All Order Linearized Hydrodynamics from Fluid/Gravity Correspondence

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based on **Yanyan Bu and M.L., arXiv:1406.7222 (PRD), arXiv:1409.3095 (JHEP), arXiv:1502.08044 (JHEP)**
Yanyan Bu, M.L., Amir Sharon, arXiv:1504.01370

following some older works with Edward Shuryak

E. Shuryak, M.L., Phys.Rev.C76 (2007) 021901, D80 (2009) 065026, C84 (2011) 061901:
Generalize NS hydro by introducing ALL order dissipative terms in the gradient expansion of fluid stress tensor

$(\nabla\nabla\mathbf{u})$ we keep $\qquad$ $(\nabla\mathbf{u})^2$ we neglect

Extract momenta-dependent viscosity function $\eta(\omega, \mathbf{q})$ by matching two-point correlation functions of the stress tensor with the correlation functions computed from BH AdS/CFT
We postulated a problem, but at the time failed to solve it completely.
(We have done it now!)
We performed some phenomenological studies of the effects of all-order gradients on entropy/multiplicity production in HI collisions

Motivation: Experiments (RHIC,LHC) probe systems with finite gradients.
Phenomenologically observed low viscosity is an "effective" viscosity measured at momentum typical for process in study.
High order gradients are very big in early stages of HI collisions
Small perturbations/correlations on top of global explosion are sensitive to high gradients. This is where our results are most applicable
Relativistic Navier-Stokes hydrodynamics is non-causal/non-stable.
Causality is supposed to be restored after summation of all orders

Relativistic Hydrodynamics

Energy momentum tensor

$$\langle \mathbf{T}^{\mu\nu} \rangle = (\epsilon + \mathbf{P})\, \mathbf{u}^{\mu}\, \mathbf{u}^{\nu} + \mathbf{P}\, \mathbf{g}^{\mu\nu} + \mathbf{\Pi}^{\mu\nu}$$

$$\mathbf{u_v} = -1/\sqrt{1-\beta^2}, \qquad \mathbf{u_i} = \beta_\mathbf{i}/\sqrt{1-\beta^2}$$

$\mathbf{\Pi}^{\mu\nu}$ – tensor of dissipations (ideal fluid: $\mathbf{\Pi}^{\mu\nu} = 0$)

Landau frame choice: $\mathbf{u}_\mu\, \mathbf{\Pi}^{\mu\nu} = \mathbf{0}.$

Navier Stokes hydro (expanding up to first order in the velocity gradient)

$$\mathbf{\Pi_{ij}} = -\,\eta_0\, \sigma_\mathbf{ij}\,, \qquad \sigma_\mathbf{ij} = \frac{1}{2}\left(\partial_\mathbf{i}\beta_\mathbf{j} + \partial_\mathbf{j}\beta_\mathbf{i} - \frac{2}{3}\delta_\mathbf{ij}\partial\beta\right), \qquad \mathbf{\Pi_{vv}} = \mathbf{\Pi_{vj}} = \mathbf{0}.$$

$$\nabla_\mu \langle \mathbf{T}^{\mu\nu} \rangle = \mathbf{0} \qquad \longrightarrow \qquad \text{Navier} - \text{Stokes Eqns.}$$

Linearized Hydro to all orders

Shuryak and M. L.: Introduce all order gradient expansion of $\langle T^{\mu\nu} \rangle$:

$$\Pi_{ij} = -\eta[\nabla^2, (u\nabla)]\,\sigma_{ij} - \zeta[\nabla^2, (u\nabla)]\,\pi_{ij}$$

where π_{ij} is a third order tensor structure

$$\pi_{ij} = \partial_i \partial_j \partial \beta - \frac{1}{3}\delta_{ij}\partial^2 \partial \beta$$

In Fourier space: $\nabla^2 \to \omega^2 - q^2$ and $(u\,\nabla) \to -i\,\omega$.

$$\eta \to \eta(\omega, q^2)\,; \qquad \zeta \to \zeta(\omega, q^2)$$

We keep the nonlinear dispersion to all orders, but

We neglect nonlinear interactions (though some terms could be recovered).

