MMNLP-Le - Mathematical Methods of Nonlinear Physics

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Mathematical Methods of Nonlinear Physics - MMNLP

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Keywords

nonlinear integrable partial differential/difference equations, singularities in hydrodynamical systems, classical and quantum continous/ discrete dynamical systems, nonlinear physics.

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Angelelli Mario (B. Konopelchenko)
? ? Assegnista

Attivitá

1. Boiti and Pempinelli

   Solution of the direct and inverse spectral problem for the KPII equation in the case of perturbed multi-soliton solutions (A. Pogrebkov, Steklov, Moscow).

2. Konopelchenko

   Multi-dimensional integrable systems associate with finite-dimensional Grassamannians (L.Bogdanov, Moscow, Russia)

   Hogomological and Poisson structures of Birkhoff strata of Sato Grassmannian (G.Ortenzi, Milano) Gradient catastrophies, Thom's catastrophies, instabilities in vortex filament dynamics, their regularization (G.Ortenzi, MIlano)

   Special Seiberg-Witten curves arising in gauge/string dualities (L. Martinez Alonso, Madrid, Spain)

3 L Martina

Symmetries ans integrable sectors in the Skyrme - Faddeev Models (M. Pavlov, Lebedev Inst Moscow),

Waves in NematoAcoustic (G. De Matteis (Newcastle, UK)),

Dual Monopoles and exotic symmeries (P Horvathy, Tours, France),

Symmetry invariant discretization and discrete surfaces (D. Levi , Rome 3; M Grundland and P Winternitz (Montreal, Canada))

4 Landolfi e Gianfreda

Integrability, Symmetries and Entanglement of Quantum Systems (Paris , MIlano,C. Bender, St. Louis, USA),

characterization of thermodynamical systems linked to the generalized Burgers equations. (A Moro, Newcastle, UK)

PT-symmetric systems and phase transitions (C. Bender, St. Louis, USA; I Barashenkov, Cape Town, South Africa; N. Hatano, Tokio, Japan))

5 Prinari e Vitale

Well-posedness issues of the IST for the defocusing scalar NLS when non-zero boundary conditions (F Demontis and C van der Mee Cagliari)

Initial-value problem for the defocusing NLS with piecewise constant initial conditions (G. Biondini, Buffalo)

Propagation of optical pulses in an optical medium with coherent three-level atomic transitions (M. Ablowitz , Boulder , USA)

6 Renna

Chaotic systems and applications.

Partecipanti esterni stranieri

1. Steklov Mathematical Institute of Moscow, Russia (A. Pogrebkov)
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11. CRM, Univ. de Montreal, (Qué) Canada (A.M. Grundland, P. Winternitz)
12. Washington University, St. Louis, (Mo USA) (C.M. Bender)
13. Northumbria University (Newcastle, UK) (A. Moro, G. De Matteis)

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5. Dip. Fisica, Univ. Roma III (Roma, Italia) (D. Levi)
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Pubblicazioni

1. L. V. Bogdanov and B. Konopelchenko, Grassmannians Gr (N-1,N+1), closed differential N-1 forms and N-dimensional integrable systems, J. Phys. A: Math Theor. 46 (2013) 085201
2. B. G. Konopelchenko, G. Ortenzi, Quasi-classical approximation in vortex filament dynamics. Integrable systems, gradient catastrophe and flutter, Stud. Appl. Math. , 130, 167-199 (2013).
3. F. Demontis, B. Prinari, C. van der Mee, F. Vitale: The inverse scattering transfom for the defocusing nonlinear Schroedinger equation with nonzero boundary conditions, Stud App Math, 40 (2013) 1
4. L V Bogdanov and B G Konopelchenko 2014 J. Phys.: Conf. Ser. 482 012005 doi:10.1088/1742-6596/482/1/012005
5. G. Dean, T. Klotz, B. Prinari and F. Vitale: Dark-dark and dark-bright soliton interactions in the two-component defocusing nonlinear Schroedinger equation, Applicable Analysis, 92 pp. 379-397 (2013)

6 S. Chakravarty, B. Prinari and M.J. Ablowitz, ŞInverse Scattering Transform for 3-level coupled Maxwell-Bloch equations with inhomogeneous broadeningŤ, *Physica* **D**, **278** pp. 58-78 (2014)

7 G. Biondini and B. Prinari, On the spectrum of the Dirac operator and the existence of discrete eigenvalues for the defocusing nonlinear Schroedinger equation, *Stud App Math*, **132** pp. 138-159 (2014)

8 M.Boiti, F. Pempinelli, A Pogrebkov: IST of KPII equation for perturbed multisoliton equations arXiv: 1212.6793, to appear in AMS Translations, Ser.2. (2014).

