Entanglement entropy in a holographic model of the Kondo effect

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Overview

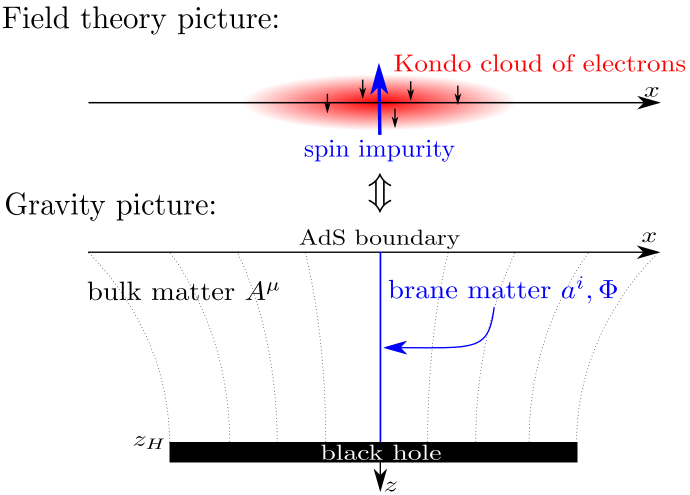
- Part I: The holographic Kondo model
  - The Kondo effect
  - Bottom up bulk model
- Part II: Including backreaction
  - Israel junction conditions
  - General results for $AdS_3/BCFT_2$
  - Including Chern-Simons fields
- Part III: Entanglement entropy for Kondo model
  - Numerical results
  - Qualitative discussion

Part I: The holographic Kondo model

- Field theory side:
  - Spin-spin interaction of electrons with a localised magnetic impurity.
  - Can be mapped to $1+1$ dimensional conformal system [Affleck et. al. 1991].
  - At low temperature, electrons form a bound state around impurity, the *Kondo cloud*.

- Holographic gravity side: [Erdmenger et. al.: 1310.3271]
  - Dual gravity model has $2+1$ (bulk-) dimensions.
  - Localised spin impurity is represented by co-dimension one hypersurface ("brane") extending from boundary into the bulk.
  - Finite T is implemented by BTZ black hole background.
  - Kondo cloud is described by condensation of scalar field Φ.

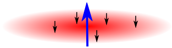
The holographic Kondo model



$$S = S_{CS}[A] - \int d^3x \delta(x) \sqrt{-g} \left(\frac{1}{4} f^{mn} f_{mn} + \gamma^{mn} (D_m\Phi)^\dagger D_n\Phi + V(\Phi^\dagger\Phi) \right)$$

The holographic Kondo model

How can we obtain information about the Kondo cloud from our model?

- Kondo cloud is formed by anti-aligned spins 

- $\Rightarrow$ expect imprint on *entanglement entropy* $\mathcal{S}_{EE}$, e.g. entanglement of state $|\Psi\rangle = \frac{1}{\mathcal{N}}\left(|\uparrow\downarrow\downarrow\ldots\rangle - |\downarrow\uparrow\uparrow\ldots\rangle\right)$ does not vanish.

- $\mathcal{S}_{EE}$ is determined by spacelike geodesics [Ryu, Takayanagi, 2006]
$\Rightarrow$ to calculate it, we need *backreaction* on the geometry.

- What is the backreaction of an infinitely thin hypersurface carrying energy-momentum? *Israel junction conditions!*

Part II: Including backreaction

In electromagnetism: To describe field around an infinitely thin charged surface Σ, integrate Maxwells equations in a box around Σ:

$$\Rightarrow \vec{E}_{||} \text{ continuous, } \ \vec{E}_{\perp} \text{ discontinuous on } \Sigma$$

In gravity: To describe backreaction of an infinitely thin massive surface, integrate Einsteins equations in a box
$\Rightarrow$ *Israel junction conditions* [Israel, 1966]:

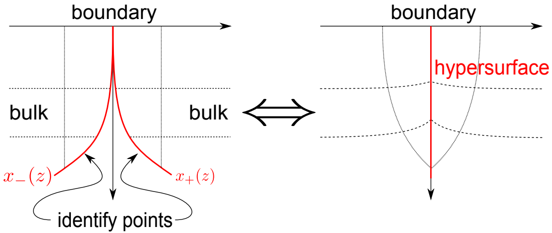
$$(K^+_{ij} - \gamma_{ij} K^+) - (K^-_{ij} - \gamma_{ij} K^-) = -\kappa\, S_{ij}$$

S_{ij}: energy momentum tensor on the brane, γ_{ij}: induced metric,
$K^{\pm}$: extrinsic curvatures depending on embedding.

Israel junction conditions

With mirror symmetry ($K^+ = -K^-$): $$K^+_{ij} - \gamma_{ij} K^+ = -\tfrac{\kappa}{2} S_{ij} \qquad (*)$$

$\Rightarrow$ Embedding (location of the brane) will not be $x \equiv 0$ anymore, but a dynamical function $x(z)$ with $(*)$ its own equations of motion.



With $(*)$ we arrive at a *general* setting for the study of AdS/boundary CFT correspondence proposed by Takayanagi et. al.: [Takayanagi 2011, Fujita et. al. 2011, Nozaki et. al. 2012].

Israel junction conditions

$$K^+_{ij} - \gamma_{ij} K^+ = -\frac{\kappa}{2} S_{ij}$$

curvature = energy momentum

Geometric equations of a similar form as Einstein equations, *extrinsic* curvature tensors (K^+_{ij}) instead of *intrinsic* ones ($R_{\mu\nu}$).

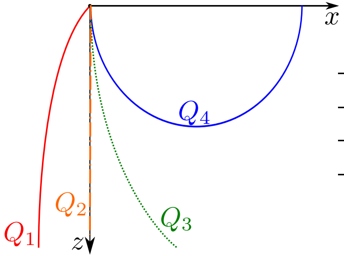
General questions:

- Impact of energy conditions on possible geometries?
- Find exact solutions for simple toy models of S_{ij}?
- Investigate Kondo model?

Answers in [Erdmenger, M.F., Newrzella: 1410.7811].

Possible geometries

Utilising the *barrier theorem* [Engelhardt, Wall: 1312.3699], we can constrain the possible geometries allowed by different energy conditions.

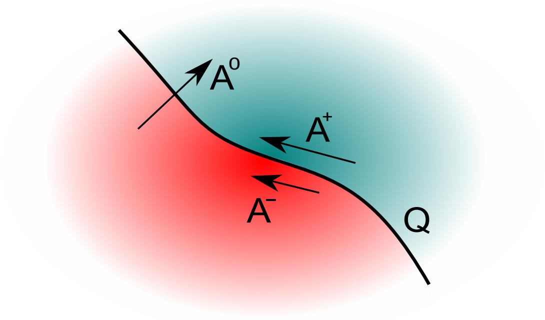


	NEC	WEC	SEC	comment
Q_1	yes	yes	no	
Q_2	yes	yes	yes	$S_{ij} = 0$
Q_3	yes	no	yes	
Q_4	yes	yes	yes	$\cup$ shaped

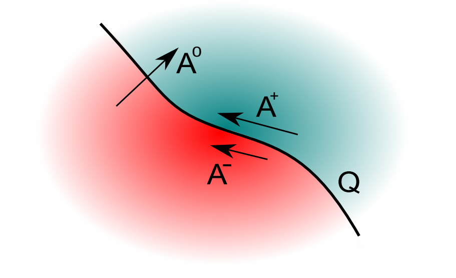
Whether or not a brane Q bends back to the boundary or goes deep into the bulk depends on whether S_{ij} satisfies or violates WEC and SEC.