Fluids in a weakly curved 4d background

Most general constitutive relation with weakly ($h_{\mu\nu} \sim u_i$) curved metric

$$\mathbf{\Pi}_{\mu\nu} = -\eta \nabla_\mu \mathbf{u}_\nu - \zeta \nabla_\mu \nabla_\nu \nabla \mathbf{u} + \kappa \mathbf{u}^\alpha \mathbf{u}^\beta \mathbf{C}_{\mu\alpha\nu\beta} + \rho \mathbf{u}^\alpha \nabla^\beta \mathbf{C}_{\mu\alpha\nu\beta} + \xi \nabla^\alpha \nabla^\beta \mathbf{C}_{\mu\alpha\nu\beta} - \theta \mathbf{u}^\alpha \nabla_\alpha \mathbf{R}_{\mu\nu},$$

$C_{\mu\alpha\nu\beta}$, $R_{\mu\nu}$ **are the Weyl and Ricci tensors of** $h_{\mu\nu}$**.**

$\kappa(\omega, q^2)$, $\rho(\omega, q^2)$, $\xi(\omega, q^2)$, $\theta(\omega, q^2)$ **are Gravitational Susceptibilities of the Fluid.**

All GSFs contribute to two-point correlators

GSFs carry info about zero temperature limit (pair production)

Results from the Fluid/Gravity correspondence

Analytical results in the hydrodynamic regime $\omega, \mathbf{q} \ll 1$ $(\pi\,\mathbf{T} = 1)$:

$$\eta(\omega, \mathbf{q}^2) = \mathbf{2} + (\mathbf{2} - \ln \mathbf{2})\mathbf{i}\omega - \frac{\mathbf{1}}{\mathbf{4}}\mathbf{q}^2 - \frac{\mathbf{1}}{\mathbf{24}}\left[\mathbf{6}\pi - \pi^2 + \mathbf{12}\left(\mathbf{2} - \mathbf{3}\ln\mathbf{2} + \ln^2\mathbf{2}\right)\right]\omega^2 + \cdots$$

$$\zeta(\omega, \mathbf{q}^2) = \frac{\mathbf{1}}{\mathbf{12}}(\mathbf{5} - \pi - \mathbf{2}\ln\mathbf{2}) + \cdots$$

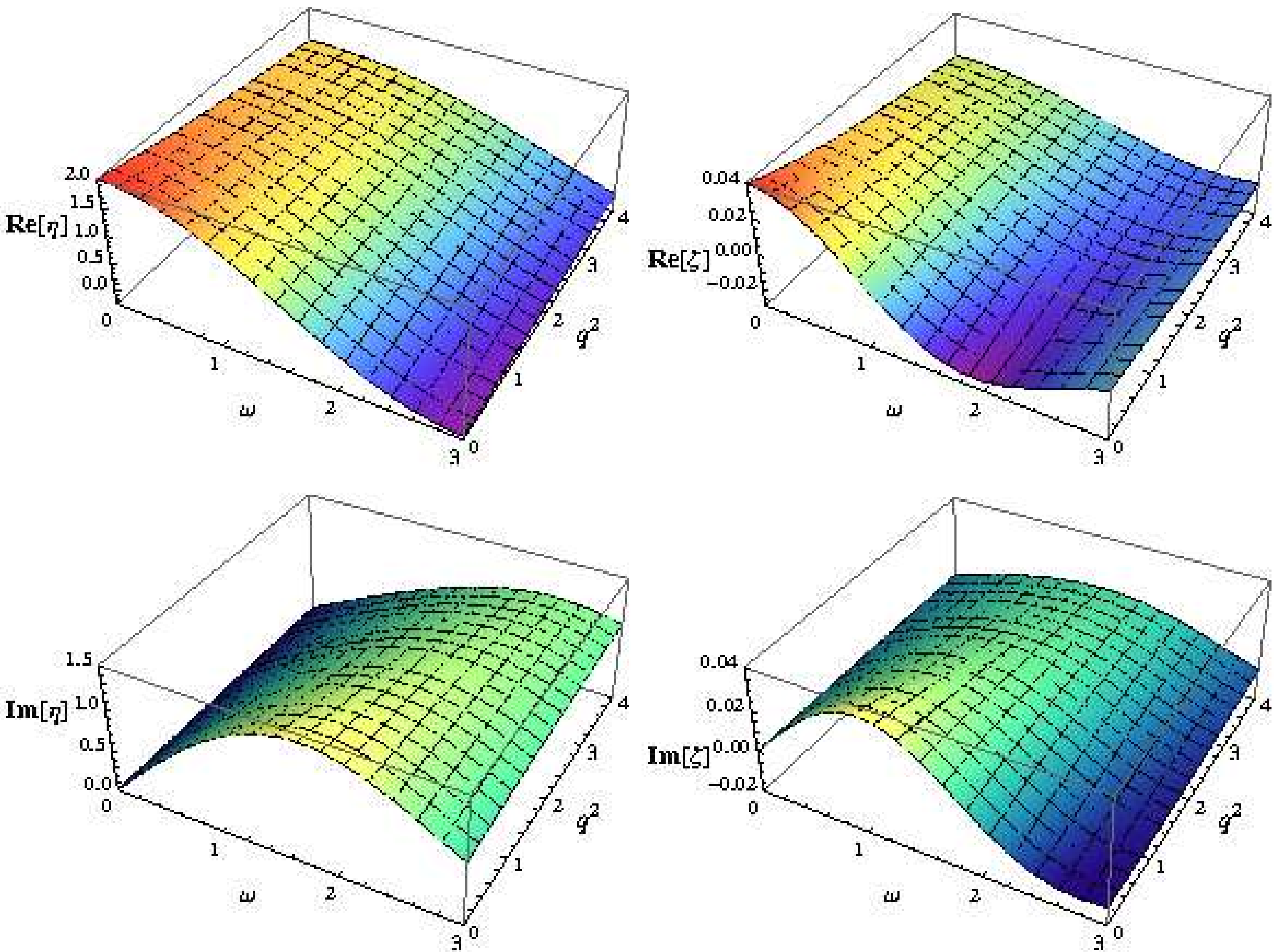
Blue terms are new!