9 L. Martina, M.V. Pavlov and S. Zykov: Waves in the Skyrme–Faddeev model and Integrable reductions, J. Phys. A: Math. Theor. 46 275201 (2013)

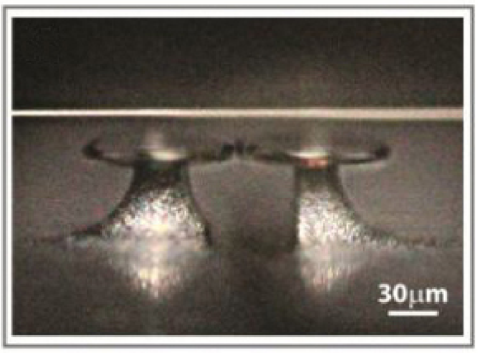
10 L. Martina, M.V. Pavlov : Magnetic domains and waves in the Skyrme - Faddeev model, J. Phys. Conf Ser. **482** (2014) 012031

11 C M Bender and M Gianfreda, Nonuniqueness of the operator in -symmetric quantum mechanics,J. Phys. A: Math. Theor. 46 (2013) 275306

12 C.M. Bender and M. Gianfreda, *Twofold transition in $\mathcal{PT}$-symmetric coupled oscillators*, Journ. Phys. A: Math. Theor. (2013).

13 Igor V Barashenkov, MG Gianfreda: An exactly solvable PT -symmetric dimer from a Hamiltonian system of nonlinear oscillators with gain and loss, J. Phys A: Math Theor **47** 2820001 (2014)

14 Bo Peng *et al* : Parity-Time symmetric whispering-gallery microcavities, *Nature Phys.* **10** 394 (April 2014)



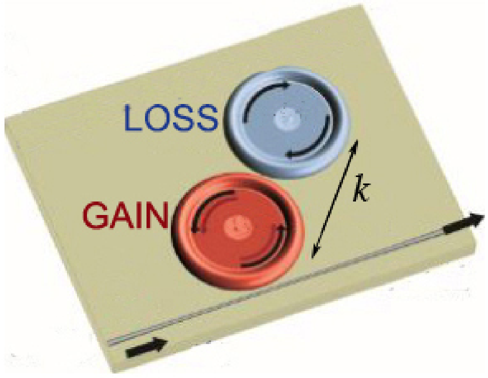


Figura : Foto di due microcavitAtoroidali chiamate whispering gallery mode e illustrazione schematica dell'apparato sperimentale: I due risonatori sono accoppiati tra loro con accoppiamento proporzionale alla distanza k; il risonatore attivo accoppiato ad una fibra ottica ed dopato con ioni di erbio (gain), mentre il risonatore passivo (loss) non contiene alcun dopante.

$$H = pq - \gamma(xp - yq) + (\omega^2 - \gamma^2)xy + k(x^2 + y^2)/2. \quad (1)$$

