

Non-perturbative renormalization of the energy-momentum tensor in SU(3) Yang-Mills theory

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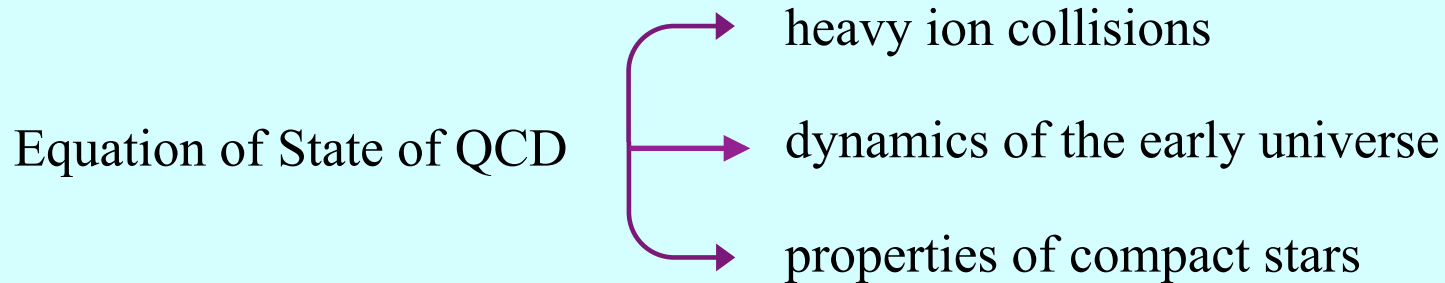
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PLAN OF THE TALK

- Introduction
- A new perspective from relativistic thermodynamics
- The lattice setup and the numerical results
- Conclusions and perspectives

Introduction

- Thermal quantum field theory: a bridge between particle and nuclear physics




- Quantum field theory described by an action S in a space-time volume $V \times \beta$

$$\mathcal{Z}(\beta, V) = \int \mathcal{D}\phi e^{-S(\phi)} \quad \langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \mathcal{O}(\phi) e^{-S(\phi)} \quad \text{temperature: } T = \frac{1}{\beta}$$

$$\text{free energy density: } f = -\frac{T}{V} \log \mathcal{Z}(\beta, V) \quad \text{pressure: } p = T \frac{\partial \log \mathcal{Z}(\beta, V)}{\partial V} = -f$$

$$\text{energy density: } e = \frac{T^2}{V} \frac{\partial \log \mathcal{Z}(\beta, V)}{\partial T} \quad \text{entropy density: } s = \frac{e + p}{T} = \frac{\partial p}{\partial T}$$

 $\mathcal{Z}(\beta, V)$ contains all the information about the system but it cannot be computed directly

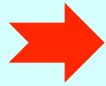


Engels, Fingberg, Karsch,
Miller, and Weber
Phys. Lett. B (1990)

$$\log \mathcal{Z}(\beta, V) \longrightarrow \int_{x_0}^x dx' \frac{\partial \log \mathcal{Z}}{\partial x'} \quad \text{usually } x = \frac{2N}{g_0^2} \quad \longrightarrow \quad \frac{f}{T^4} - \frac{f_0}{T_0^4} = \beta^4 \int_{x_0}^x \frac{dx'}{x'} [\langle s \rangle_0 - \langle s \rangle]$$

New problems for an old solution

theory of special
relativity



relativistic thermodynamics



thermodynamics and
statistical mechanics

$$T = \gamma T_0$$

$$s = \gamma s_0$$

$$e = \gamma^2 (e_0 + p_0 \vec{v}^2)$$

$$f = \gamma^2 f_0$$

$$V = \frac{V_0}{\gamma}$$

$$p = p_0$$

$$w = e + p = \gamma^2 w_0$$

$$\gamma(\vec{v}) = \frac{1}{\sqrt{1 - \vec{v}^2}}$$

energy-momentum
tensor

$$T_{\mu\nu} = \begin{pmatrix} e_0 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix}$$



$$T_{ij} = \gamma^2 w_0 v_i v_j + p \delta_{ij}$$

$$T_{00} = \gamma^2 w_0 - p = e$$

$$T_{0k} = \gamma^2 w_0 v_k$$

Landau, Lifschiz,
vol 6, "Fluid mechanics"

rewriting:

$$s_0 = \frac{w_0}{T_0} = \frac{T_{0k}}{T_0 \gamma^2 v_k}$$



$$\frac{s_0}{T_0^3} = \frac{T_{0k}}{T_0^4 \gamma^2 v_k} = \frac{\beta^4 (1 - \vec{v}^2)^3}{v_k} T_{0k}$$

$$\frac{1}{T_0} = \beta_0 = \beta \sqrt{1 - \vec{v}^2}$$



$$T_{0k} = \frac{v_k}{1 + v_k^2} (T_{00} + T_{kk})$$

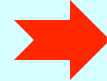
the entropy density in the rest frame is related to the momentum T_{0k} and the time extent β in the moving frame argument is general: applies to any system at equilibrium

