Pion resonances in holographic QCD

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Summary

Introduction : Chiral symmetry breaking (χSB) in QCD

Left and right flavour currents

$$J^a_{L\hat{\mu}} = \bar{q}_L \gamma_{\hat{\mu}} t^a q_L \quad , \quad J^a_{R\hat{\mu}} = \bar{q}_R \gamma_{\hat{\mu}} t^a q_R \, , \tag{1}$$

corresponding to the $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry.

χSB **in QCD** : $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V$

Chiral condensate : $\sigma_q = \langle \bar{q}_R q_L \rangle \neq 0$, responsible for spontaneous χSB in the chiral limit. The corresponding **Goldstone bosons** are the **ground-state pions** $\hat{\pi}^{a,0}$.

Partially conserved axial current (**PCAC**) relation :

$$\partial_\mu \langle J^{\hat{\mu}}_{a,V} \rangle = 0 \quad , \quad \partial_\mu \langle J^{\hat{\mu}}_{a,A} \rangle = f_{\pi^0} m^2_{\pi^0} \, \hat{\pi}^{a,0} \, , \tag{2}$$

where m_{π^0} and f_{π^0} are the **mass and decay constant** of the ground state pions.

Pion resonances

Pion resonances in lattice QCD :

$$m_{\pi^n} \gg m_{\pi^0} \quad , \quad f_{\pi^n} \ll f_{\pi^0} \qquad (n = 1, 2, \dots)\,. \tag{3}$$

UKQCD Collaboration 2006 , Hadron Spectrum Collaboration 2014

Nonperturbative QCD prediction :

$$f_{\pi^n} \to 0 \quad \text{when} \quad m_q \to 0 \quad (\text{chiral limit})\,. \tag{4}$$

A. Holl, A. Krassnigg and C. D. Roberts 2004

How do f_{π^n} behave at finite m_q ? Is the PCAC relation modified ?

Gell-Mann-Oakes-Renner (GOR) relationship :

$$f_{\pi^0}^2\, m_{\pi^0}^2 = 2 m_q\, \sigma_q\,. \tag{5}$$

Is there a GOR-like relationship for pion resonances ?

χSB in holographic QCD

Bottom-up approach to confinement : A hard cutoff in 5-d Anti-de-Sitter spacetime :

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right)\,, \quad 0 < z \leq z_0\,. \tag{6}$$

Polchinski-Strassler 2002

χSB **fields**: $\quad A_m^{L(R)} \leftrightarrow J_\mu^{L(R)} \quad , \quad X \leftrightarrow \bar{q}_R q_L.$

The 5-d action :

$$S = \int d^5x \sqrt{|g|}\,\mathrm{Tr}\left[|D_m X|^2 + 3|X|^2 - \frac{1}{4g_5^2}\left(F_L^{mn}F_{mn}^L + F_R^{mn}F_{mn}^R\right)\right], \tag{7}$$

Erlich-Katz-Son-Stephanov 2005
Da Rold-Pomarol 2005

where $\quad D_m X = \partial_m X - iA_m^L X + iXA_m^R \quad , \quad g_5^2 = \frac{12\pi^2}{N_c}.$

$N_f = 2$ (good approximation for χSB) .

Classical fields :

$$A_m^{L(R)} = 0\,, \quad X_0 = \frac{1}{2}\left[\zeta M z + \frac{\Sigma}{\zeta} z^3\right]\,, \tag{8}$$

where $\quad M = m_q \mathbb{1} \quad , \quad \Sigma = \sigma_q \mathbb{1} \quad , \quad \eta = \sqrt{N_c}/(2\pi)$.
Perturbing the classical fields :

$$A_m^{L(R)} = (A_m^{a,V} \pm A_m^{a,A}) t^a\,, \quad X = X_0\, e^{2i\pi^a t^a}\,. \tag{9}$$

Expanding the action (7) up to quadratic order :

$$S^{\mathrm{Kin}} = \int d^5x \sqrt{|g|}\left[\frac{v(z)^2}{2}(\partial_m \pi^a - A_m^{a,A})^2 - \frac{1}{4g_5^2}(f_{a,V}^{mn} f_{mn}^{a,V} + f_{a,A}^{mn} f_{mn}^{a,A})\right], \tag{10}$$

where $f_{mn}^{a,V(A)} = \partial_m A_n^{a,V(A)} - \partial_n A_m^{a,V(A)}$, $v(z) = \zeta m_q z + \frac{\sigma_q}{\zeta} z^3$.

