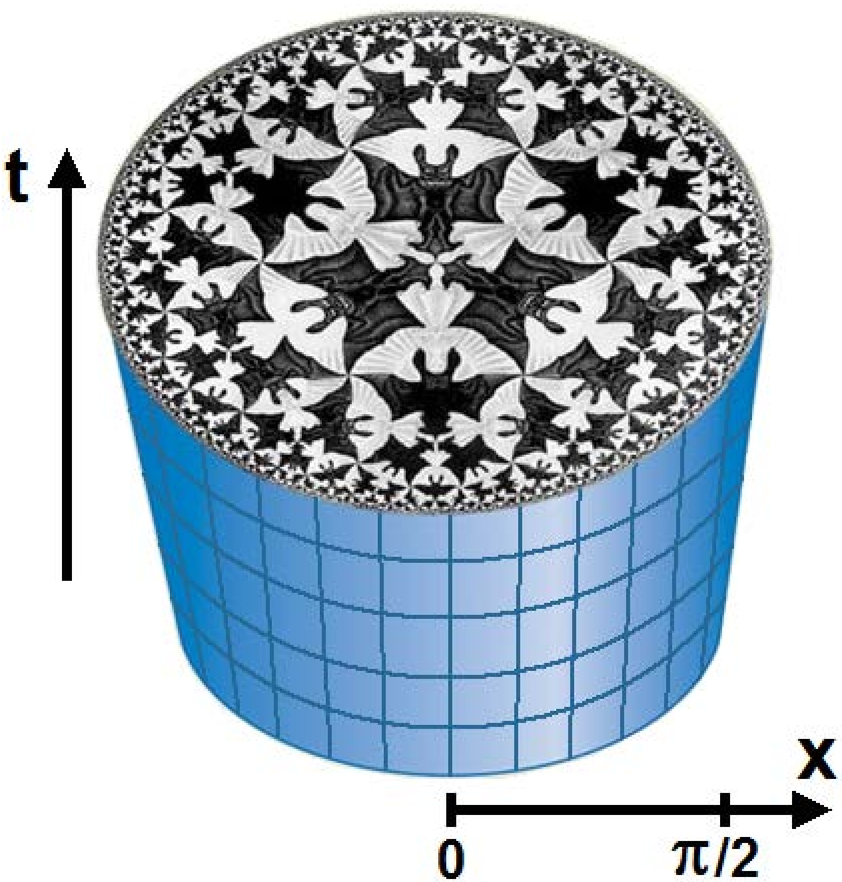
with support in $t \in [0, \delta t]$

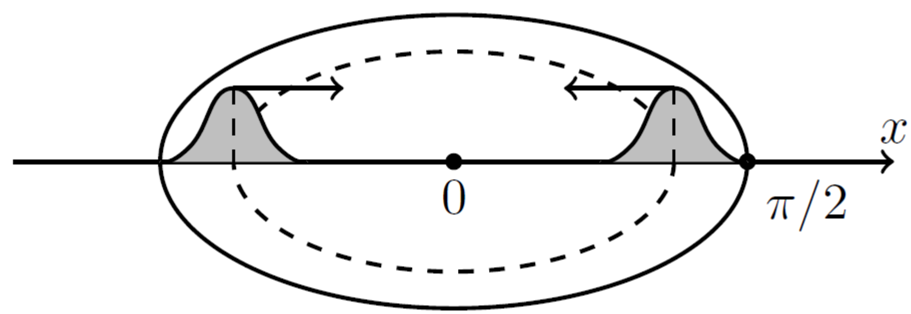
treat as small parameter

results in formation of black brane with temperature $T \sim \dfrac{\epsilon^{2/d}}{\delta t}$

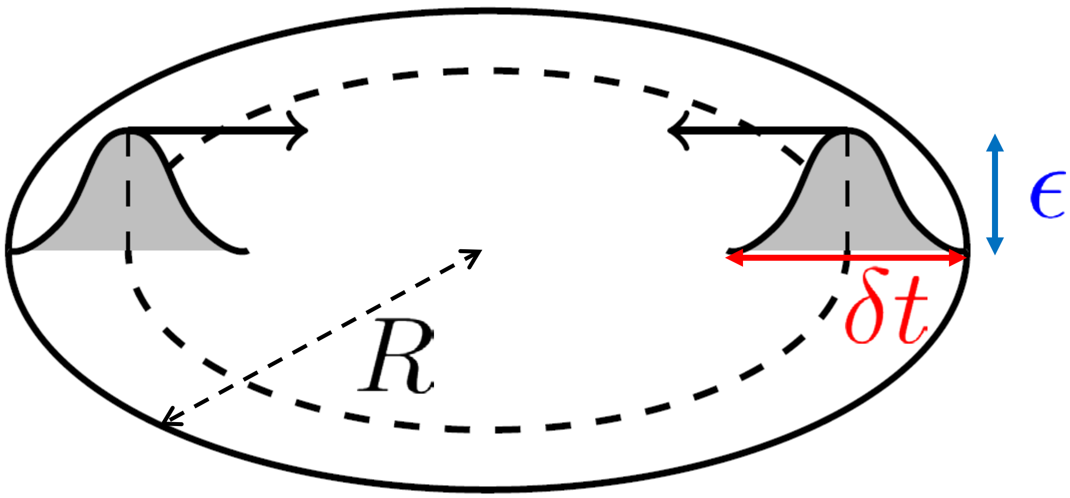
[Bhattacharyya, Minwalla 2009]

Global AdS: do spherical shells collapse?