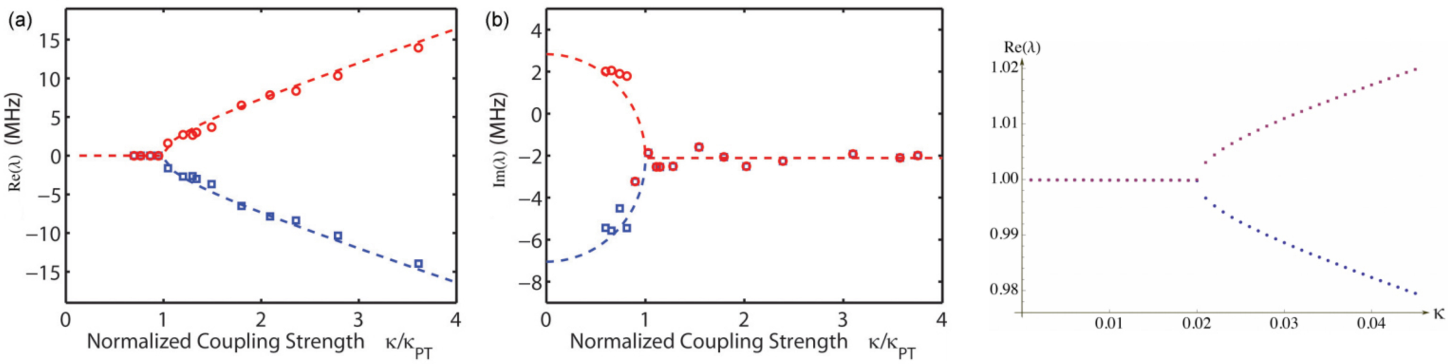


Figura : (sinistra): Plot della parte reale e immaginaria della frequenza λ nella soluzione $x = e^{i\lambda t}$ di (1) per γ vicino alla transizione di fase ad $\epsilon \approx 0.02$. (destra): Misure sperimentali

Invariant discretization

1. Discretization procedure of PDEs leaving invariant the Lie-point
2. Example: the Liouville equation

$$z_{xy} = e^z, \tag{2}$$

$$u\, u_{xy} - u_x\, u_y = u^3, \qquad u = e^z, \tag{3}$$

3. not use a preconceived constant lattice.
4. Construct an invariant set of equations defining both the lattice and system of difference equations.
5. The group acts on the solutions of the equation and on the lattice.
6. The study of symmetries of genuinely discrete phenomena.
7. The third aspect of this program fits into the general field of **geometrical integration** .
8. Improve numerical methods of solving specific ordinary and partial differential equations, by incorporating important qualitative features of these equations into their discretization. Such features may be point symmetries, integrability, linearizability, Lagrangian or Hamiltonian formulation.

$$z = \ln 2 \frac{\phi_x \phi_y}{\phi^2}, \qquad \phi_{x\,y} = 0.$$

$$z = \ln 2 \frac{\phi_{1,x} \phi_{2,y}}{(\phi_1 + \phi_2)^2}.$$

$\phi(x,y) = \phi_1(x) + \phi_2(y)$
$\mathrm{X}(f(x)) = f(x)\,\partial_x - f_x(x)\; u\,\partial_u,$
$\mathrm{Y}(g(y)) = g(y)\,\partial_y - g_y(y)\; u\,\partial_u,$

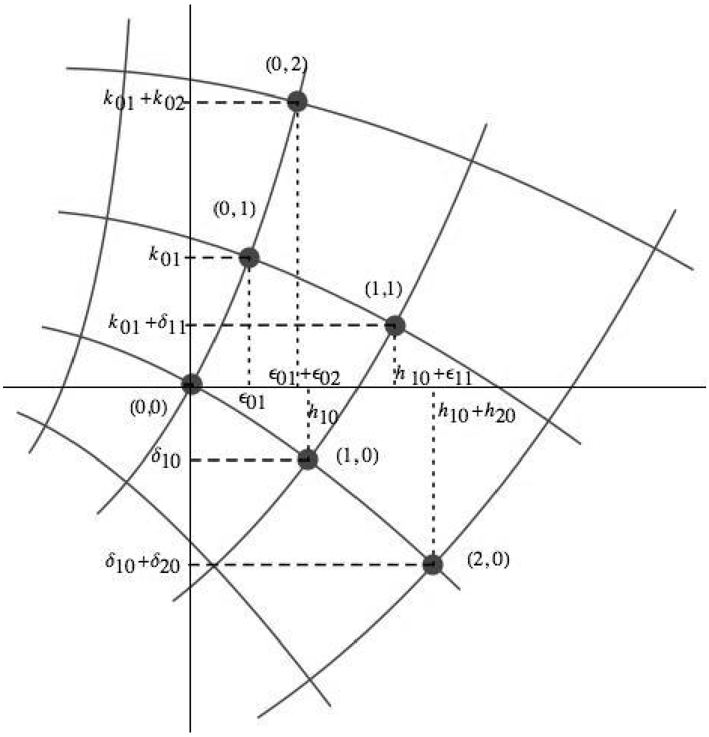
$$\left[X(f), X(\tilde{f})\right] = X\left(f\tilde{f}_x - \tilde{f} f_x\right), \quad [Y(g), Y(\tilde{g})] = Y(g\,\tilde{g}_y - \tilde{g}\,g_y), \quad [X(f$$