Equations of motion and holographic currents

Writing $S^{\rm Kin} = \int d^5x\, \mathcal{L}^{\rm Kin}$ its variation takes the general form

$$\begin{aligned} \delta S^{\rm Kin} &= \int d^5x \left[\left(\frac{\partial \mathcal{L}^{\rm Kin}}{\partial A_\ell^{a,V}} - \partial_m P^{m\ell}_{a,V} \right) \delta A_\ell^{a,V} \right. \\ &\quad \left. + \left(\frac{\partial \mathcal{L}^{\rm Kin}}{\partial A_\ell^{a,A}} - \partial_m P^{m\ell}_{a,A} \right) \delta A_\ell^{a,A} + \left(\frac{\partial \mathcal{L}^{\rm Kin}}{\partial \pi^a} - \partial_m P^m_{\pi,a} \right) \delta \pi^a \right] \\ &\quad + \int d^5x\, \partial_m \left(P^{m\ell}_{a,V}\, \delta A_\ell^{a,V} + P^{m\ell}_{a,A}\, \delta A_\ell^{a,A} + P^m_{\pi,a}\, \delta\pi^a \right), \end{aligned} \tag{11}$$

where

$$P^{m\ell}_{a,V(A)} = \frac{\partial \mathcal{L}^{\rm Kin}}{\partial(\partial_m A_\ell^{a,V(A)})} \quad , \quad P^m_{\pi,a} = \frac{\partial \mathcal{L}^{\rm Kin}}{\partial(\partial_m \pi^a)} \,. \tag{12}$$

For $m = 0$ these quantities reduce to the usual conjugate momenta.

Using the explicit form of (10) we find

$$\frac{\partial \mathcal{L}^{\rm Kin}}{\partial A^{a,V}_{\ell}} = \frac{\partial \mathcal{L}^{\rm Kin}}{\partial \pi^a} = 0, \quad , \quad \frac{\partial \mathcal{L}^{\rm Kin}}{\partial A^{a,A}_{\ell}} = -v(z)^2 \sqrt{|g|} \left(\partial^{\ell}\pi_a - A^{\ell}_{a,A}\right), \quad (13)$$

$$P^{m\ell}_{a,V(A)} = -\frac{1}{g_5^2}\sqrt{|g|}\, f^{m\ell}_{a,V(A)}, \quad , \quad P^{m}_{a,\pi} = v(z)^2 \sqrt{|g|} \left(\partial^{m}\pi_a - A^{m}_{a,A}\right). \quad (14)$$

Imposing $\delta S^{\rm Kin} = 0$, we find from the bulk terms in (11) the field equations

$$\partial_m \left(\sqrt{|g|}\, f^{mn}_{a,V}\right) = 0, \quad (15)$$

$$\partial_m \left(\sqrt{|g|}\, f^{mn}_{a,A}\right) - g_5^2 v(z)^2 \sqrt{|g|} \left(\partial^{n}\pi^a - A^{n}_{a,A}\right) = 0, \quad (16)$$

$$\partial_m \left[v(z)^2 \sqrt{|g|} \left(\partial^{m}\pi^a - A^{m}_{a,A}\right)\right] = 0. \quad (17)$$

The boundary terms at $z = \epsilon$ can be written as

$$\delta S^{\rm Kin}_{\rm Bdy} = -\int d^4x \left[\langle J^{\hat\mu}_{V,a} \rangle \, \left(\delta V^a_{\hat\mu}\right)_{z=\epsilon} + \langle J^{\hat\mu}_{A,a} \rangle \, \left(\delta A^a_{\hat\mu}\right)_{z=\epsilon} + \langle J_{\pi,a} \rangle \, \left(\delta\pi^a\right)_{z=\epsilon} \right], \quad (18)$$

where we find the **holographic currents**

$$\begin{aligned} \langle J^{\hat\mu}_{a,V} \rangle &= P^{z\mu}_{a,V}|_{z=\epsilon} = -\frac{1}{g_5^2} \left(\sqrt{|g|} f^{z\mu}_{a,V} \right)_{z=\epsilon} , \\ \langle J^{\hat\mu}_{a,A} \rangle &= P^{z\mu}_{a,A}|_{z=\epsilon} = -\frac{1}{g_5^2} \left(\sqrt{|g|} f^{z\mu}_{a,A} \right)_{z=\epsilon} , \\ \langle J_{a,\pi} \rangle &= P^{z}_{a,\pi}|_{z=\epsilon} = \left[\sqrt{|g|} v(z)^2 \left(\partial^z \pi_a - A^z_{a,A} \right) \right]_{z=\epsilon} = \partial_\mu \langle J^\mu_{a,A} \rangle . \end{aligned} \quad (19)$$