This approach can be applied to a thermal quantum field theory in the euclidean path-integral formulation

L. Giusti and H. Meyer,
PRL 2011, JHEP 2011 and 2013

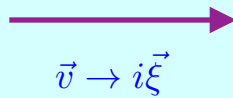
$$\phi(\beta, \vec{x}) = \phi(0, \vec{x} - \beta \vec{\xi})$$

The energy-momentum tensor contains the currents associated to translation invariance

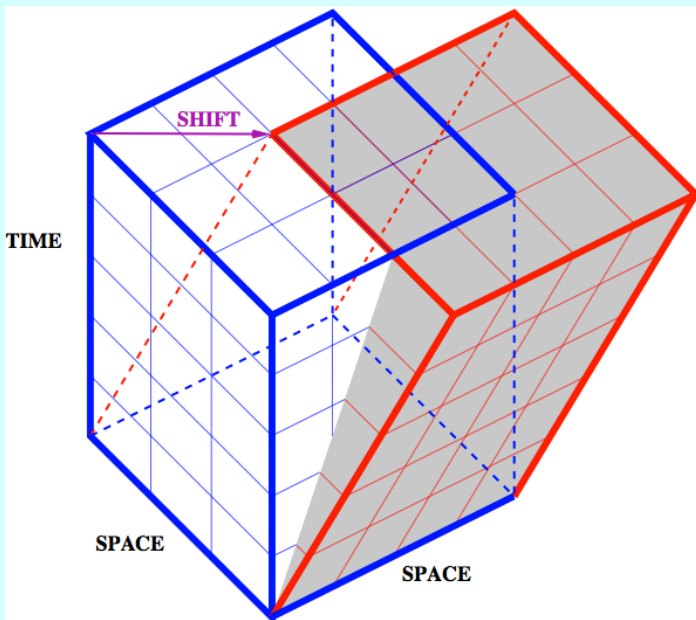


Use the machinery of quantum fields to produce Ward identities generated by Lorentz invariance

$$\mathcal{Z}(\beta, \vec{v}) = \text{Tr} \left(e^{-\beta(H - \vec{v} \cdot \vec{P})} \right)$$



$$\mathcal{Z}(\beta, \vec{\xi}) = \text{Tr} \left(e^{-\beta(H - i\vec{\xi} \cdot \vec{P})} \right)$$



$$f(\beta \sqrt{1 + \vec{\xi}^2}, 0) = -\frac{1}{\beta V} \log \mathcal{Z}(\beta, \vec{\xi})$$

$$\frac{s_0}{T_0^3} = -\frac{\beta^4 (1 + \vec{\xi}^2)^3}{\xi_k} \langle T_{0k} \rangle_{\xi} Z_T$$

$$s_0 = -\frac{1}{V \gamma^3 \xi_k} \frac{\partial}{\partial \xi_k} \log \mathcal{Z}(\beta, \vec{\xi})$$

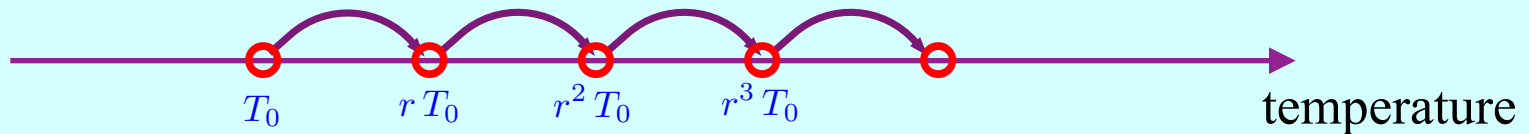
$$\langle T_{0k} \rangle_{\xi} = \frac{\xi_k}{1 - \xi_k^2} \langle T_{00} - T_{kk} \rangle_{\xi} Z_T$$

lattice regularization:
breaking of translation
invariance

Equation of State: entropy vs. temperature

- **Strategy 1:** two shifts ξ and ξ' , measure the entropy density step-scaling function

$$\Sigma(T, r) = \frac{T^3 s(T')}{T'^3 s(T)} = \frac{(1 + \vec{\xi}'^2)^3 \xi_k \langle T_{0k} \rangle_{\xi'}}{(1 + \vec{\xi}^2)^3 \xi'_k \langle T_{0k} \rangle_{\xi}} \quad r = \frac{T'}{T} = \frac{\sqrt{1 + \vec{\xi}^2}}{\sqrt{1 + \vec{\xi}'^2}} \quad \text{step in the temperature}$$



the renormalization factor Z_T drops.