Junction conditions for Chern-Simons field

- Our Kondo model contains both the metric field and a *Chern-Simons field* in the bulk. Assume CS field to be $U(1)$ in simplest case.
- Similarly to the metric, we get junctions conditions for the CS field along the hypersurface Q (located at $\eta \equiv 0$) if it carries a current in its worldvolume.
- Split up field: $A \sim \theta(\eta)A^{+} + \theta(-\eta)A^{-} + \delta(\eta)A^{0}$.



Junction conditions for Chern-Simons field



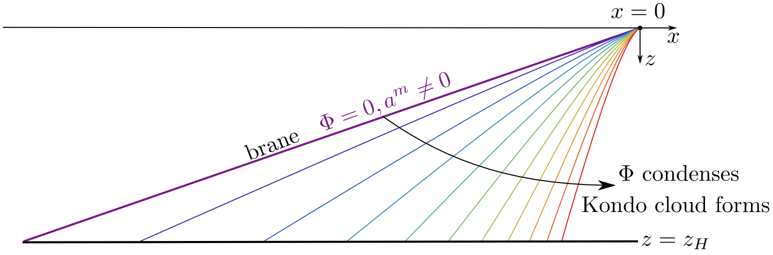
With	$D_m \equiv (A_m^{+\|\|} + A_m^{-\|\|})/2$	(projected mean value),
	$C_m \equiv A_m^{+\|\|} - A_m^{-\|\|}$	(projected discontinuity),
and	$A^0_\mu = A^0 n_\mu$	(component localised on Q is normal)

we find: $\epsilon^{im}\left(C_i + \partial_i A^0\right) = 2\pi J^m\left[\gamma, \Phi, a, D\right]$

Part III: Entanglement entropy for Kondo model

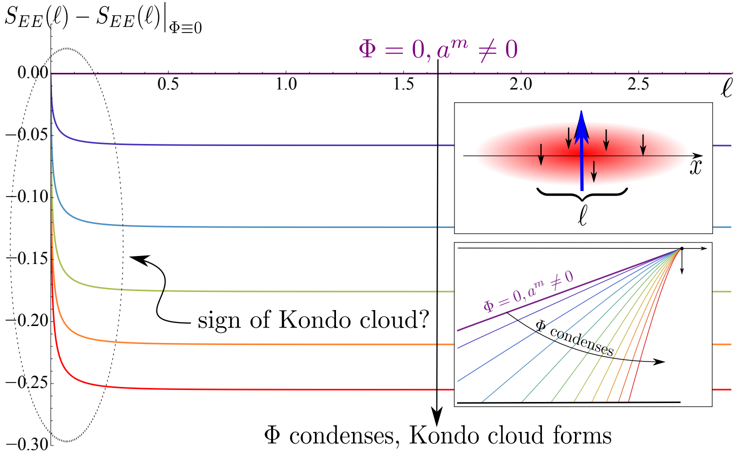
$$S_{brane}[a^m, \Phi] = -\int dV_{brane} \left(\frac{1}{4} f^{mn} f_{mn} + \gamma^{mn} (D_m \Phi)^\dagger D_n \Phi + V(\Phi^\dagger \Phi)\right)$$

- Due to Yang-Mills field a^m, SEC is violated everywhere in the bulk.
- Hence brane starts at boundary and falls into black hole, does *not* turn around and bend back to boundary.
- Preliminary numerical results:



Numerical results

Preliminary results on entanglement entropy: Difference of $S_{EE}(\ell)$ relative to solution with $\Phi \equiv 0$.



[Erdmenger, M.F., Hoyos, Newrzella, O'Bannon, Wu: work in progress]

Discussion

Some of the features of these results follow directly from the energy conditions and geometric considerations.