It depends on amplitude ϵ and width δt



CFT on $\mathbb{R} \times S^{d-1}$

radius R

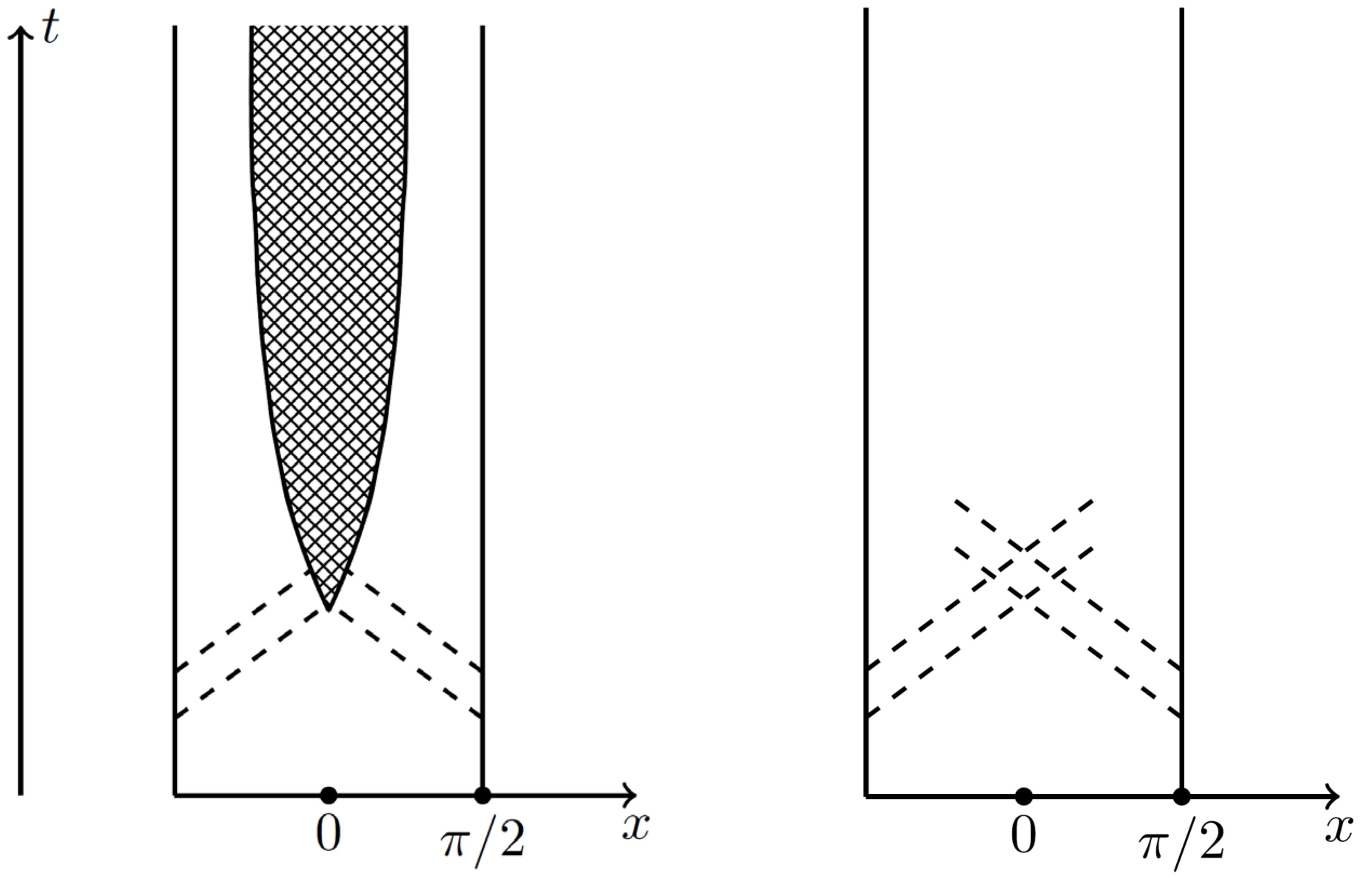
$x \equiv \frac{\delta t}{R}$, consider $x \ll 1$

- $x \ll \epsilon^{2/d}$: large BH $\longrightarrow$ cf. planar AdS
- $\epsilon^{2/d} \ll x \ll \epsilon^{\frac{1}{d-1}}$: small BH
- $\epsilon^{\frac{1}{d-1}} \ll x$: scattering

cf. Minkowski

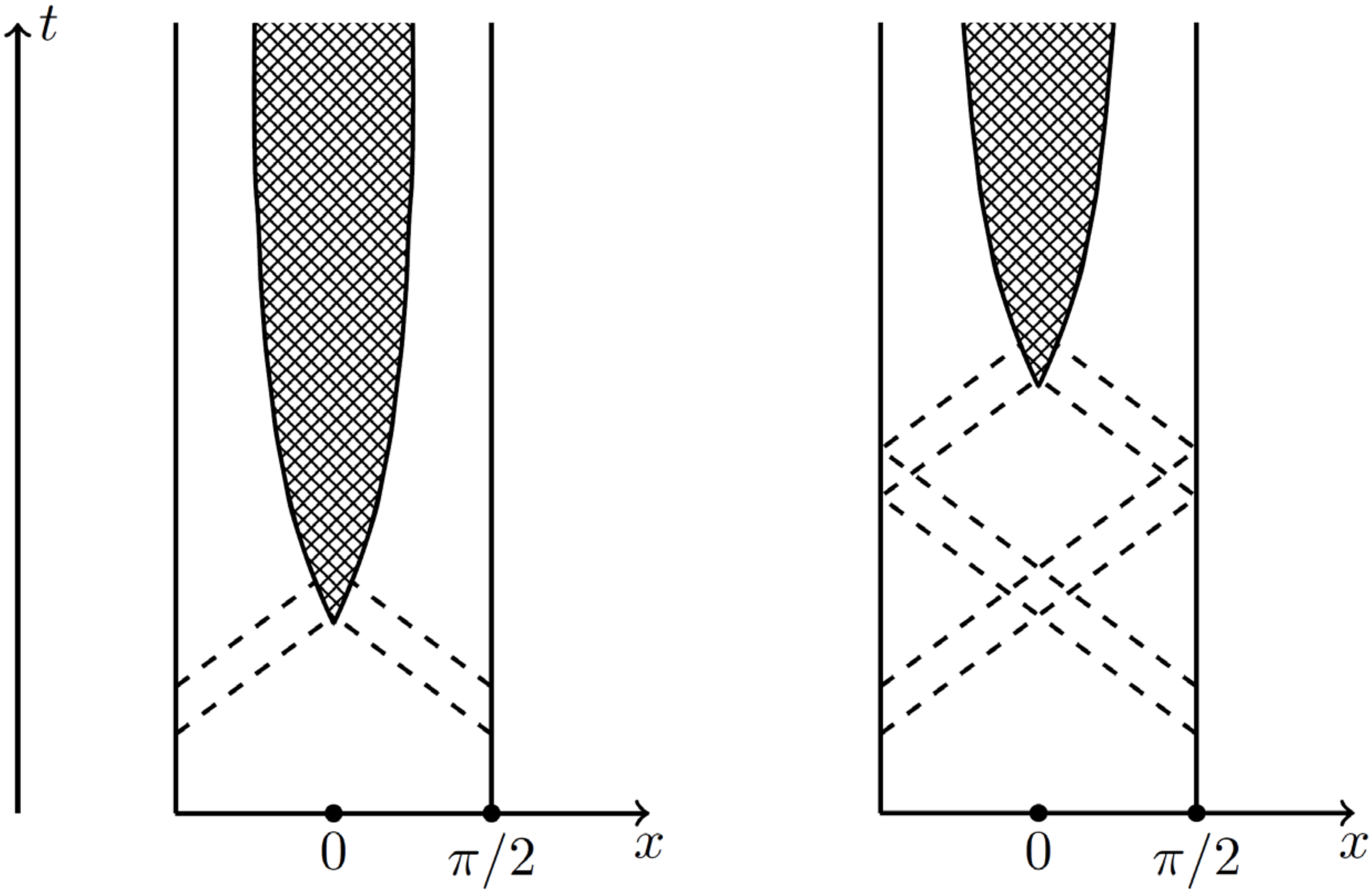
[Bhattacharyya, Minwalla 2009]