$$\kappa = 2 + \frac{1}{4}(5 + \pi - 6\log 2)\, i\omega + \cdots, \quad \rho = 2 + \cdots, \quad \xi = \ln 2 - \frac{1}{2} + \cdots, \quad \theta = \frac{3}{2}\zeta.$$

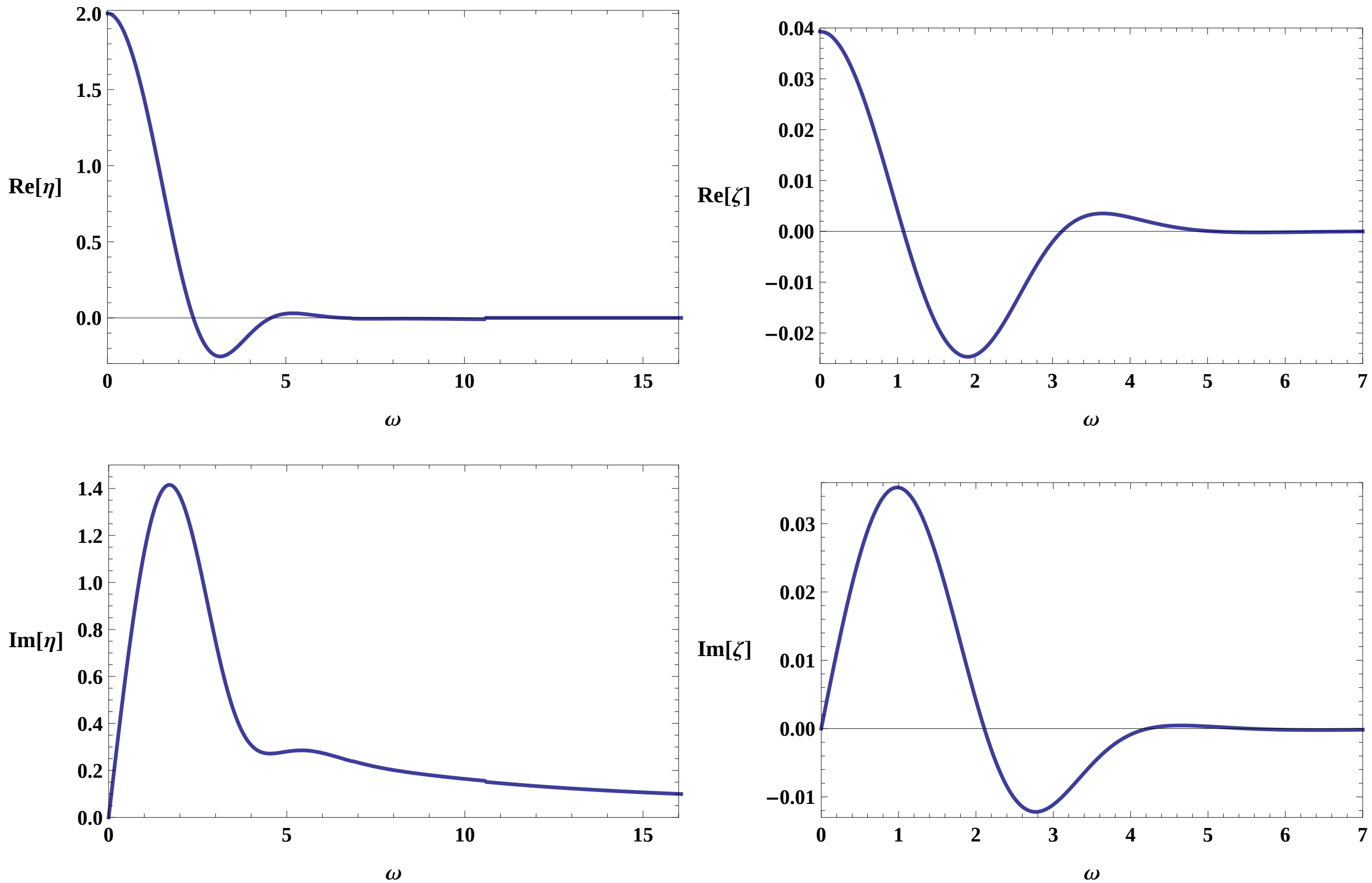
R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, JHEP 0804, 100 (2008)