$\mathrm{pr}^{(2)} X(f) = f\partial_x - f' u\,\partial_u + 2u_x\,\partial_{u_x} + u_y\,\partial_{u_y} + 2u_{xy}\,\partial_{u_{xy}} + 3u_{xx}\,\partial_{u_{xx}} + u_{yy}\,\partial_{u_{yy}}$
- f'' u $\partial_{u_x} + u_y \partial_{u_{xy}} + 3u_x \partial_{u_{xx}} - f''' u \partial_{u_{xx}}$

$$\mathrm{pr}^{(2)} X(f)\; I_1 = \mathrm{pr}^{(2)} Y(g)\; I_1 = 0$$

$$\mathrm{pr}^{(2)} X(f)\; I_2 = \frac{2f_{xxx}\left(3u_y^2 - 2uu_{yy}\right)}{4}, \quad \mathrm{pr}^{(2)} Y(g)\; I_2 = \frac{2g_{yyy}\left(3u_x^2 - 2uu_{xx}\right)}{4}.$$

$$E_\alpha\left(x_{m+i,n+j}, y_{m+i,n+j}, u_{m+i,n+j}\right) = 0,$$
$$\alpha = 1, \ldots, N, \quad i_{min} \le i \le i_{max}, \quad j_{min} \le j \le j_{max}.$$



4-point stencil

$s_4^0 \equiv \{(0,0), (0,1), (1,0), (1,1)\}$

$$\hat{Z} = \xi(x,y,u)\partial_x + \eta(x,y,u)\partial_y + \phi(x,y,u)\partial_u \qquad (4)$$

$$\text{pr}\hat{Z} = \sum_{i,j}(\xi_{i,j}\partial_{x_{i,j}} + \eta_{i,j}\partial_{y_{i,j}} + \phi_{i,j}\partial_{u_{i,j}}).$$

pr $\mathrm{I}_s = 0$,
pr $\mathrm{I}_w \,|_{I_w=0} = 0$,
$X^D(f) = \mathrm{pr}\, X(f) = \sum_{(m,n)\in s_4^{m,n}} \left[f(x_{m,n})\,\partial_{x_{mn}} - f'(x_{m,n})\; u_{mn}\,\partial_{u_{mn}}\right],$
$Y^D(g) = \mathrm{pr}\, Y(g) = \sum_{(m,n)\in s_4^{m,n}} \left[g(y_{m,n})\,\partial_{y_{mn}} - \dot{g}(y_{m,n})\; u_{mn}\,\partial_{u_{mn}}\right].$

4-point stencil invariants

$$\xi_1 = \frac{(x_{0,1}-x_{0,0})(x_{1,1}-x_{1,0})}{(x_{0,0}-x_{1,0})(x_{0,1}-x_{1,1})} = \frac{\epsilon_{0,1}\epsilon_{1,1}}{h_{1,0}(h_{1,0}+\epsilon_{1,1}-\epsilon_{0,1})},$$
$$\eta_1 = \frac{(y_{0,0}-y_{1,0})(y_{0,1}-y_{1,1})}{(y_{0,1}-y_{0,0})(y_{1,1}-y_{1,0})} = \frac{\delta_{1,0}\delta_{1,1}}{k_{0,1}(k_{0,1}+\delta_{1,1}-\delta_{1,0})}$$
$$H_1 = u_{0,0}u_{0,1}\epsilon_{0,1}^2 k_{0,1}^2$$
$$H_2 = u_{1,0}u_{1,1}\epsilon_{1,1}^2(k_{0,1}+\delta_{1,1}-\delta_{1,0})^2$$

$$H_3 = \frac{u_{1,0}(h_{1,0}-\epsilon_{0,1})^2(k_{0,1}-\delta_{1,0})^2}{u_{0,0}\,\epsilon_{0,1}^2\,k_{0,1}^2}$$