Global AdS: large amplitude shells collapse, small amplitude shells scatter



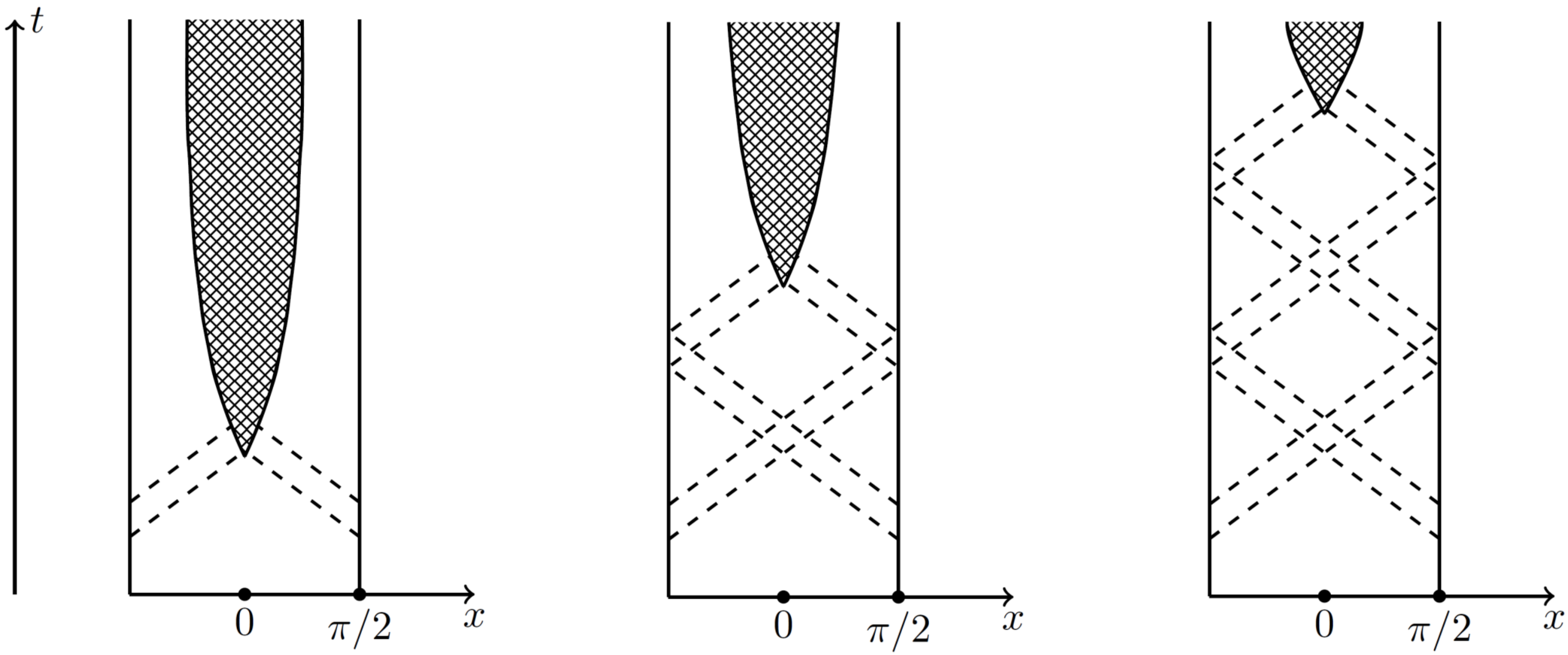
[Bhattacharyya, Minwalla 2009]

Some shells collapse after two attempts



[Bizon, Rostworowski 2011]

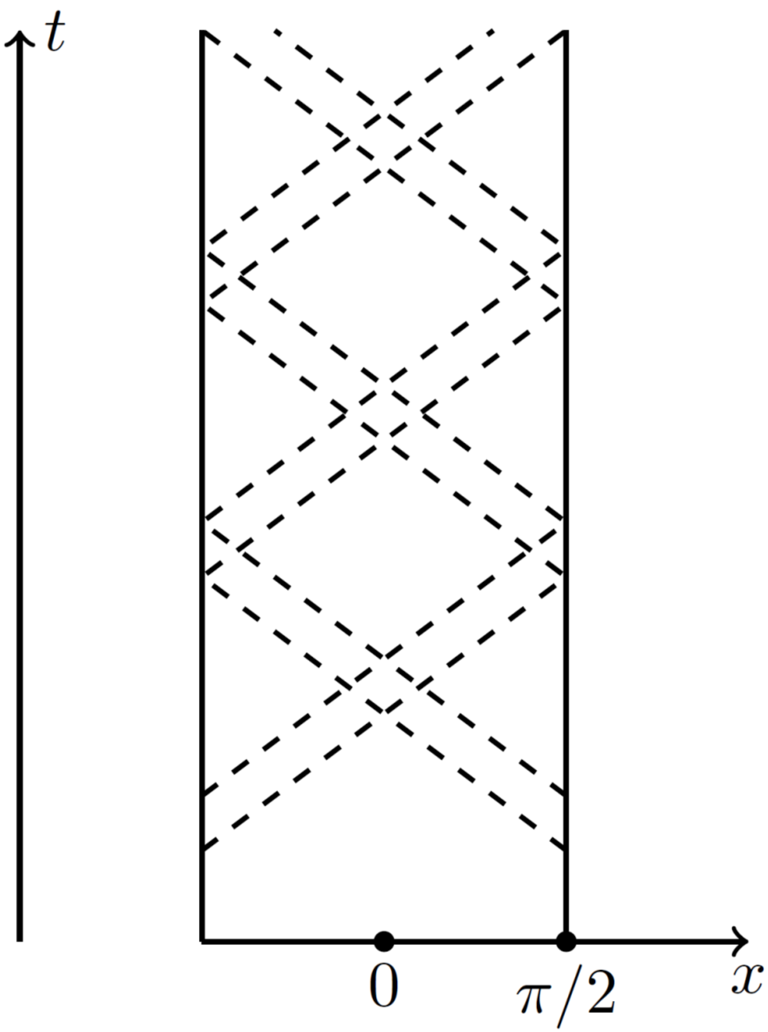
Some shells collapse after many attempts



For initial conditions $\phi, \dot{\phi} \sim \epsilon$, time scale for collapse $\sim 1/\epsilon^2$

[Bizon, Rostworowski 2011]

Other shells do not seem to collapse: islands of stability?