The boundary terms at $z = z_0$ vanish under Neumann boundary conditions

$$P^{z\mu}_{a,V}|_{z=z_0} = P^{z\mu}_{a,A}|_{z=z_0} = P^{z}_{a,\pi}|_{z=z_0} = 0 \, . \quad (20)$$

The Kaluza-Klein expansion

Using the explicit form of the AdS metric (6), the action (10) takes the form

$$S^{\mathrm{Kin}} = \int d^4x \int \frac{dz}{z} \left\{ \frac{v(z)^2}{2z^2} \left[-\left(\partial_z \pi^a - A_z^{a,A}\right)^2 + \left(\partial_{\hat{\mu}} \pi^a - A_{\hat{\mu}}^{a,A}\right)^2 \right] \right.$$
$$\left. -\frac{1}{4g_5^2} \left[-2(f_{z\hat{\mu}}^{a,V})^2 + (f_{\hat{\mu}\hat{\nu}}^{a,V})^2 - 2(f_{z\hat{\mu}}^{a,A})^2 + (f_{\hat{\mu}\hat{\nu}}^{a,A})^2 \right] \right\}. \tag{21}$$

The action (21) is invariant under the gauge transformations

$$\begin{array}{lcl} A_m^{a,V} & \rightarrow & A_m^{a,V} - \partial_m \lambda_V^a \\ A_m^{a,A} & \rightarrow & A_m^{a,A} - \partial_m \lambda_A^a \quad , \quad \pi^a \rightarrow \pi^a - \lambda_A^a . \end{array} \tag{22}$$

Because of this, we can fix the gauge as $A_z^{a,V} = A_z^{a,A} = 0$

The axial-vector field $A^{a,A}_{\hat{\mu}}$ can be decomposed into transverse and longitudinal parts:

$$A^{a,A}_{\hat{\mu}} = A^{\perp,a,A}_{\hat{\mu}} + \partial_{\hat{\mu}}\phi^a \quad , \quad \partial_{\hat{\mu}} A^{\hat{\mu}}_{\perp,a,A} = 0\,. \tag{23}$$

The transverse part will describe the **axial-vector mesons** whereas the longitudinal part will be associated with the **pions**.

Then the action (21) reduces to

$$\begin{aligned} S^{\mathrm{Kin}} &= \int d^4x \int \frac{dz}{z} \left\{ \frac{v(z)^2}{2z^2} \left[-(\partial_z \pi^a)^2 + (\partial_{\hat{\mu}}\pi^a - \partial_{\hat{\mu}}\phi^a)^2 + \left(A^{\perp,a,A}_{\hat{\mu}}\right)^2 \right] \right. \\ & \quad - \frac{1}{4g_5^2} \left[-2\left(\partial_z A^{a,V}_{\hat{\mu}}\right)^2 + \left(f^{a,V}_{\hat{\mu}\hat{\nu}}\right)^2 \right. \\ & \quad \left. \left. -2\left(\partial_z A^{a,A}_{\hat{\mu}}\right)^2 - 2\left(\partial_z \partial_{\hat{\mu}}\phi^a\right)^2 + \left(f^{\perp,a,A}_{\hat{\mu}\hat{\nu}}\right)^2 \right] \right\}. \end{aligned} \tag{24}$$

Kaluza-Klein expansions for the 5-d fields :

$$A_{\hat{\mu}}^{a,V} = g_5 \sum_{n=0}^{\infty} v^{a,n}(z) \hat{V}_{\hat{\mu}}^{a,n}(x) \quad , \quad A_{\hat{\mu}}^{\perp,a,A} = g_5 \sum_{n=0}^{\infty} a^{a,n}(z) \hat{A}_{\hat{\mu}}^{a,n}(x),$$
$$\pi^a = g_5 \sum_{n=0}^{\infty} \pi^{a,n}(z) \hat{\pi}^{a,n}(x) \quad , \quad \phi^a = g_5 \sum_{n=0}^{\infty} \phi^{a,n}(z) \hat{\pi}^{a,n}(x). \qquad (25)$$