$$H_4 = \frac{u_{1,1}\epsilon_{1,1}^2(k_{0,1}-\delta_{1,1}-\delta_{1,0})^2}{u_{0,0}\,h_{1,0}^2\,\delta_{1,0}^2}$$

$$\xi_1 = 0, \qquad \eta_1 = 0. \tag{4}$$

$$x_{m,n} = x_m, \qquad y_{m,n} = y_n, \tag{4}$$

$$\mathrm{X}^D(x^3)\xi_1 = (x_{1,1}-x_{0,0})(x_{1,0}-x_{0,1})\xi_1 |_{\xi_1=0} = 0$$
$$\mathrm{X}^D(x^3)\eta_1 = 0.$$

$$J_1 = H_1 H_3 = u_{0,1} u_{1,0} h_{1,0}^2 k_{0,1}^2,$$
$$J_2 = \frac{1}{\xi_1^2} \frac{H_2}{H_3} = u_{0,0} u_{1,1} h_{1,0}^2 k_{0,1}^2.$$

Discretized Liouville Equation

$$J_2 - J_1 = a|\, J_1\,|^{3/2} + bJ_1|\, J_2\,|^{1/2} + c|\, J_1\,|^{1/2} J_2 + d|\, J_2\,|^{3/2},$$
$$\xi_1 = 0, \quad \eta_1 = 0, \qquad a + b + c + d = 1.$$

$$J_2 - J_1 - \left[aJ_1^{3/2} + bJ_1 I_2^{1/2} + cJ_1^{1/2} J_2 + dJ_2^{3/2}\right] =$$
$$= h_{1,0}^3 k_{0,1}^3 \left[uu_{xy} - u_x u_y - u^3\right] + h_{1,0}^4 k_{0,1}^3 \left[\tfrac{1}{2} u_y u_{xx}(u-1) - \tfrac{3}{2} u^2 u_x\right] +$$
$$+ h_{1,0}^3 k_{0,1}^4 \left[\tfrac{1}{2} u_x u_{yy}(u-1) - \tfrac{3}{2} u^2 u_y\right] + \mathcal{O}(h_{1,0}^4 k_{0,1}^4),$$

Numerics

Standard discretization

$$u_{1,1}u_{0,0} - u_{0,1}u_{1,0} = hk\ u_{0,0}^3. \tag{5}$$

Symmetry invariant discretization

$$u_{1,1} = \frac{u_{0,1}u_{1,0}\left(ahk\sqrt{u_{0,1}u_{1,0}}+1\right)}{u_{0,0}\left((a-1)hk\sqrt{u_{0,1}u_{1,0}}+1\right)} \qquad (a \neq 0,1)\,, \tag{6}$$

$$s_1 = \frac{2\beta\gamma\delta}{(\beta^2x^2+1)(\delta^2y^2+1)(\tan^{-1}(\beta x)+\gamma\tan^{-1}(dy)+\alpha)^2},$$

$$s_2 = \frac{2As^2e^{s(x+y)}}{(Ae^{sy}+e^{sx})^2},$$

$$s_3 = \frac{8\left(1-4\left(x+\frac{1}{2}\right)\right)(1-4y)\exp\left(-4\left(x+\frac{1}{2}\right)^2+2\left(x+\frac{1}{2}\right)-4y^2+2y\right)}{\left(e^{2\left(x+\frac{1}{2}\right)-4\left(x+\frac{1}{2}\right)^2}+e^{2y-4y^2}+1\right)^2},$$