- **Strategy 2:** only one shift and measure $\langle T_{0k} \rangle_{\xi}$ and Z_T

$$\frac{s_0}{T_0^3} = -\frac{\beta^4 (1 + \vec{\xi}^2)^3}{\xi_k} \langle T_{0k} \rangle_{\xi} Z_T$$

need to know Z_T for many values of g_0^2 .

Strategy 1: entropy step-scaling function

we consider the shifts: $\xi=(1,1,1)$ and $\xi'=(1,0,0) \rightarrow r = \frac{T'}{T} = \frac{\sqrt{1+\xi^2}}{\sqrt{1+\xi'^2}} = \sqrt{2}$

good compromise: small step (fine T dependence) and big step (broad range in T)

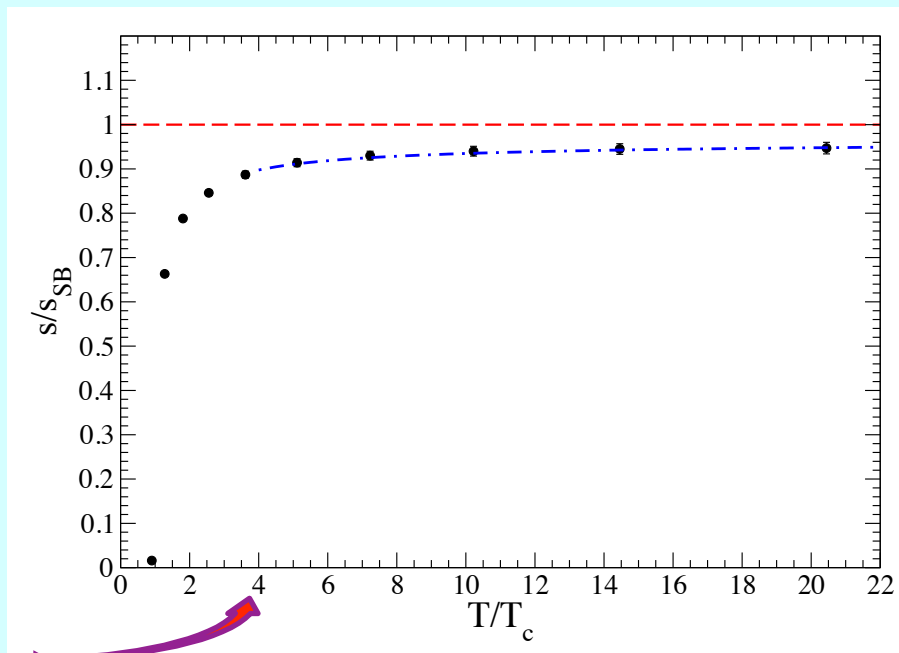
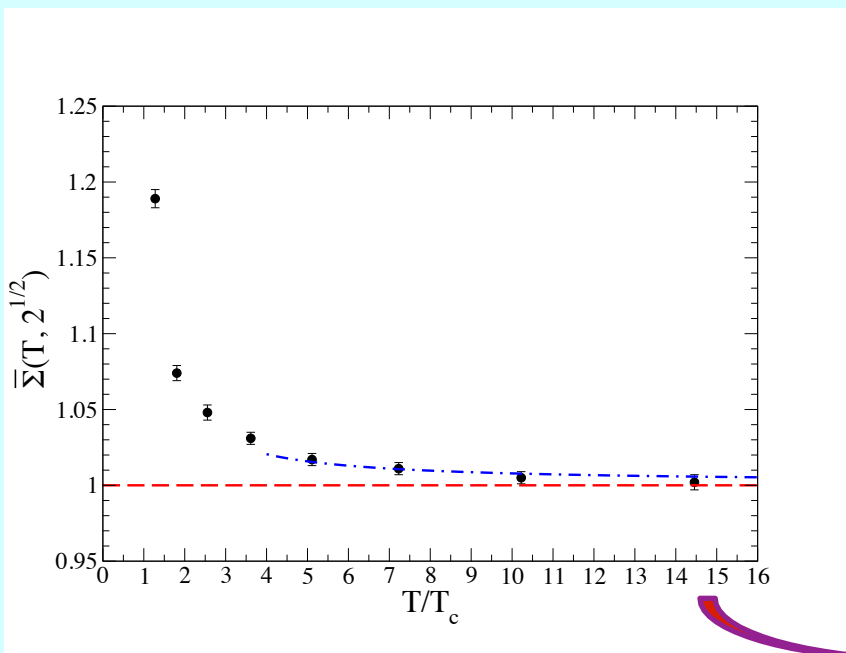
$$\Sigma(T, \sqrt{2}) = \frac{1}{8} \frac{\langle T_{0k} \rangle_{\xi=(1,0,0)}}{\langle T_{0k} \rangle_{\xi=(1,1,1)}}$$