Using these expansions the action (24) takes the form

$$\begin{aligned} S^{\mathrm{Kin}} \;=\; & \sum_{n,m=0}^{\infty} \int d^4x \left\{ \frac{1}{2} \Delta_{\pi}^{a,nm} \, \partial_{\hat{\mu}} \hat{\pi}^{a,n} \partial^{\hat{\mu}} \hat{\pi}^{a,m} - \frac{1}{2} M_{\pi}^{a,nm} \, \hat{\pi}^{a,n} \, \hat{\pi}^{a,m} \right. \\ & - \frac{1}{4} \Delta_{V}^{a,nm} \, \hat{v}_{\hat{\mu}\hat{\nu}}^{a,n} \, \hat{v}_{a,m}^{\hat{\mu}\hat{\nu}} + \frac{1}{2} M_{V}^{a,nm} \, \hat{V}_{\hat{\mu}}^{a,n} \, \hat{V}_{a,m}^{\hat{\mu}} \\ & \left. - \frac{1}{4} \Delta_{A}^{a,nm} \, \hat{a}_{\hat{\mu}\hat{\nu}}^{a,n} \, \hat{a}_{a,m}^{\hat{\mu}\hat{\nu}} + \frac{1}{2} M_{A}^{a,nm} \, \hat{A}_{\hat{\mu}}^{a,n} \, \hat{A}_{a,m}^{\hat{\mu}} \right\}, \end{aligned} \qquad (26)$$

where

$$\Delta_\pi^{a,nm} = \int \frac{dz}{z} \left\{ \partial_z \phi^{a,n} \partial_z \phi^{a,m} + \beta(z) \left[\pi^{a,n} - \phi^{a,n} \right] \left[\pi^{a,m} - \phi^{a,m} \right] \right\},$$
$$\Delta_V^{a,nm} = \int \frac{dz}{z} v^{a,n} v^{a,m} \quad , \quad \Delta_A^{a,nm} = \int \frac{dz}{z} a^{a,n} a^{a,m},$$
$$M_\pi^{a,nm} = \int \frac{dz}{z} \beta(z) \partial_z \pi^{a,n} \partial_z \pi^{a,m} \quad , \quad M_V^{a,nm} = \int \frac{dz}{z} \partial_z v^{a,n} \partial_z v^{a,m},$$
$$M_A^{a,nm} = \int \frac{dz}{z} \left\{ \partial_z a^{a,n} \partial_z a^{a,m} + \beta(z) a^{a,n} a^{a,m} \right\}, \tag{27}$$

and $\beta(z) = \frac{g_5^2}{z^2} v(z)^2$. The goal is to obtain a 4-d action for the **vector mesons** $\hat{V}_{\hat{\mu}}^{a,n}$, **axial-vector mesons** $\hat{A}_{\hat{\mu}}^{a,n}$ and **pions** $\hat{\pi}^{a,n}$. This can be achieved imposing the conditions

$$\Delta_\pi^{a,nm} = \Delta_V^{a,nm} = \Delta_A^{a,nm} = \delta^{nm}, \tag{28}$$
$$M_\pi^{a,nm} = m_{\pi^{a,n}}^2 \delta^{nm} \; , \; M_V^{a,nm} = m_{V^{a,n}}^2 \delta^{nm} \; , \; M_A^{a,nm} = m_{A^{a,n}}^2 \delta^{nm}. \tag{29}$$

The conditions (28) are **normalization rules**. The conditions (29) can be obtained from (28) as long as we impose the equations

$$\frac{\beta(z)}{z}\left[\pi^{a,n}-\phi^{a,n}\right]=-\partial_z\left[\frac{1}{z}\partial_z\phi^{a,n}\right], \tag{30}$$

$$\beta(z)\,\partial_z\pi^{a,n}=m^2_{\pi^{a,n}}\partial_z\phi^{a,n}, \tag{31}$$

$$-\partial_z\left[\frac{1}{z}\partial_z v^{a,n}\right]=\frac{m^2_{V^{a,n}}}{z}v^{a,n}, \tag{32}$$

$$\left[-\partial_z\left(\frac{1}{z}\partial_z\right)+\frac{1}{z}\beta(z)\right]a^{a,n}=\frac{m^2_{A^{a,n}}}{z}a^{a,n}. \tag{33}$$

These are **on-shell** conditions for the Kaluza-Klein modes $v^{a,n}$, $a^{a,n}$, $\pi^{a,n}$ and $\phi^{a,n}$.