[Bizon, Rostworowski 2011]

[Dias, Horowitz, Santos 2011]

[Dias, Horowitz, Marolf, Santos 2012]

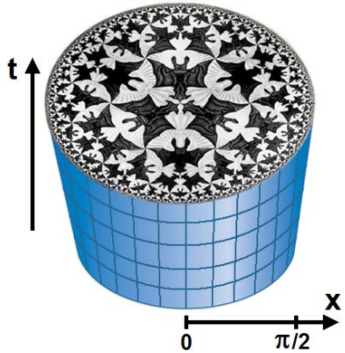
[Maliborski, Rostworowski 2013]

[Buchel, Liebling, Lehner 2013]

[Maliborski, Rostworowski 2014]

[Dimitrakopoulos, Freivogel, Lippert, Yang 2014]

Study scalar in AdS perturbatively



Spherically symmetric perturbations $\phi = \phi(x,t)$ and

$$ds^2 = \frac{L^2}{\cos^2 x}\left(\frac{dx^2}{A(x,t)} - A(x,t)e^{-2\delta(x,t)}dt^2 + \sin^2 x\, d\Omega^2_{d-1}\right)$$

Metric determined by constraints → Solve e.o.m. for ϕ

Perturbative expansion $\phi = \epsilon\phi_{(1)} + \epsilon^3\phi_{(3)} + \ldots$

Expansion in normal modes $e_n(x)$ with $\omega_n = d + 2n$

$$\phi_{(1)}(x,t) = \sum_{n=0}^{\infty} a_n \cos(\omega_n t + b_n) e_n(x), \quad \phi_{(3)}(x,t) = \sum_{n=0}^{\infty} c_n(t) e_n(x)$$

→ $\ddot{c}_n + \omega_n^2 c_n = C_{ijkn} a_i a_j a_k \cos((\omega_i \pm \omega_j \pm \omega_k)t + (b_i \pm b_j \pm b_k)) + \ldots$

specific complicated integrals of AdS mode functions

[Bizon, Rostworowski 2011]

Secular terms invalidate perturb. theory

$$\phi(x,t) = \sum_{n=0}^{\infty} \left[\epsilon a_n \cos(\omega_n t + b_n) + \epsilon^3 c_n(t) + \ldots\right] e_n(x)$$

$$\ddot{c}_n + \omega_n^2 c_n = C_{ijkn} a_i a_j a_k \cos((\omega_i \pm \omega_j \pm \omega_k)t + (b_i \pm b_j \pm b_k)) + \ldots$$

Resonant if $\pm\omega_n = \omega_i \pm \omega_j \pm \omega_k$

Integer normal mode spectrum $\omega_n = d + 2n$ → many resonances!

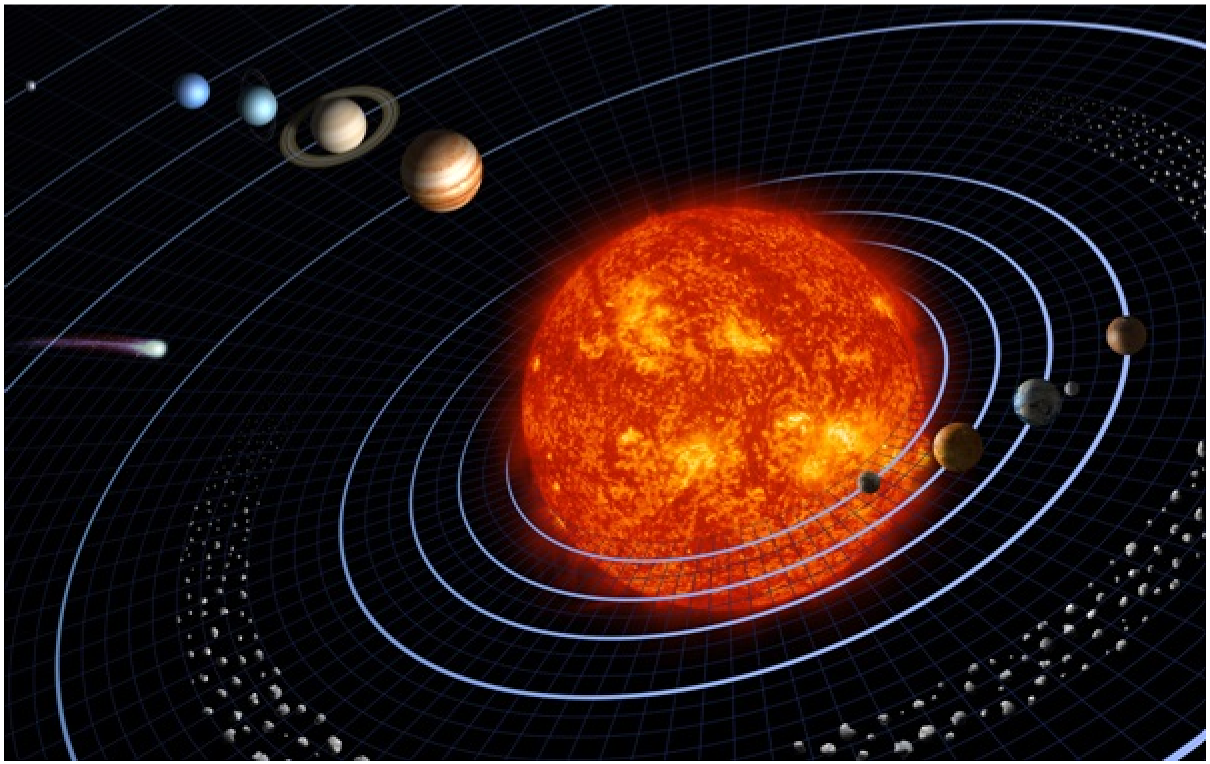
Resonances lead to secular terms

$$c_n(t) = C_{ijkn} a_i a_j a_k\, t \sin(\omega_n t + (b_i \pm b_j \pm b_k)) + \ldots$$

They become important on time scales $t \sim 1/\epsilon^2$

[Bizon, Rostworowski 2011]

Secular terms were first studied in celestial mechanics



Stability of solar system: can perturbatively small corrections accumulate to give large effects on very long time scales?