Modified sound dispersion:

$$\omega = \pm\frac{1}{\sqrt{3}}q - \frac{i}{6}q^2 \pm \frac{1}{24\sqrt{3}}(3 - 2\ln 2)\, q^3 + \frac{i}{288}\left(8 - \frac{\pi^2}{3} + 4\ln^2 2 - 4\ln 2\right) q^4 +$$



Re[η]
2.0
1.5
1.0
0.5
0.0
0
1
2
3
ω
0
1
2
3
4
q²
Re[ζ]
0.04
0.02
0.00
−0.02
Im[η]
1.5
1.0
0.5
0.0
Im[ζ]
0.04
0.02
0.00
−0.02



$q^2 = 0$

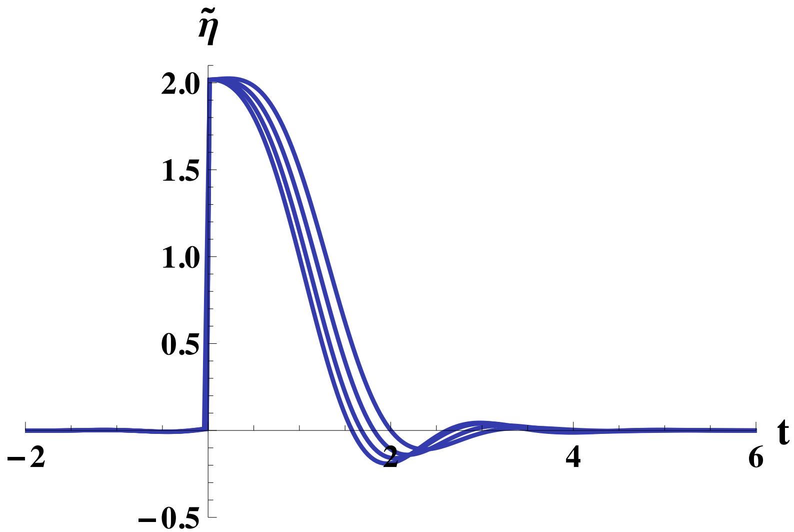
- **Real parts of the viscosities are decreasing functions of momenta. Oscillations are consistent with the expectations about the viscosities have infinitely many complex poles.**
- **Imaginary parts have a clear maximum near $\omega \sim 2$, introducing a (new?) transition scale.**
- **Viscosity vanish at large momenta, which is what is required to restore causality.**

- **ζ is always subleading vs η.**

Memory function

$$\mathbf{\Pi}_{\mu\nu}(\mathbf{t}) = -\int_{-\infty}^{\infty} \mathbf{dt}' \left[\mathbf{2}\tilde{\eta}(\mathbf{t}-\mathbf{t}', \mathbf{q}^2)\partial_\mu \mathbf{u}_\nu(\mathbf{t}') + \tilde{\zeta}(\mathbf{t}-\mathbf{t}', \mathbf{q}^2)\partial_\mu\partial_\nu\partial^\alpha \mathbf{u}_\alpha(\mathbf{t}') \right]$$

$$\tilde{\eta}(\mathbf{t}, \mathbf{q}^2) \equiv \int_{-\infty}^{\infty} \frac{\mathbf{d}\omega}{\sqrt{\mathbf{2}\pi}} \eta(\omega, \mathbf{q}^2) \mathbf{e}^{-\mathbf{i}\omega\mathbf{t}},$$

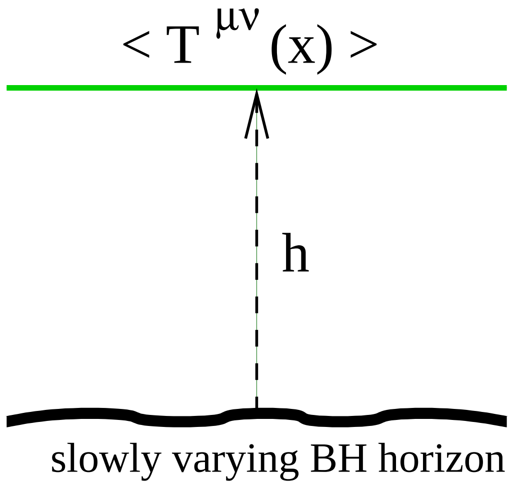


Causality: $\tilde{\eta}(\mathbf{t}-\mathbf{t}') \sim \theta(\mathbf{t}-\mathbf{t}')$

$$\mathbf{\Pi}_{\mu\nu}(\mathbf{t}) = \int_{-\infty}^{\mathbf{t}} \mathbf{dt}' \left[\tilde{\eta}(\mathbf{t}-\mathbf{t}', \mathbf{q}^2)\partial_\mu \mathbf{u}_\nu(\mathbf{t}') + \tilde{\zeta}(\mathbf{t}-\mathbf{t}', \mathbf{q}^2)\partial_\mu\partial_\nu\partial^\alpha \mathbf{u}_\alpha(\mathbf{t}') \right]$$