- Entanglement entropy for given ℓ *decreases* as Kondo cloud forms, because Φ satisfies NEC, brane bends to the right.
- As $\ell \to \infty$, curves go to a *constant*.
- The fall-off towards this constant value is for large ℓ *exponential*, due to simple geometric arguments:

$$\Delta S_{EE}(\ell) \xrightarrow{\sim} c_0 + c_1(T)T\left(1 + 2e^{-4\pi\ell T} + ...\right)$$

Qualitative agreement with results of field theory calculations [Affleck et. al. 2007, 2009; Eriksson, Johannesson 2011]:

$$\Delta S_{EE}(\ell) = \tilde{c}_0 + \frac{\pi^2 \xi_K T}{6v} \coth\left(\frac{2\pi\ell T}{v}\right) \to \tilde{c}_0 + \frac{\pi^2 \xi_K T}{6v}\left(1 + 2e^{-\frac{4\pi\ell T}{v}} + ...\right)$$

v: Fermi velocity, ξ_K: Kondo scale

Summary and Outlook

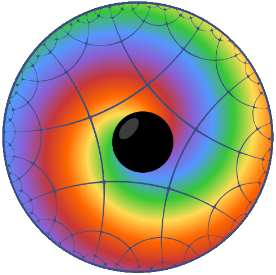
- We studied a holographic model of the Kondo effect.
- Gravity dual involves thin brane carrying energy-momentum.
- Backreaction of the brane is described by Israel junction conditions.
- We obtained general results constraining possible geometries of the brane by energy conditions [Erdmenger et. al. 1410.7811].
- These results may also be applicable to holographic duals of BCFTs [Takayanagi, 2011] or the Hall effect [Melnikov et. al, 2012] involving thin branes.
- Specific Kondo model will be solved numerically, results on entanglement entropy can be compared to field theory literature.

Thank you for your attention

Munich 2013

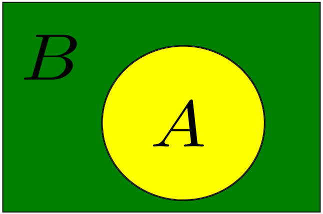
Florence 2015

Back up slides...



Entanglement entropy

Entanglement entropy $\mathcal{S}_{EE}(A)$ defines the entropy of a subsystem A with respect to the total system $A \cup B$.



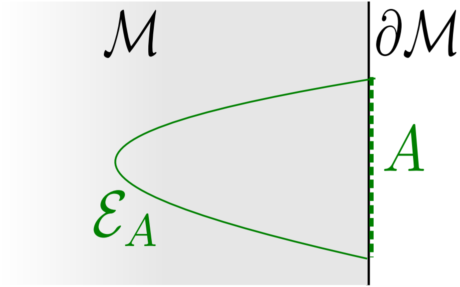
$$\mathcal{S}_{EE}(A) = -\mathrm{Tr}_A[\rho_A \log(\rho_A)]$$

with reduced density matrix $\rho_A \equiv \mathrm{Tr}_B[\rho_{A\cup B}]$.

[see e.g. Nielsen, Chuang: *Quantum Computation and Quantum Information*]

Holographic entanglement entropy

In AdS/CFT correspondence: bulk spacetime $\mathcal{M}$, boundary $\partial\mathcal{M}$.



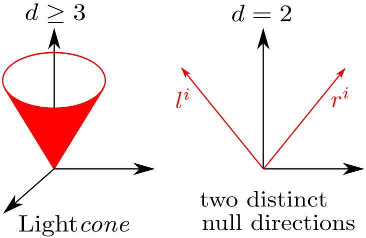
$\mathcal{S}_{EE}(A) = \frac{\text{Area}(\mathcal{E}_A)}{4G_N}$ where $\mathcal{E}_A$ is a spacelike extremal surface in the bulk.

$\rightarrow$ Generalisation of Bekenstein-Hawking entropy formula

[Ryu, Takayanagi, 2006]

A simple form

Our brane has $1+1$ dimensions, hence there are only two distinct null directions.