$$T_{\mu\nu} = -\frac{2}{g_0^2} \left[\text{Tr} (F_{\mu\alpha}(x) F_{\nu\alpha}(x)) - \frac{1}{4} \delta_{\mu\nu} \text{Tr} (F_{\alpha\beta}(x) F_{\alpha\beta}(x)) \right]$$

$$F_{\mu\nu}(x) = \frac{1}{8} [Q_{\mu\nu}(x) - Q_{\nu\mu}(x)]_{\text{traceless}}$$

$$Q_{\mu\nu}(x) = \sum \text{Diagram}$$

L. Giusti, M. P.
Phys. Rev. Lett. (2014)



$$v_0(T_0) = \frac{s(T_0)}{T_0^3}$$

$$v_{k+1}(T_{k+1}) = \Sigma(T_k, \sqrt{2}) v_k(T_k)$$

$$T_k = (\sqrt{2})^k T_0$$

K.Kajantie, M. Laine, K.
Rummukainen and Y.
Schroder Phys. Rev. D (2003)

Strategy 2: direct measure of the entropy

We consider only one shift: $\xi=(1,0,0)$

$$\frac{s_0}{T_0^3} = -\frac{\beta^4(1 + \vec{\xi}^2)^3}{\xi_k} \langle T_{0k} \rangle_{\xi} Z_T$$

More appealing: no constraint on the temperature, any value is fine

- measurement of $\langle T_{0k} \rangle_{\xi}$: OK, no news.
- measurement of Z_T : need to know $Z_T(g_0^2)$ not only some values

conserved charge:
$$s_0 = -\frac{1}{V\gamma^3\xi_k} \frac{\partial}{\partial \xi_k} \log \mathcal{Z}(\beta, \vec{\xi})$$

$$Z_T = \frac{1}{V} \frac{1}{\langle T_{0k} \rangle_{\xi}} \frac{\partial \log \mathcal{Z}(\beta, \vec{\xi})}{\partial \xi_k} \quad \rightarrow \quad Z_T = \frac{1}{2V} \frac{1}{\langle T_{0k} \rangle_{\xi}} \log \frac{\mathcal{Z}(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{\mathcal{Z}(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})}$$

D. Robaina and H. Meyer,
PoS Lattice2013 (2013).

L. Giusti and M.P.,
Phys. Rev. Lett. 2014.

ratio of partition functions: very challenging to measure, hard for large volumes

$$\frac{Z_1}{Z_2} = \frac{Z_1}{Z_{1+\epsilon}} \frac{Z_{1+\epsilon}}{Z_{1+2\epsilon}} \frac{Z_{1+2\epsilon}}{Z_{1+3\epsilon}} \cdots \frac{Z_{2-2\epsilon}}{Z_{2-\epsilon}} \frac{Z_{2-\epsilon}}{Z_2}$$

P. de Forcrand, M. D'Elia, M. P.,
Phys. Rev. Lett. 2001.

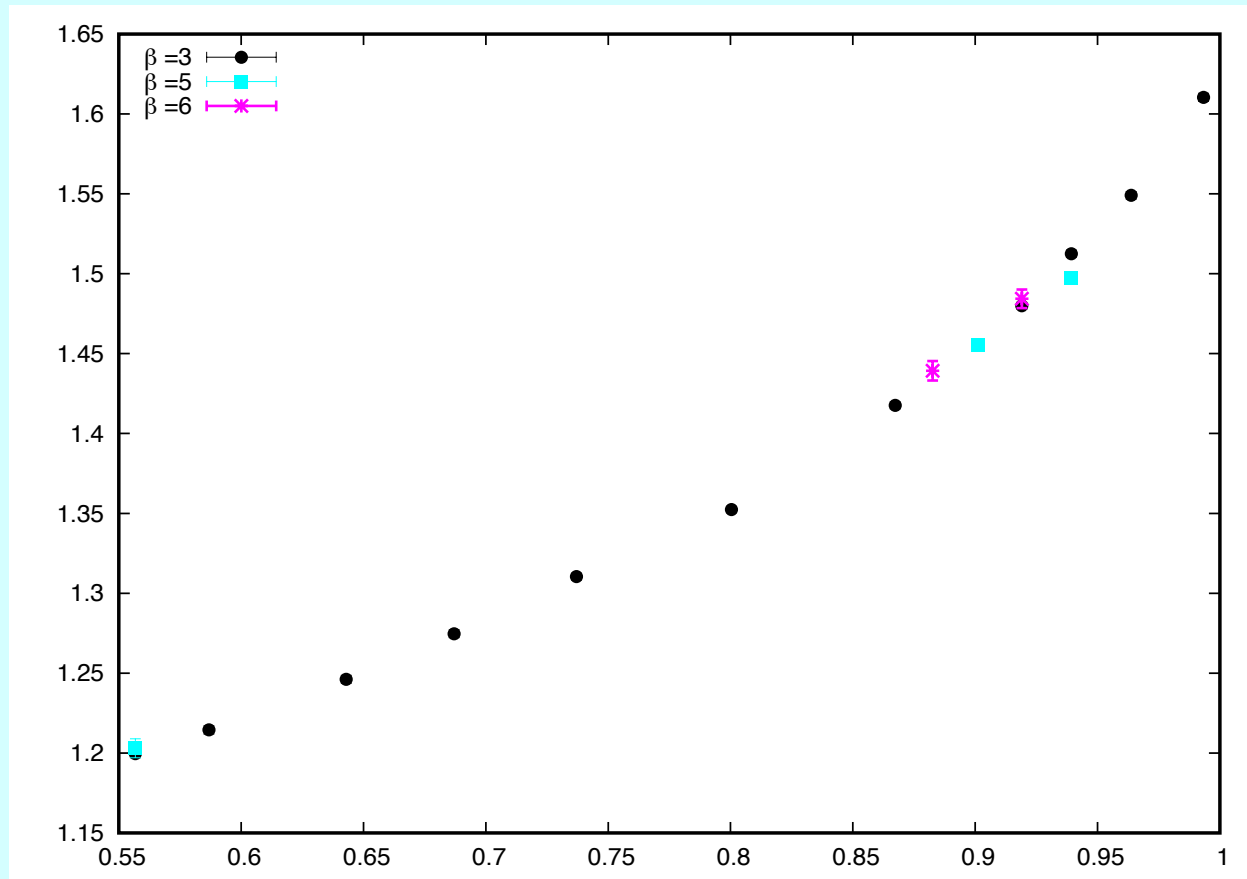
M. Della Morte, L. Giusti,
Comp. Phys. Comm. 2009.