"Gravity", starring S. Bhattacharyya, V. E Hubeny, S. Minwalla, M. Rangamani

5d GR with negative cosmological constant:



$$\mathbf{S} = \frac{\mathbf{1}}{\mathbf{16}\pi \mathbf{G_N}} \int \mathbf{d^5x} \sqrt{-\mathbf{g}} \, (\mathbf{R} + \mathbf{12}) \, ,$$

Einstein Equations

$$\mathbf{E_{MN}} \equiv \mathbf{R_{MN}} - \frac{\mathbf{1}}{\mathbf{2}} \mathbf{g_{MN}} \, \mathbf{R} - \mathbf{6} \, \mathbf{g_{MN}} = \mathbf{0}.$$

Solution: Boosted Black Brane in asymptotic AdS$_5$

$$\mathbf{ds^2} = -\mathbf{2u_\mu dx^\mu dr} - \mathbf{r^2 f\,(br)\, u_\mu u_\nu dx^\mu dx^\nu} + \mathbf{r^2} \mathcal{P}_{\mu\nu} \mathbf{dx^\mu dx^\nu},$$

$$\mathbf{f(r) = 1 - 1/r^4} \quad \text{and} \quad \mathcal{P}_{\mu\nu} = \eta_{\mu\nu} + \mathbf{u_\mu u_\nu}$$

Hawking temperature

$$\mathbf{T} = \frac{\mathbf{1}}{\pi \mathbf{b}},$$

S. Bhattacharyya, V. E Hubeny, S. Minwalla, M. Rangamani, JHEP 0802:045,2008:

Promote β_i and b into a slowly varying functions of boundary coordinates x^α

$$\mathbf{ds^2 = -2u_\mu(x^\alpha)dx^\mu dr - r^2 f\,(b(x^\alpha)r)\,u_\mu(x^\alpha)u_\nu(x^\alpha)dx^\mu dx^\nu + r^2}\mathcal{P}_{\mu\nu}\mathbf{(x^\alpha)dx^\mu dx^\nu},$$

Use gradient expansion of the fields $\mathbf{u(x)\ =\ u_0 + \delta x\,\nabla u}$ and $\mathbf{b(x) = b + \delta x \nabla b}$ to set up a perturbative procedure

The resulting energy momentum tensor

$$\langle \mathbf{T}^{\mu\nu} \rangle \ = \ \mathbf{T}^{\mu\nu}_{\mathbf{ideal}} \ + \ \mathbf{\Pi}^{\mu\nu}_{\mathbf{NS}} \ + \ \tau_{\mathbf{R}}\,(\mathbf{u}\,\nabla)\,\mathbf{\Pi}^{\mu\nu}_{\mathbf{NS}} \ + \ \mathbf{O}[\ (\nabla\ \mathbf{u})^{\mathbf{2}}\ ]$$

$$\frac{\eta_0}{\mathbf{s}} \ = \ \frac{\mathbf{1}}{\mathbf{4}\,\pi}, \qquad\qquad \tau_{\mathbf{R}} \ = \ \mathbf{2} \ - \ \log(\mathbf{2})$$

also obtained in other refs

We do it somewhat differently, linearizing in the velocity amplitude

$$\mathbf{u}_\mu(\mathbf{x}^\alpha) = (-\mathbf{1}, \epsilon\beta_{\mathrm{i}}(\mathbf{x}^\alpha)) + \mathcal{O}(\epsilon^2), \quad \mathbf{b}(\mathbf{x}^\alpha) = \mathbf{b_0} + \epsilon\mathbf{b_1}(\mathbf{x}^\alpha) + \mathcal{O}(\epsilon^2),$$