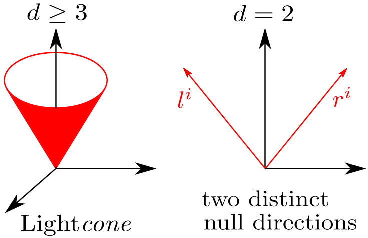
$l^i l_i = 0 = r^i r_i$
$l^i r_i = -1$

Define a basis of symmetric (0,2)-tensors:

$$S_{ij} \equiv \frac{S}{2}\gamma_{ij} + S_L l_i l_j + S_R r_i r_j = \text{trace part} + \text{traceless parts}$$

Static case: no energy flux from left to right, hence $S_L = S_R \equiv S_{L/R}$.

A simple form



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Doing this decomposition on both sides, the tensorial equation

$$K^+_{ij} - \gamma_{ij} K^+ = -\tfrac{\kappa}{2}\, S_{ij}$$

becomes the set of scalar equations

$$\mathcal{K} = \tfrac{\kappa}{2} S, \quad \mathcal{K}_L = \tfrac{\kappa}{2} S_L, \quad \mathcal{K}_R = \tfrac{\kappa}{2} S_R$$

Energy conditions

Null energy condition (NEC)

$$S_{ij} m^i m^j \geq 0 \;\; \forall \; m^i m_i = 0 \;\Rightarrow\; S_L, S_R \geq 0$$

Weak energy condition (WEC)

$$S_{ij} m^i m^j \geq 0 \;\; \forall \; m^i m_i < 0 \Rightarrow S_L, S_R \geq 0, \quad S \leq 2\sqrt{S_L S_R}$$

Strong energy condition (SEC)

$$(S_{ij} - S\gamma_{ij}) m^i m^j \geq 0 \;\; \forall \; m^i m_i < 0 \;\Rightarrow\; S_L, S_R \geq 0, \quad S \geq -2\sqrt{S_L S_R}$$

This SEC will be of much phenomenological importance.

Energy conditions and conservation of energy momentum

In the static case ($S_L = S_R \equiv S_{L/R}$), SEC reads:

$$S_L, S_R \geq 0, \quad S + 2S_{L/R} \geq 0.$$

Energy-momentum conservation $\nabla_i S^{ij} = 0$ implies for embeddings in Poincaré background:

$$\partial_z \left(S + 2S_{L/R} \right) = \frac{4}{z} S_{L/R}.$$

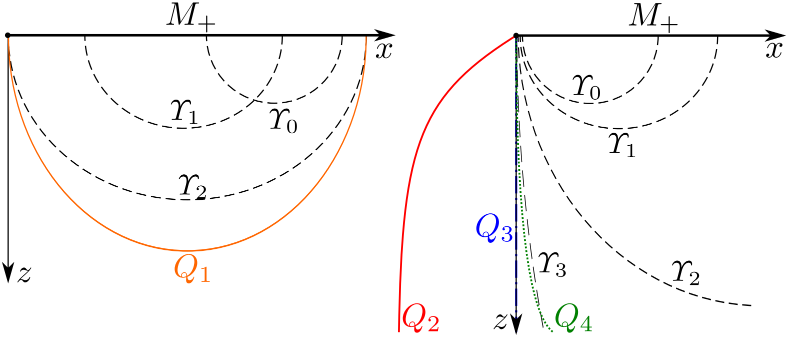
By NEC, the right hand side is positive, hence $S + 2S_{L/R}$ can only grow with z.

When NEC holds and the SEC is satisfied near the boundary $z = 0$, it is satisfied everywhere in the bulk.

Possible geometries

Barrier Theorem (Engelhardt, Wall arXiv:1312.3699)

Let Q be a hypersurface splitting the spacetime N in two parts $N_\pm$ with boundaries $M_\pm$ such that $K^+_{ij} v^i v^j \leq 0$ for any vector field v^i on Q. Then any spacelike extremal surface Υ which is anchored in M_+ remains in N_+.