$$S[U, r_i] = r_i S[U, \vec{\xi} + \frac{a}{\beta} \hat{k}] + (1 - r_i) S[U, \vec{\xi} - \frac{a}{\beta} \hat{k}] \quad \rightarrow \quad \log \frac{\mathcal{Z}(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{\mathcal{Z}(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = \sum_{i=0}^{n-1} \log \frac{\mathcal{Z}(\beta, r_i)}{\mathcal{Z}(\beta, r_{i+1})}$$

$$r_i = \frac{i}{n} \quad i = 0, \dots, n \quad \log \frac{\mathcal{Z}(\beta, r_i)}{\mathcal{Z}(\beta, r_{i+1})} = \langle \exp [S[U, r_{i+1}] - S[U, r_i]] \rangle_{r_i}$$

we have used the SU(3) Wilson action on a lattice with L=16

Z_T



g_0^2

The method works well but:

- increasing the spatial volume is very challenging
- we need $Z_T(g_0^2)$ accurately for many values of g_0^2

Compute Z in perturbation theory \rightarrow function of g_0^2

$$\frac{1}{2V} \log \frac{\mathcal{Z}(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{\mathcal{Z}(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = c_0 + c_1 g_0^2 + c_2 (g_0^2)^2 + \dots$$

$$Z_T = \frac{c_0 + c_1 g_0^2 + c_2 (g_0^2)^2 + \dots}{a_0 + a_1 g_0^2 + a_2 (g_0^2)^2 + \dots} - \frac{c_0}{a_0} + 1$$

$$\langle T_{0k} \rangle_{\xi} = a_0 + a_1 g_0^2 + a_2 (g_0^2)^2 + \dots$$

G. Parisi and Y. Wu, *Sci. Sin.* (1981)

Numerical Stochastic Perturbation Theory

F. DiRenzo, E. Onofri, G. Marchesini, P. Marchesoni,
Nucl. Phys. B (1994)

$$\phi(x) \rightarrow \phi(x, \tau)$$

$$\phi(x, \tau) = \sum_{n=0} \phi_n(x, \tau) g_0^n$$

$$\frac{\partial \phi(x, \tau)}{\partial \tau} = -\frac{\delta S[\phi]}{\delta \phi(x, \tau)} + \eta(x, \tau)$$

equal order in $g_0 \rightarrow$ cascade of eqs. for ϕ_n

$$\mathcal{O}[\phi(x)] \rightarrow \mathcal{O}[\phi(x, \tau)] = \sum_{n=0} \mathcal{O}_n[\phi(x, \tau)] g_0^n \rightarrow \langle \mathcal{O} \rangle = \langle \mathcal{O}_0 \rangle + \langle \mathcal{O}_1 \rangle g_0 + \langle \mathcal{O}_2 \rangle g_0^2 + \dots$$