The **meson spectrum** is obtained by solving these equations with Dirichlet BC at $z=\epsilon$ and Neumann BC at $z=z_0$.

Extended PCAC

Using the KK expansions (25) in the currents (19) we find

$$\langle J_V^{\hat{\mu}}(x)\rangle = \sum_{n=0}^{\infty}\left[\frac{1}{g_5 z}\partial_z v^n\right]_{z=\epsilon}\hat{V}_n^{\hat{\mu}}(x)\,,$$
$$\langle J_A^{\hat{\mu}}(x)\rangle = \sum_{n=0}^{\infty}\left[\frac{1}{g_5 z}\partial_z a^n\right]_{z=\epsilon}\hat{A}_n^{\hat{\mu}}(x) + \sum_{n=0}^{\infty}\left[\frac{1}{g_5 z}\partial_z \phi^n\right]_{z=\epsilon}\partial^{\hat{\mu}}\hat{\pi}^n(x)\,,$$
$$\partial_{\hat{\mu}}\langle J_A^{\hat{\mu}}(x)\rangle = \langle J_\Pi(x)\rangle = -\sum_{n=0}^{\infty}\left[\frac{\beta(z)}{g_5 z}\partial_z \pi^n(z)\right]_{z=\epsilon}\hat{\pi}_n(x). \qquad (34)$$

From the expansions (34) we extract the **decay constants** :

$$g_{V^n} = \left[\frac{1}{g_5 z}\partial_z v^n\right]_{z=\epsilon}\,, \quad g_{A^n} = \left[\frac{1}{g_5 z}\partial_z a^n\right]_{z=\epsilon}\,, \qquad (35)$$
$$f_{\pi^n} = \left[-\frac{1}{g_5 z}\partial_z \phi^n\right]_{z=\epsilon}. \qquad (36)$$

These results are consistent with the standard definition of meson decay constants:

$$\begin{aligned} \langle 0|J_V^{\hat{\mu}}(0)|V_n(p,\lambda)\rangle &= \epsilon^{\hat{\mu}}(p,\lambda) g_{V^n} \,, \\ \langle 0|J_A^{\hat{\mu}}(0)|A_n(p,\lambda)\rangle &= \epsilon^{\hat{\mu}}(p,\lambda) g_{A^n} \,, \\ \langle 0|J_A^{\hat{\mu}}(0)|\pi_n(p)\rangle &= i p^{\hat{\mu}} f_{\pi^n} . \end{aligned} \tag{37}$$

Using eq. (31) and the prescription (36) we find a GOR-like relationship

$$f_{\pi^n} m_{\pi^n}^2 = -\frac{1}{g_5} \left[\frac{\beta(z)}{z} \partial_z \pi^n \right]_{z=\epsilon} := 2 m_q \, \rho_{\pi^n} \,. \tag{38}$$

The divergence of the axial current takes the form of an **extended PCAC relation**

$$\partial_\mu \langle J_A^{\hat{\mu}}(x) \rangle = \sum_{n=0}^{\infty} f_{\pi^n} m_{\pi^n}^2 \, \hat{\pi}_n(x). \tag{39}$$

Normalization

Combining (30) and (31) we find the equation

$$(z\partial_z)^2 \,\Pi^n(z) + A(z) z\partial_z \Pi^n(z) + B_n(z)\Pi^n(z) = 0\,, \tag{40}$$

where $\Pi^n := \partial_z \pi^n$ and

$$A(z) = z\partial_z \ln\beta(z) - 2 \quad , \quad B_n(z) = 1 + z^2\left[\partial_z^2 \ln\beta(z) + m_{\pi^n}^2 - \beta(z)\right]. \tag{41}$$

Expanding $\Pi^n(z)$ in powers of z near the boundary we find

$$\Pi^n(z) = C_n\left[-z + \frac{1}{4}\left(m_{\pi^n}^2 + \frac{32\pi^2\sigma_q}{3m_q} - 3m_q^2\right)z^3 + \cdots\right] =: C_n\,\Pi_U^n(z). \tag{42}$$