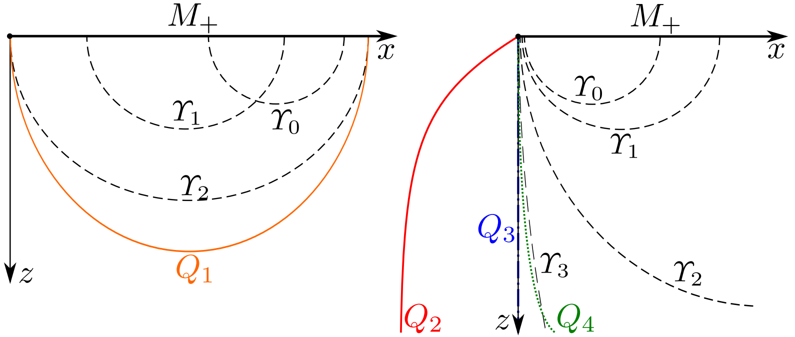


$K^+_{ij} v^i v^j \leq 0$ for Q_1, Q_3. We call Q_1, Q_2, Q_3 *extremal surface barriers.*

Possible geometries

With the junction conditions, we can express the assumption made in the barrier theorem in terms of energy conditions:

$$\text{WEC and SEC satisfied on } Q \;\Rightarrow K^{+}_{ij} v^i v^j \leq 0 \;\forall v^i$$



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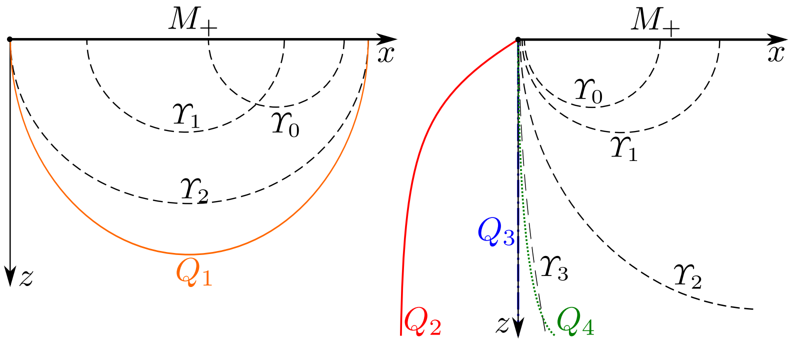
Q_2 violates SEC, Q_4 violates WEC. For Q_3, $S_{ij} = 0$.

Possible geometries



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Q_2 violates SEC, Q_4 violates WEC. For Q_3, $S_{ij} = 0$.

Whether or not a brane Q bends back to the boundary or goes deep into the bulk depends on whether S_{ij} satisfies or violates WEC and SEC.

Exact analytical solutions

We first studied simple models for S_{ij} and obtained some exact analytical solutions to the junction conditions for:

- Perfect fluids:

$$S_{ij} = (\rho + p)u_i u_j + p\gamma_{ij} \text{ with } p = a \cdot \rho,\ a \in \mathbb{R}.$$

- As the special case thereof with $a = 1$: The free massless scalar ϕ with

$$S_{ij} = \partial_i\phi\partial_j\phi - \frac{1}{2}\gamma_{ij}(\partial\phi)^2.$$

- The $U(1)$ Yang-Mills field a_i in the absence of sources:

$$S_{ij} = -\frac{1}{4}f^{mn}f_{mn}\gamma_{ij} + \gamma^{mn}f_{mi}f_{nj} = -\frac{1}{2}\gamma_{ij}C^2.$$

All of these were studied in AdS and BTZ backgrounds.

Exact analytical solutions

For the free massless scalar ϕ with $S_{ij} = \partial_i \phi \partial_j \phi - \frac{1}{2}\gamma_{ij}(\partial\phi)^2$, we obtain

$$x(z) = \frac{cz^3}{3} {}_2F_1\left(\frac{1}{2}, \frac{3}{4}; \frac{7}{4}; c^2 z^4\right).$$

with ${}_2F_1(a, b; c; d)$ the hypergeometric function. WEC and SEC are satisfied, hence the brane bends back to the boundary.