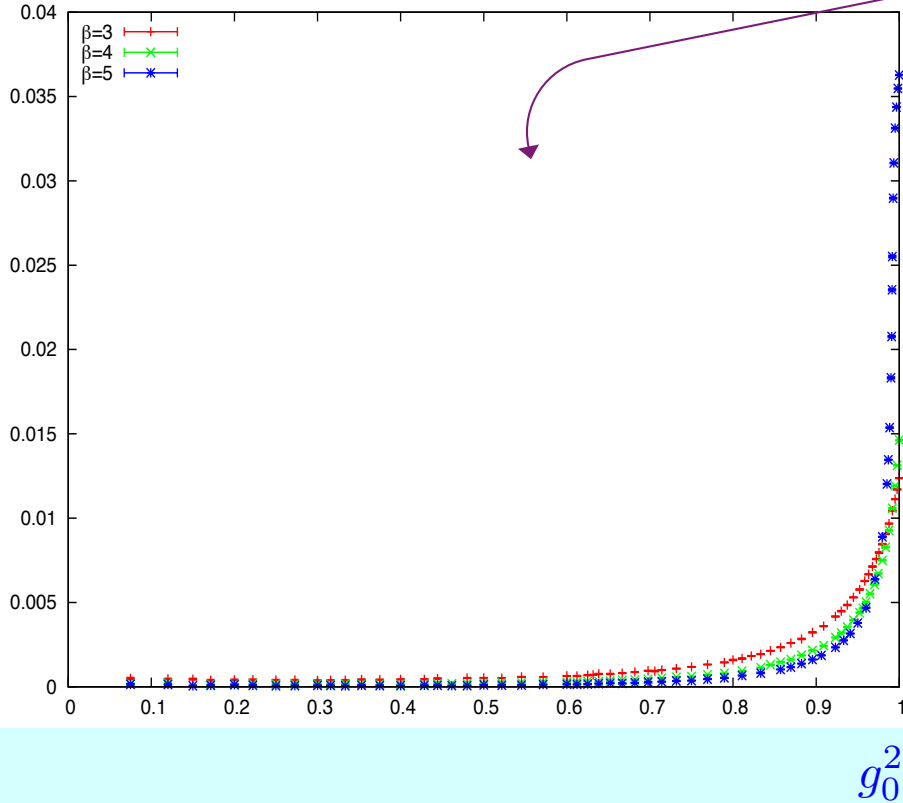
Calculation of: $a_0, a_1, a_2, c_0, c_1, c_2$ at $\beta=3$ and $L=24$ and 48

very strong finite size corrections

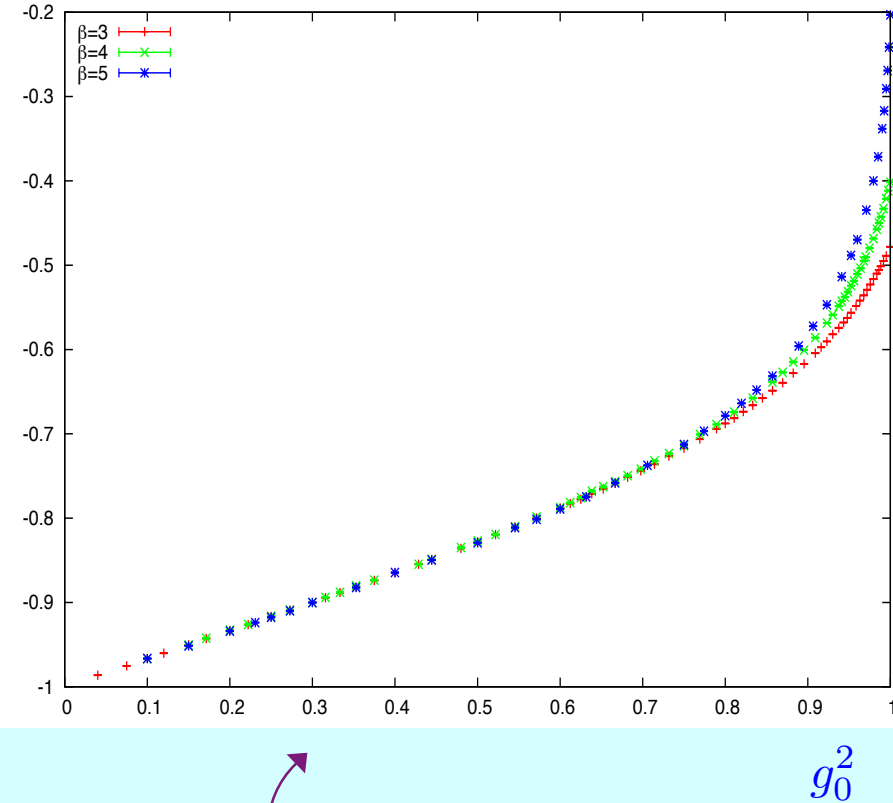
relevant higher order corrections

alternative approach: non-perturbative measure of Z + function of g_0^2

$$\frac{1}{2V} \log \frac{\mathcal{Z}(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{\mathcal{Z}(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = c_0 + \frac{1}{2V} \int_0^{g_0^2} dg^2 \frac{\partial}{\partial g^2} \log \frac{\mathcal{Z}(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{\mathcal{Z}(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = c_0 + \int_0^{g_0^2} dg^2 \frac{18\beta}{(g^2)^2} \left(\langle \square \rangle_{\vec{\xi} - \frac{a}{\beta} \hat{k}} - \langle \square \rangle_{\vec{\xi} + \frac{a}{\beta} \hat{k}} \right)$$