"seed" metric, i.e., a linearized version of the BH metric

$$\mathbf{ds}^2_{\mathrm{seed}} = \mathbf{2drdv} - \mathbf{r^2f(r)dv^2} + \mathbf{r^2d\vec{x}^2} - \epsilon\left[\mathbf{2}\beta_{\mathrm{i}}(\mathbf{x}^\alpha)\mathbf{drdx^i} + \frac{\mathbf{2}}{\mathbf{r^2}}\beta_{\mathrm{i}}(\mathbf{x}^\alpha)\mathbf{dvdx^i} + \frac{\mathbf{4}}{\mathbf{r^2}}\mathbf{b_1}(\mathbf{x}^\alpha)\mathbf{dv^2}\right] + \mathcal{O}(\epsilon^2),$$

$$\mathbf{ds^2} = \mathbf{ds}^2_{\mathrm{seed}} + \mathbf{ds}^2_{\mathrm{corr}}[\beta] \qquad \text{gauge fix} \quad \mathbf{g_{rr}} = \mathbf{0}, \quad \mathbf{g}_{\mathbf{r}\mu} \propto \mathbf{u}_\mu$$

$$\mathbf{ds}^2_{\mathrm{corr}} = \epsilon\left(-\mathbf{3h\,drdv} + \frac{\mathbf{k}}{\mathbf{r^2}}\mathbf{dv^2} + \mathbf{r^2\,h\,d\vec{x}^2} + \frac{\mathbf{2}}{\mathbf{r^2}}\mathbf{j_i\,dvdx^i} + \mathbf{r^2}\,\alpha_{\mathbf{ij}}\,\mathbf{dx^idx^j}\right)$$

$\mathbf{h}[\beta]$, $\mathbf{k}[\beta]$, $\mathbf{j}[\beta]$, $\alpha[\beta]$ are to be found by solving the Einstein equations.

Boundary cond: no singularities, no modification to AdS asymptotics at $r \to \infty$

$$\mathbf{h} < \mathcal{O}(\mathbf{r^0}), \quad \mathbf{k} < \mathcal{O}(\mathbf{r^4}), \quad \mathbf{j_i} < \mathcal{O}(\mathbf{r^4}), \quad \alpha_{\mathbf{ij}} < \mathcal{O}(\mathbf{r^0}).$$

Stress tensor from the Holographic Dictionary

We consider a hypersurface $\boldsymbol{\Sigma}$ at constant $\mathbf{r}$.

Vector $\mathbf{n_M}$ normal to $\boldsymbol{\Sigma}$: $$\mathbf{n_M} = \frac{\nabla_\mathbf{M}\mathbf{r}}{\sqrt{\mathbf{g^{MN}}\nabla_\mathbf{M}\mathbf{r}\nabla_\mathbf{N}\mathbf{r}}}.$$

Induced metric $\gamma_\mathbf{MN}$ on $\boldsymbol{\Sigma}$: $$\gamma_\mathbf{MN} = \mathbf{g_{MN}} - \mathbf{n_M n_N}$$

Extrinsic curvature tensor $\mathcal{K}_\mathbf{MN}$:

$$\mathcal{K}_\mathbf{MN} = \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{n^A}\,\partial_\mathbf{A}\,\gamma_\mathbf{MN} + \gamma_\mathbf{MA}\,\partial_\mathbf{N}\,\mathbf{n^A} + \gamma_\mathbf{NA}\,\partial_\mathbf{M}\,\mathbf{n^A}\right).$$

The stress tensor for the dual fluid

$$\langle \mathbf{T}^\mu_\nu \rangle = \lim_{\mathbf{r}\to\infty} \tilde{\mathbf{T}}^\mu_\nu(\mathbf{r})\,; \qquad \tilde{\mathbf{T}}^\mu_\nu(\mathbf{r}) \equiv \mathbf{r}^4\left(\mathcal{K}^\mu_\nu - \mathcal{K}\gamma^\mu_\nu + \mathbf{3}\gamma^\mu_\nu - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{G}^\mu_\nu\right),$$

where G^μ_ν is associated with $\gamma_{\mu\nu}$. The last two terms are counter-terms which remove divergences near the boundary $r = \infty$.

$$\tilde{\mathbf{T}}^{0}_{0} = -3(1-4\epsilon b_1) + \frac{\epsilon}{2r}\Big\{-6rk + 4r^4\partial\beta - 4\partial j - r^3\partial_i\partial_j\alpha_{ij} + 18(r^5-r)h$$
$$+6(r^6-r^2)\partial_r h + 2r^3\partial^2 h + 6r^4\partial_v h\Big\}\,,$$

$$\tilde{\mathbf{T}}^{0}_{i} = \frac{\epsilon}{2r^4}\Big\{2\left[4r^4\beta_i - 4(r^4-1)j_i + r^7\partial_v\beta_i - r^3\partial_i k + (r^5-r)\partial_r j_i\right]$$
$$-r^2\left(-\partial^2 j_i + \partial_i\partial j + r^4\partial_v\partial_k\alpha_{ik} - 2r^4\partial_v\partial_i h - 3r^5\partial_i h\right)\Big\}\,,$$