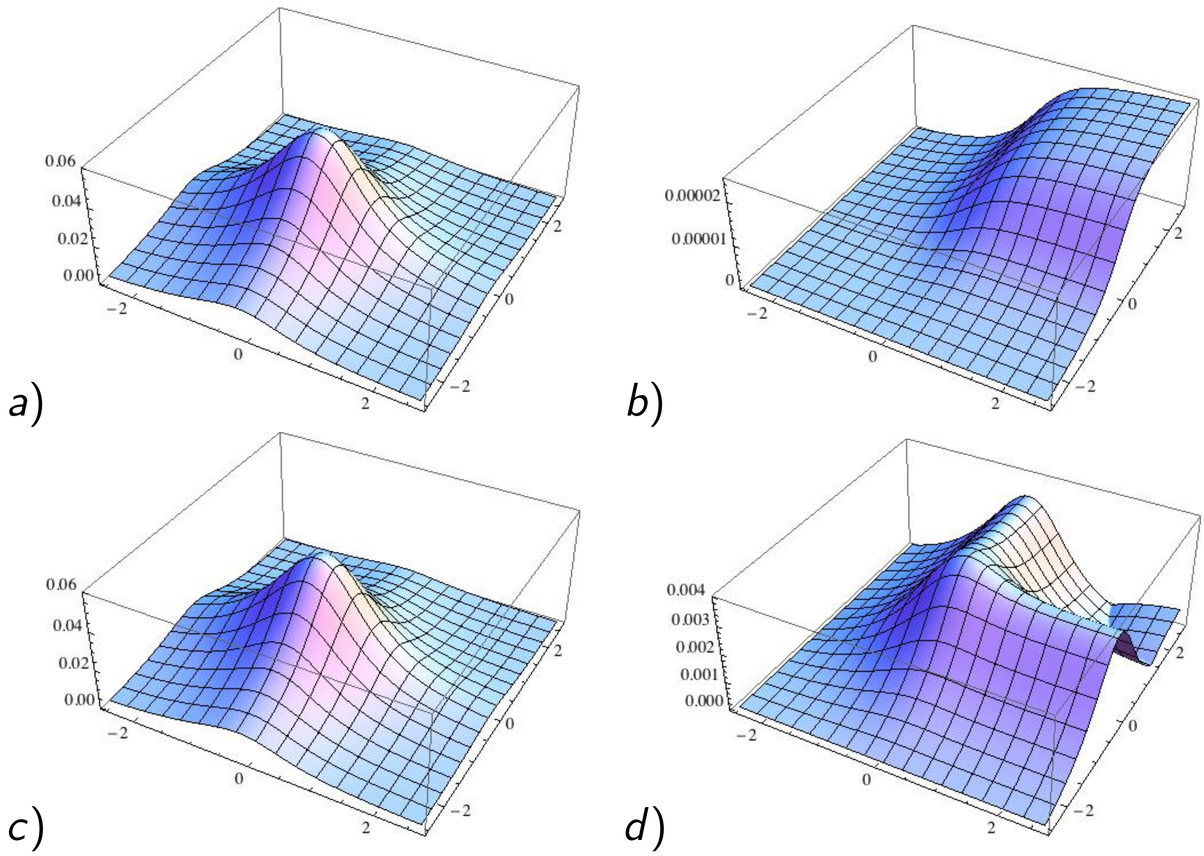

0.06
0.04
0.02
0.00
−2
0
2
a)
0.00002
0.00001
0
b)
0.004
0.003
0.002
0.001
0.000
c)
d)

Figura :

	χ_{Inv}	χ_{stand}
s_1	6.4×10^{-16}	7.2×10^{-5}
s_2	1.6×10^{-7}	7.0×10^{-1}
s_3	1.7×10^{-2}	6.0×10^{-1}