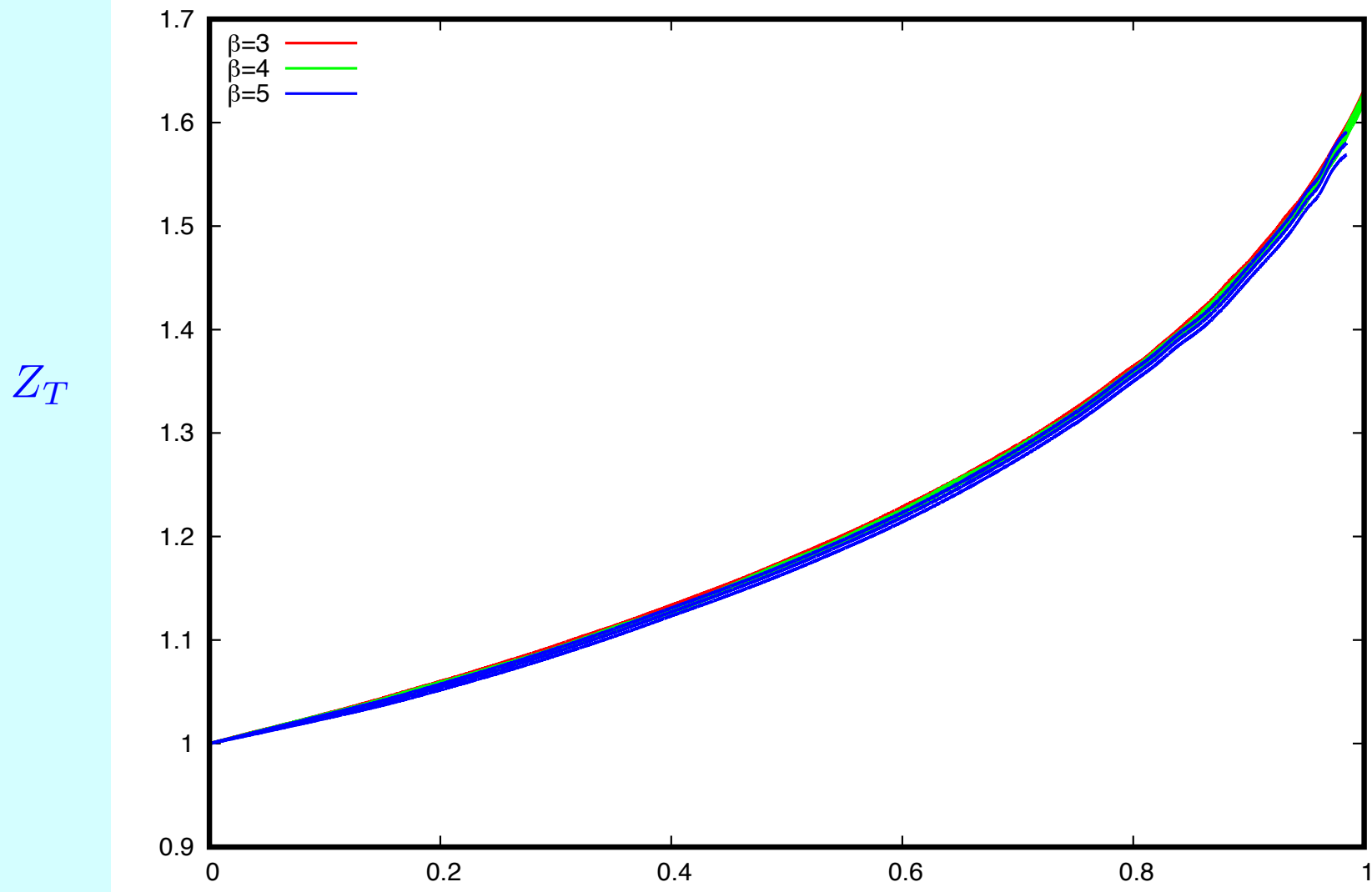


Spatial size $L=48$



$$Z_T = \frac{1}{2V} \frac{1}{\langle T_{0k} \rangle_\xi} \log \frac{\mathcal{Z}(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{\mathcal{Z}(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})}$$