$$\tilde{\mathbf{T}}^{i}_{0} = -\frac{\epsilon}{2r^3}\Big\{2\left[4r^3\beta_i - 4r^3 j_i + r^6\partial_v\beta_i - r^2\partial_i k + (r^4-1)\partial_r j_i\right]$$
$$+r\left[\partial^2 j_i - \partial_i\partial j - r^4\partial_v\partial_k\alpha_{ik} - 2r^4\partial_v\partial_i h - 3(r^6-r^2)\partial_i h\right]\Big\}\,,$$

$$\tilde{\mathbf{T}}^{i}_{j} = \delta^i_j(1-4\epsilon b_1) + \frac{\epsilon}{2r^4}\delta^i_j\Big\{r^2\left[-\partial^2 k + (1-r^4)\partial_k\partial_l\alpha_{kl} + 2\partial_v\partial j\right]$$
$$-2\left[(1-r^4)k - 2r^7\partial\beta + 2r^3\partial j - r^3\partial_v k + (r^5-r)\partial_r k\right] + r^6\partial^2 h$$
$$-2r^6\partial_v^2 h + 2\left[\left(3-12r^4+9r^5\right)h + (r^3-r^7)\partial_v h + (2r-4r^5+2r^9)\partial_r h\right]\Big\}$$
$$+\frac{\epsilon}{2r^2}\Big\{-2r\left[2r^4\partial_{(i}\beta_{j)} - 2\partial_{(i}j_{j)} + r^4\partial_v\alpha_{ij} + (r^6-r^2)\partial_r\alpha_{ij}\right] - r^4\partial_i\partial_j h$$
$$+\left[\partial_i\partial_j k + (1-r^4)\partial^2\alpha_{ij} + 2(r^4-1)\partial_k\partial_{(i}\alpha_{j)k} - 2\partial_v\partial_{(i}j_{j)} + r^4\partial_v^2\alpha_{ij}\right]\Big\}\,,$$

Approaching the boundary $\mathbf{r} \to \infty$

$$\mathbf{j_i} \to -\mathbf{i}\omega\mathbf{r}^3\beta_\mathbf{i} - \frac{\mathbf{1}}{\mathbf{3}}\mathbf{r}^2\partial_\mathbf{i}\partial\beta + \mathcal{O}\left(\frac{\mathbf{1}}{\mathbf{r}}\right),$$

$$\alpha_\mathbf{ij} \to \left(\frac{\mathbf{2}}{\mathbf{r}} - \frac{\eta(\omega, \mathbf{q}^2)}{\mathbf{4r^4}}\right)\sigma_\mathbf{ij} - \frac{\zeta(\omega, \mathbf{q}^2)}{\mathbf{4r^4}}\pi_\mathbf{ij} + \mathcal{O}\left(\frac{\mathbf{1}}{\mathbf{r^5}}\right).$$

$$\mathbf{k} \to \frac{\mathbf{2}}{\mathbf{3}}\left(\mathbf{r}^3 + \mathbf{i}\omega\mathbf{r}^2\right)\partial\beta + \mathcal{O}\left(\frac{\mathbf{1}}{\mathbf{r^2}}\right),$$

$$\mathbf{h} = \mathbf{0}$$

The dissipative part of the stress tensor

$$\mathbf{\Pi_{ij}} = -\left[\eta(\omega, \mathbf{q}^2)\sigma_\mathbf{ij} + \zeta(\omega, \mathbf{q}^2)\pi_\mathbf{ij}\right]$$

Einstein equations for the metric corrections

Dynamical equations:

$$\mathbf{E_{rr} = 0 :} \qquad 5\,\partial_r h + r\partial_r^2 h = 0\,.$$

$$\mathbf{E_{rv} = 0 :} \qquad 3\,r^2\partial_r k = 6\,r^4\,\partial\beta + r^3\,\partial_v\partial\beta - 2\,\partial j - r\,\partial_r\partial j - r^3\,\partial_i\partial_j\alpha_{ij}$$

$$\mathbf{E_{ri} = 0 :} \qquad -\partial_r^2 j_i = \left(\partial^2\beta_i - \partial_i\partial\beta\right) + 3r\,\partial_v\beta_i - \tfrac{3}{r}\partial_r j_i + r^2\partial_r\partial_j\alpha_{ij}.$$

$$\mathbf{E_{ij} = 0 :}$$

$$\begin{aligned}
&(r^7 - r^3)\partial_r^2\alpha_{ij} + (5r^6 - r^2)\partial_r\alpha_{ij} + 2r^5\partial_v\partial_r\alpha_{ij} + 3r^4\partial_v\alpha_{ij} \\
&+r^3\left\{\partial^2\alpha_{ij} - \left(\partial_i\partial_k\alpha_{jk} + \partial_j\partial_k\alpha_{ik} - \frac{2}{3}\delta_{ij}\partial_k\partial_l\alpha_{kl}\right)\right\} \\
&+\left(\partial_i j_j + \partial_j j_i - \frac{2}{3}\delta_{ij}\partial j\right) - r\partial_r\left(\partial_i j_j + \partial_j j_i - \frac{2}{3}\delta_{ij}\partial j\right) \\
&+3r^4\left(\partial_i\beta_j + \partial_j\beta_i - \frac{2}{3}\delta_{ij}\partial\beta\right) + r^3\partial_v\left(\partial_i\beta_j + \partial_j\beta_i - \frac{2}{3}\delta_{ij}\partial\beta\right) = 0\,.
\end{aligned}$$