Secular terms invalidate naive perturb. theory and must be resummed

Typical result of time-dependent perturbation theory:

$$x(t) = a\cos(\omega t + b) + (\ldots)\epsilon + (\ldots)\,\epsilon t\sin(\omega t + b)$$
$$+ (\ldots)\,\epsilon t\cos(\omega t + b) + \mathcal{O}\left(\epsilon^2\right)$$

Resummation is needed:

- Poincaré-Lindstedt
- Multiscale analysis
- Renormalization group
- Averaging

Typical result of resummation:

$$x(t) = a(\epsilon t)\cos(\omega t + b(\epsilon t)) + (\ldots)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

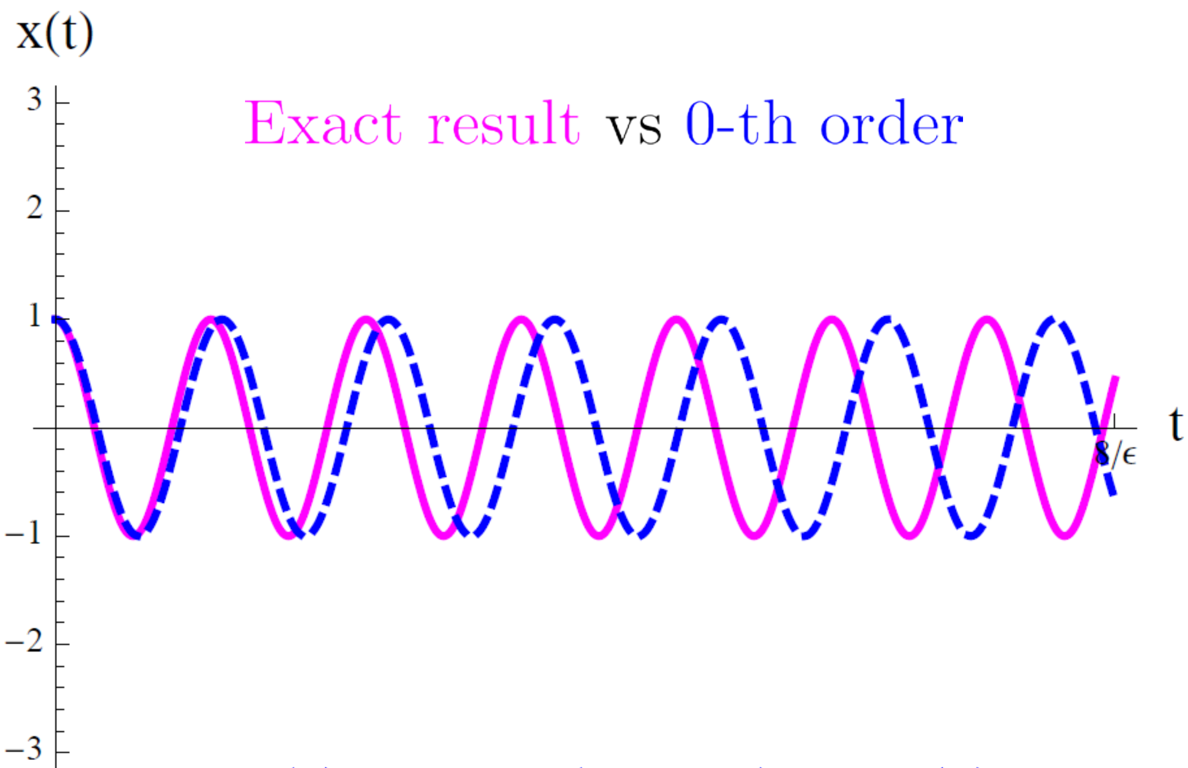
Secular terms invalidate naive perturb. theory for anharmonic oscillator

Particle in potential $V(x) = \frac{\omega^2}{2} x^2 + \frac{\epsilon}{4} x^4$

Equation of motion: $\ddot{x} + \omega^2 x + \epsilon x^3 = 0$

Perturbative expansion: $x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots$

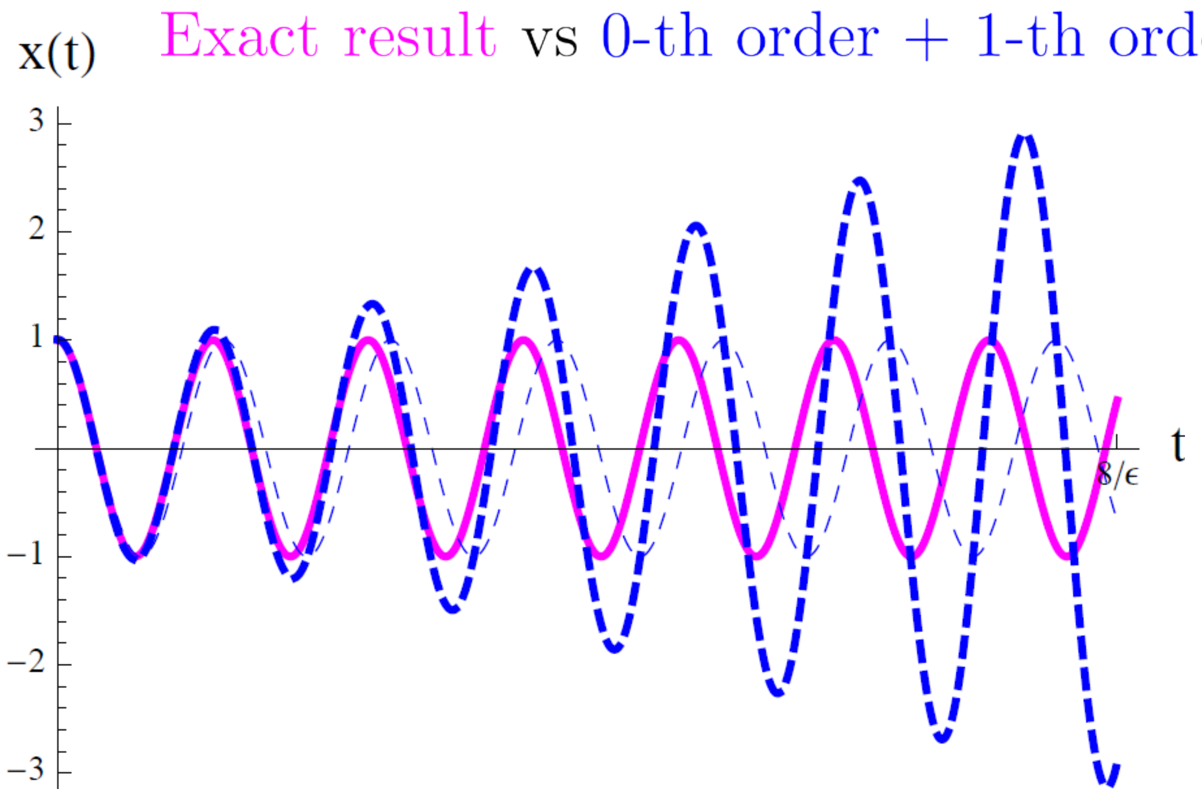
Secular terms invalidate naive perturb. theory for anharmonic oscillator