The normalization rules imply that

$$C_n = \frac{m_{\pi^n}}{N_{\pi^n}} \quad , \quad N_{\pi^n}^2 = \int \frac{dz}{z}\,\beta(z)\,\left[\Pi_U^n(z)\right]^2. \tag{43}$$

Using these results and the definition of $\beta(z)$ we find that

$$\rho_{\pi^n} = \frac{g_5\zeta^2}{2}\frac{m_q m_{\pi^n}}{N_{\pi^n}}\,. \tag{44}$$

The pion spectrum : Integrating numerically eq. (40) from $z = \epsilon$ to $z = z_0$ and imposing the appropriate BC the pion spectrum is obtained.

The mass of the ρ meson $m_\rho = 775.5$ MeV fixes the hard-wall cutoff $z_0 = (322.5\ \mathrm{MeV})^{-1}$.

At fixed σ_q the pion masses m_{π^n} and decay constants f_{π^n} become functions of m_q.

We fix $\sigma_q = (213.7\ \mathrm{MeV})^3$ in order to obtain the experimental values $m_{\pi^0} = 139.6\ \mathrm{MeV}$, $f_{\pi^0} = 92.4\ \mathrm{MeV}$ at the same value for the quark mass, $m_q = 8.31$ MeV.

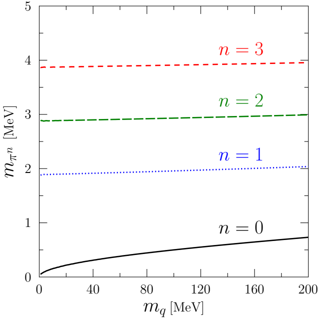

Figure 1 : Quark mass dependence of pion masses.

Quark mass dependence of N_{π^n} and ρ_{π^n}

N_{π^n} is completely determined by the unnormalized eigenfunctions $\Pi_U^n(z)$, corresponding to the eigenvalues m_{π^n}.

The function ρ_{π^n} is then obtained using eq. (44).

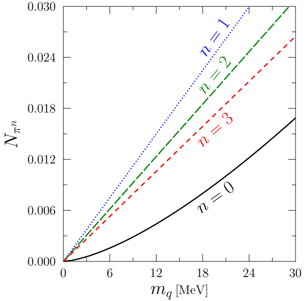

Figure 2 : m_q dependence of N_{π^n}.

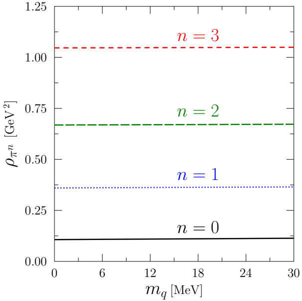

Figure 3 : m_q dependence of ρ_{π^n}.

Decay constants of the pion resonances

The results show that ρ_{π^n} is finite as $m_q \to 0$ and remarkably independent of m_q for $0 \leq m_q \leq 30$ MeV.

Using the results for ρ_{π^n} and the pion spectrum in the GOR-like relationship (38) we find the pion decay constants f_{π^n}.

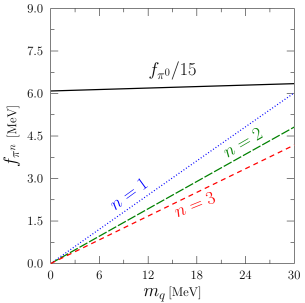

Figure 4 : m_q dependence of f_{π^n}

For $m_q = 8.31$ MeV, corresponding to $m_{\pi^0} = 139.6$ MeV and $f_{\pi^0} = 92.4$ MeV, we can estimate the decay constants of pion resonances.

n	0	1	2	3
f_{π^n} (MeV)	92.4	1.68	1.34	1.16

Table 1 : Decay constants for the pion ground-state and first excited states.

Conclusions

We have investigated the leptonic decay constants of pion resonances in a 5-d holographic model for QCD.

We have obtained a generalization of the PCAC relation that takes into account the pion resonances.

We have also found a GOR-like relationship and used it to prove numerically that the decay constants of pion resonances vanish in the chiral limit.

It would be interesting to explore other holographic models for QCD and investigate the dependence of the decay constants on the pion spectrum.