non-perturbative calculation of Z_T



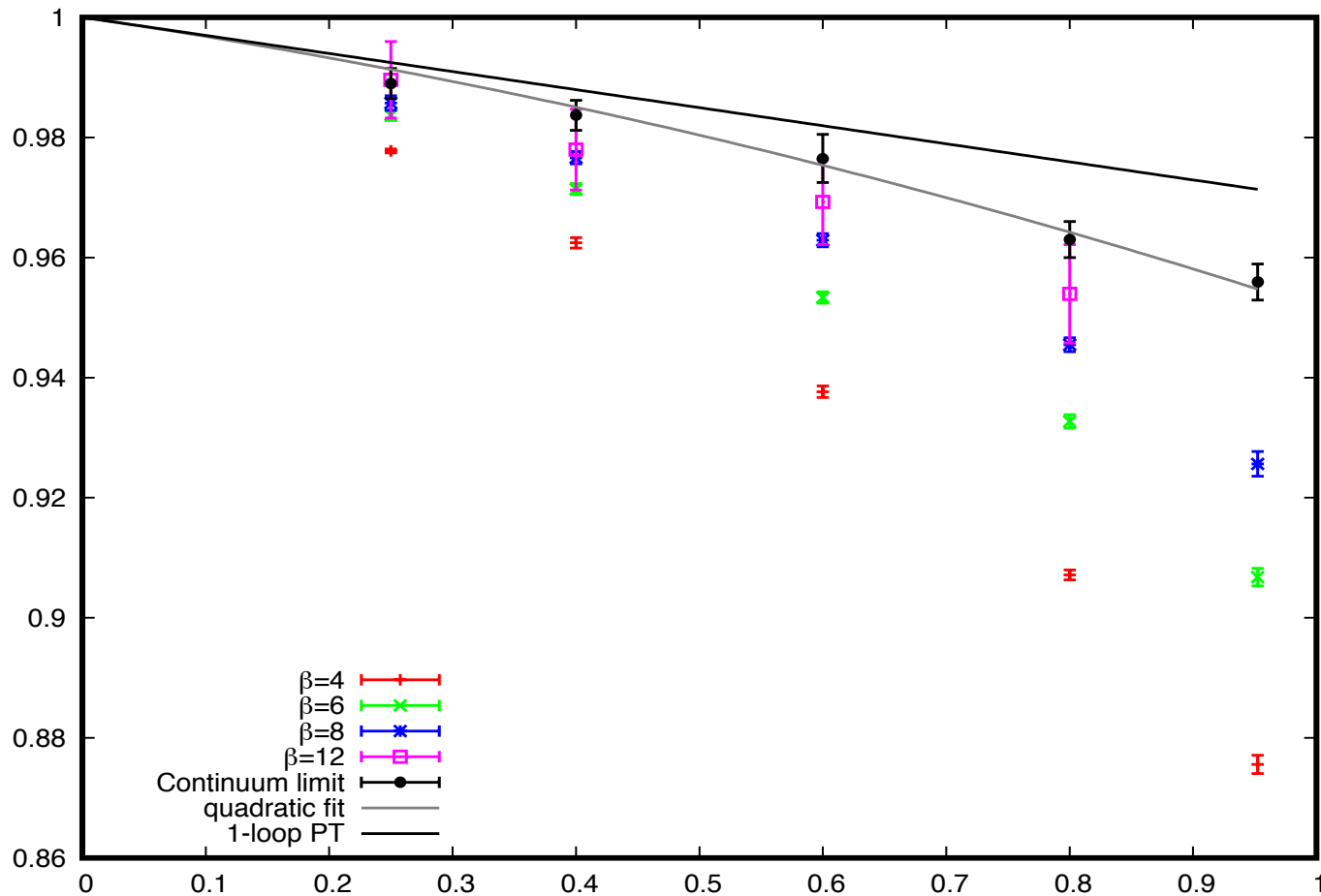
Uncertainty of order 5 per-mille

g_0^2

Non-perturbative calculation of z_T

We use consider the shift $\xi = (1/2, 0, 0)$ and the equation

$$\langle T_{0k} \rangle_\xi = \frac{\xi_k}{1 - \xi_k^2} \langle T_{00} - T_{kk} \rangle_\xi z_T$$



z_T

g_0^2

Conclusions

- A new approach to thermal field theories based on relativistic thermodynamics is investigated: far more efficient than the state-of-the-art techniques
- The step-scaling function of the entropy density has been obtained with quite moderate computational effort → entropy density vs. T with 5 per-mille accuracy
- We have shown measurements of Z_T , z_T with an accuracy of about 5 per-mille. This introduces also another method to compute the EoS.
- The approach used here is general and can be applied to any thermal field theory: in particular to QCD (extremely challenging with the standard techniques).
- Renormalization factor z_d of the trace-anomaly of the energy-momentum tensor: there is a simple equation → easy to compute numerically.

What next?

- Investigate the EoS in SU(3) YM with high accuracy using this new method
- Thermodynamics of QCD