$$x(t) = a\cos(\omega t + b) + \mathcal{O}(\epsilon)$$

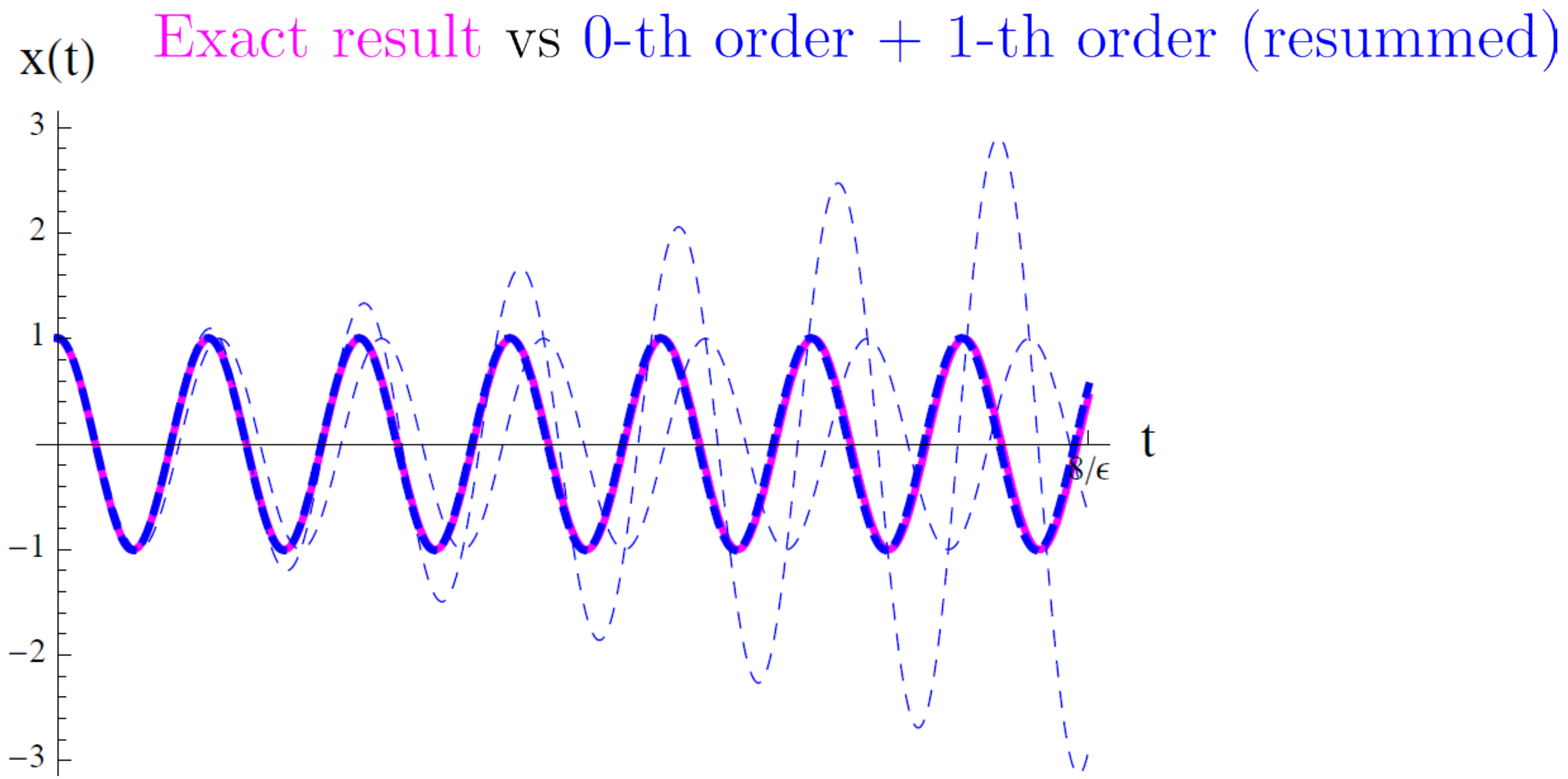
Plot: $\omega = 1,\ \epsilon = 0.2,\ a = 1,\ b = 0$

Secular terms invalidate naive perturb. theory for anharmonic oscillator



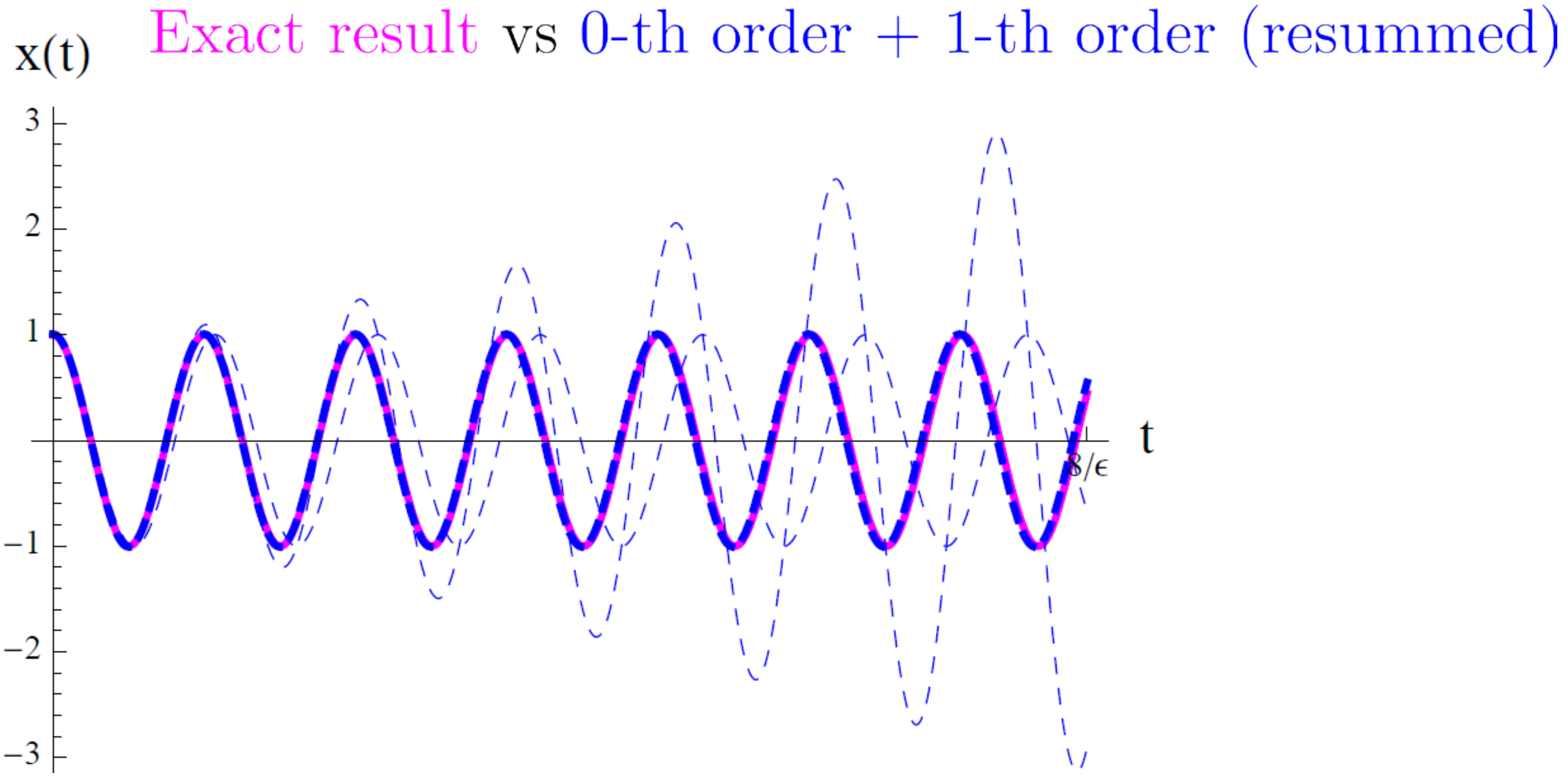
$$x(t) = a\cos(\omega t + b) + \left(\frac{a^3}{32\omega^2}\cos(3\omega t + 3b) - \frac{3a^3}{8\omega} t \sin(\omega t + b)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