Holographic RG flow-type equations

$\mathbf{j_i}$ **and** $\alpha_{\mathbf{ij}}$ **are linear functionals of** β_i**. They can be uniquely decomposed as**

$$\mathbf{j_i} = \mathbf{a}(\omega, \mathbf{q}, \mathbf{r})\, \beta_{\mathbf{i}} + \mathbf{b}(\omega, \mathbf{q}, \mathbf{r})\, \partial_{\mathbf{i}}\partial\beta$$

$$\alpha_{\mathbf{ij}} = \mathbf{2\, c}(\omega, \mathbf{q}, \mathbf{r})\, \sigma_{\mathbf{ij}} + \mathbf{d}(\omega, \mathbf{q}, \mathbf{r})\, \pi_{\mathbf{ij}},$$

The Einstein equations reduce to ordinary diff equations

$$\begin{aligned}
&\mathbf{r}\partial_{\mathbf{r}}^2\mathbf{a} - 3\partial_{\mathbf{r}}\mathbf{a} - \mathbf{q}^2\mathbf{r}^3\partial_{\mathbf{r}}\mathbf{c} - 3\mathbf{i}\omega\mathbf{r}^2 - \mathbf{q}^2\mathbf{r} = 0 \\
&\mathbf{r}\partial_{\mathbf{r}}^2\mathbf{b} - 3\partial_{\mathbf{r}}\mathbf{b} + \frac{1}{3}\mathbf{r}^3\partial_{\mathbf{r}}\mathbf{c} - \frac{2}{3}\mathbf{r}^3\mathbf{q}^2\partial_{\mathbf{r}}\mathbf{d} - \mathbf{r} = 0 \\
&(\mathbf{r}^7 - \mathbf{r}^3)\partial_{\mathbf{r}}^2\mathbf{c} + (5\mathbf{r}^6 - \mathbf{r}^2)\partial_{\mathbf{r}}\mathbf{c} - 2\mathbf{i}\omega\mathbf{r}^5\partial_{\mathbf{r}}\mathbf{c} - \mathbf{r}\partial_{\mathbf{r}}\mathbf{a} + \mathbf{a} - 3\mathbf{i}\omega\mathbf{r}^4\mathbf{c} + 3\mathbf{r}^4 - \mathbf{i}\omega\mathbf{r}^3 = 0 \\
&(\mathbf{r}^7 - \mathbf{r}^3)\partial_{\mathbf{r}}^2\mathbf{d} + (5\mathbf{r}^6 - \mathbf{r}^2)\partial_{\mathbf{r}}\mathbf{d} - 2\mathbf{i}\omega\mathbf{r}^5\partial_{\mathbf{r}}\mathbf{d} + \frac{\mathbf{q}^2}{3}\mathbf{r}^3\mathbf{d} - 3\mathbf{i}\omega\mathbf{r}^4\mathbf{d} + 2\mathbf{b} - 2\mathbf{r}\partial_{\mathbf{r}}\mathbf{b} - \frac{2}{3}\mathbf{r}^3\mathbf{c} = 0\,.
\end{aligned}$$

Navier-Stokes equations

Using dynamical Einstein equations, we have constructed an "off-shell" $\mathbf{T}^{\mu\nu}$

Constraint equations

$$\mathbf{E_{vv}} = \mathbf{0} \text{ and } \mathbf{E_{vi}} = \mathbf{0}$$

are equivalent to the stress tensor conservation law

$$\partial_\mu \mathbf{T}^{\mu\nu} = \mathbf{0}$$

which determines the temperature and velocity profiles as functions of time, provided initial configuration is specified.

Conclusions

- **We have found that all order dissipative terms of a weakly perturbed conformal fluid are fully accounted for by two viscosity functions $\eta(\omega, q^2)$ and $\zeta(\omega, q^2)$**

- **For a weakly curved background space, there are additional four transport functions called Gravitational Susceptibilities of the Fluid.**

- **We have derived a closed form $linear$ holographic RG flow-type equations for the viscosity functions and GSFs.**

- **At large momenta, the effective viscosity is a decreasing function of both frequency and momentum. The corresponding memory function has support in the past only, the behavior consistent with causality restoration.**