Resummation cures perturbation theory



$$x(t) = a\cos\left(\omega t + b + \frac{3a^2}{8\omega^2}\epsilon t\right) + \frac{a^3}{32\omega^2}\cos\left(3\omega t + 3\left(b + \frac{3a^2}{8\omega^2}\epsilon t\right)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

The secular term has been resummed into an innocent frequency shift



$$x(t) = a\cos\left(\left(\omega + \frac{3a^2}{8\omega^2}\epsilon\right)t + b\right) + \frac{a^3}{32\omega^2}\cos\left(3\left(\omega + \frac{3a^2}{8\omega^2}\epsilon\right)t + 3b\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

[Poincaré, Lindstedt]

Other secular terms can also be resummed

$$x(t) = a\cos(\omega t + b) + (\ldots)\epsilon + (\ldots)\,\epsilon t\sin(\omega t + b)$$
$$+ (\ldots)\,\epsilon t\cos(\omega t + b) + \mathcal{O}\left(\epsilon^2\right)$$

Resummation methods used for AdS (in)stability problem:

- Poincaré-Lindstedt [Bizon, Rostworowski 2011; Dias, Horowitz, Santos 2012; Maliborski, Rostworowski 2013]
- Multiscale analysis [Balasubramanian, Buchel, Green, Lehner, Liebling 2014]
- Renormalization [BC, Evnin, Vanhoof 2014]
- Averaging [Basu, Krishnan, Saurabh 2014; BC, Evnin, Vanhoof 2015]

Typical result of resummation:

$$x(t) = a(\epsilon t)\cos(\omega t + b(\epsilon t)) + (\ldots)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

Main features of our approach

- Fully analytic (no numerics, no truncations to finite number of normal modes)
- Uses several resummation methods (equivalent at first order, but yield complementary insights, e.g. accuracy theorem from averaging method)
- All-mode results bring short-wavelength regime within reach

  → will hopefully be relevant for study of (absence of) turbulence
  (see [de Oliveira, Pando Zayas, Rodrigues 2013] for numerical results on turbulence)

[BC, Evnin, Vanhoof 2014, 2015]

Renormalization leads to flow equations

Consider naive perturbation series

$$c(t) = a\cos(\omega t + b) + \epsilon^2 D(a,b)\, t \sin(\omega t + b) \\ + \epsilon^2 E(a,b)\, t\cos(\omega t + b) + \ldots$$

Introduce arbitrary time τ, write $t = (t-\tau)+\tau$, and absorb τ contribution in integration constants:

$$c(t) = a_R\cos(\omega t + b_R) + \epsilon^2 D(a_R, b_R)(t-\tau)\sin(\omega t + b_R) \\ + \epsilon^2 E(a_R, b_R)(t-\tau)\cos(\omega t + b_R) + \ldots$$

with

$a = a_R - \epsilon^2 E(a_R, b_R)\,\tau + \ldots$ and $b = b_R + \dfrac{\epsilon^2}{a_R} D(a_R, b_R)\,\tau + \ldots$

Next impose $\dfrac{\partial c}{\partial \tau} = 0$ and set $\tau = t.$ (Cf. RG in QFT.)

[Chen, Goldenfeld, Oono 1996]

Renormalization leads to flow equations

$$c(t) = a_R \cos(\omega t + b_R) + \epsilon^2 D(a_R, b_R)(t-\tau)\sin(\omega t + b_R) + \epsilon^2 E(a_R, b_R)(t-\tau)\cos(\omega t + b_R) + \ldots$$

with

$a = a_R - \epsilon^2 E(a_R, b_R)\,\tau + \ldots$ and $b = b_R + \dfrac{\epsilon^2}{a_R} D(a_R, b_R)\,\tau + \ldots$

Next impose $\dfrac{\partial c}{\partial \tau} = 0$ and set $\tau = t.$

Flow equations:
$$\begin{cases} \dfrac{da_R}{d\tau} = \epsilon^2 E(a_R, b_R) \\[2ex] a_R \dfrac{db_R}{d\tau} = -\epsilon^2 D(a_R, b_R) \end{cases}$$

Amplitude and phase acquire slow time-dependence.

[Chen, Goldenfeld, Oono 1996]

Averaging method yields precise bounds

Periodic normal form: $$\frac{d\vec{x}}{dt} = \epsilon \vec{f}(\vec{x}, t)$$

periodic in t with period 2π

Averaged version of $\vec{f}$: $$\vec{f}_{\text{avr}} = \frac{1}{2\pi} \int_0^{2\pi} dt\, \vec{f}(\vec{x}, t)$$

Averaged equation: $$\frac{d\vec{x}_{\text{avr}}}{dt} = \epsilon \vec{f}_{\text{avr}}(\vec{x}_{\text{avr}})$$

Accuracy theorem: $\forall\, T, \exists\, c, \epsilon_1$ s.t.

$$|\vec{x}(t) - \vec{x}_{\text{avr}}(t)| < c\epsilon \qquad \text{for} \quad \begin{cases} 0 < t < \dfrac{T}{\epsilon} \\ 0 < \epsilon < \epsilon_1 \end{cases}$$

→ Error of order ϵ on time interval of order $1/\epsilon$

Oscillatory system can be converted to periodic normal form

approximate as cubic

$$\boxed{\ddot{c}_j + \omega_j^2\, c_j = S_j(c)}$$

Hamiltonian form: $\dot{c}_j = \pi_j, \quad \dot{\pi}_j = -\omega_j^2\, c_j + S_j(c)$

Introduce new (complex) variables $\alpha_j(t)$:

$$\begin{cases} c_j = \epsilon(\alpha_j e^{-i\omega_j t} + \bar{\alpha}_j e^{i\omega_j t}) \\ \pi_j = -i\epsilon\omega_j(\alpha_j e^{-i\omega_j t} - \bar{\alpha}_j e^{i\omega_j t}) \end{cases}$$

$$\rightarrow \quad \boxed{\dot{\alpha}_j = \epsilon^2\, S_j(\alpha, \bar{\alpha}, t)}$$

→ Averaging will give reliable results on time interval of order $1/\epsilon^2$ (unless amplitude growth invalidates cubic approximation)

→ Same for multiscale and RG (equivalent at this order)

Many flow channels are closed

$$\phi(x,t) = \sum_{n=0}^{\infty} \left[\epsilon a_n \cos(\omega_n t + b_n) + \epsilon^3 c_n(t) + \ldots\right] e_n(x)$$

$$\ddot{c}_n + \omega_n^2 c_n = C_{ijkn} a_i a_j a_k \cos((\omega_i \pm \omega_j \pm \omega_k)t + (b_i \pm b_j \pm b_k)) + \ldots$$

Normal mode spectrum $\omega_n = d + 2n$ → resonances if

- $\cancel{\omega_i + \omega_j + \omega_k = \omega_n}$ (dynamically forbidden)
- $\boxed{\omega_i + \omega_j - \omega_k = \omega_n}$
- $\cancel{\omega_i - \omega_j - \omega_k = \omega_n}$ (dynamically forbidden)
- $\cancel{-\omega_i - \omega_j - \omega_k = \omega_n}$ (positivity)

[BC, Evnin, Vanhoof 2014; for lowest modes implicit in earlier work]

Flow equations conserve three charges

Introduce $\alpha_k = \frac{a_k}{2} e^{-ib_k}$ → flow equations: $\frac{d\alpha_j}{d\tau} = \frac{i\epsilon^2}{\omega_j} \frac{\partial W}{\partial \bar{\alpha}_j}$, with

$$W = \sum_i T_i |\alpha_i|^4 + \sum_{i,j}^{i\neq j} R^S_{ij} |\alpha_i|^2 |\alpha_j|^2 + \left(\sum_i \omega_i^2 |\alpha_i|^2\right)\left(\sum_j (A_{jj} + \omega_j^2 V_{jj}) |\alpha_j|^2\right) + \sum_{\substack{i,j,k,l \\ \{i,j\}\cap\{k,l\}=\emptyset \\ \omega_i+\omega_j=\omega_k+\omega_l}} S_{ijkl} \alpha_i \alpha_j \bar{\alpha}_k \bar{\alpha}_l$$

(coefficients are specific complicated integrals of AdS mode functions)

Can be obtained from $L = \sum_k i\omega_k \left(\bar{\alpha}_k \frac{d\alpha_k}{d\tau} - \alpha_k \frac{d\bar{\alpha}_k}{d\tau}\right) + 2\epsilon^2 W$

(related to original scalar field Lagrangian by averaging)

Symmetries of this Lagrangian lead to 3 conserved charges

$\alpha_n \mapsto e^{i\omega_n\theta}\alpha_n$ → $E = \sum_n \omega_n^2 |\alpha_n|^2$ observed previously by [Balasubramanian, Buchel, Green, Lehner, Liebling 2014]

$\alpha_n \mapsto e^{i\theta}\alpha_n$ → $J = \sum_n \omega_n |\alpha_n|^2$ (closed flow channels crucial!)

$\tau \mapsto \tau + \tau_0$ → W (quartic “interaction energy”)

[BC, Evnin, Vanhoof 2015; cf. Basu, Krishnan, Saurabh 2014 for probe scalar field]

E and J conservation implies dual cascades

$$J = \sum_n \omega_n |\alpha_n|^2 \equiv \sum_n N_n$$

"particle number"

$$E = \sum_n \omega_n^2 |\alpha_n|^2 = \sum_n \omega_n N_n$$

"free motion energy"

Transferring all energy to higher-n modes (which have more energy per particle) would decrease J

→ some of the energy must flow to lower-n modes!

This was indeed observed in [Balasubramanian, Buchel, Green, Lehner, Liebling 2014]

[Buchel, Green, Lehner, Liebling 2015]

Deeper reason for closed flow channels?

- Channels are open in other models:
  - ➢ spherical cavity in 4d Minkowksi space with D boundary conditions

    [Maliboriski 2012]
  - ➢ hard wall model in AdS_4 with N boundary conditions

    [BC, Lindgren, Taliotis, Vanhoof, Zhang 2014]

- Toy model: probe self-interacting scalar

  [Basu, Krishnan, Saurabh 2014]
  - ➢ selection rules beyond spherical symmetry [Yang 2015]

- Hidden SU(d) symmetry of AdS_{d+1} mode functions

  [Evnin, Krishnan 2015]

Holographic thermalization in finite volume

- entanglement entropy oscillations

  [Abajo-Arrastia, da Silva, Lopez, Mas, Serantes 2014]

- revivals of the initial state

  [da Silva, Lopez, Mas, Serantes 2014]

- pre-thermalization: small BH as intermediate state

  [Dimitrakopoulos, Freivogel, Lippert, Yang 2014]

- cf. thermalization (or not!) in infinite-volume hard wall model

  [BC, Kiritsis, Rosen, Taliotis, Vanhoof, Zhang 2014;
  BC, Lindgren, Taliotis, Vanhoof, Zhang 2014]

Conclusions

- We have introduced a fully analytic framework for the study of AdS (in)stability.
- Resummation of secular terms leads to flow equations.
- The first-order flow equations are reliable on the $1/\epsilon^2$ time scale set by gravitational interactions, unless amplitude growth makes higher order corrections important.
- Many flow channels are closed, and the flow equations exhibit three conservation laws which restrict resonant energy transfer.
- Deeper (symmetry) reason for conservation laws?
- Quantitative study of short-wavelength regime?