9-point scheme

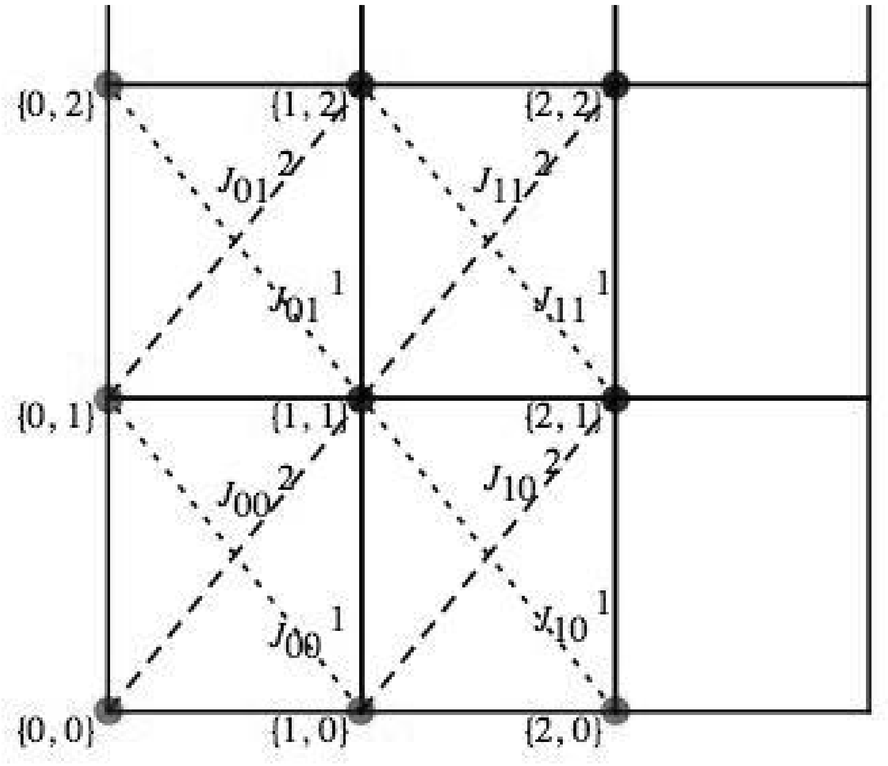


Figura :

$$J^1_{0,0}\, J^1_{1,1} \;=\; J^2_{0,1}\, J^2_{1,0},$$

(4 A+3 B)
$\Delta = (J^1_{0,0} - J^2_{0,0}) + A(J^2_{0,1} - J^1_{0,1}) + A(J^2_{1,0} - J^1_{1,0}) + B(J^2_{1,1} - J^1_{1,1}) -$
$\sum_{m,m',n,n'=0,1}{}'_{\alpha,\beta=1,2}\; a^{\alpha,\beta}_{m,n,m',n'}\; J^{\alpha}_{m,n} \left(J^{\beta}_{m',n'}\right)^{\frac{1}{2}}$

$$\Delta \to -2\,(A+B)\left(u\,u_{x\,y} - u_x\,u_y - u^3\right) +$$

$\mathrm{h}^2 q_1 + k^2 q_2 + hk\, q_3 + O\left(h^3, \ldots\right), \}$

Symmetries in Nonlinear models: Vortices and Waves

Physical Origin

1. Spin-Charge Separation of the pure Yang-Mills theory in Infrared background
2. $^{3}He - A$ superfluid $(M_L = 1, M_S = 0)$
3. 2-band superconductor (Nb-doped $SrTiO_3$, MgB_2)
4. charged condensates of tightly bounded fermion pairs

- Stability of the order parameter configurations
- Knotted and/or linked quasi-1-dimensional configurations
- Coexistence/Competition of short/long (UV/IR) wave modes
- Properties of knots and tangles
- Topological ordering in disordered background

The Skyrme Faddeev- Model

$$E[\vec{n}] = \int_{\mathbb{R}^3} \left\{ (\partial_a \vec{n})^2 + \left(\frac{1}{2} (\vec{n} \cdot \partial_a \vec{n} \times \partial_b \vec{n}) \right)^2 \right\} d^3x,$$

$$\partial_a^2 \vec{n} - (\partial_a \mathcal{F}_{ab}) (\vec{n} \times \partial_b \vec{n}) = \left(\vec{n} \cdot \partial_a^2 \vec{n} \right) \vec{n}.$$

$$\lim_{|\vec{x}| \to \infty} \vec{n}(\vec{x}) = \vec{n}_\infty = \pm \vec{z} \Rightarrow \vec{n} : \mathbb{S}^3 \to \mathbb{S}^2, \quad O(3) \hookrightarrow O(2)$$

$$E[\vec{n}] \geq c\, |N[\vec{n}]|^{3/4}, \quad c \approx (3/16)^{3/8}$$

hedgehog solution

$$\vec{n} \cdot \vec{\sigma} = U(\vec{n}_\infty \cdot \vec{\sigma}) U^\dagger$$

$$U = \exp[i\chi(r)\, \vec{\nu}(\vartheta, \varphi) \cdot \vec{\sigma}] = \cos\chi(r)\, I + i \sin\chi(r)\, \vec{\nu}(\vartheta, \varphi) \cdot \vec{\sigma}$$

Approximated solutions by rational f.

$$g_{rat}(r) = \frac{1 + a_1 r + a_2 r^2}{1 + a_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4},$$

$$a_1 = 0.216, \quad a_2 = 0.230, \quad b_2 = 0.752, \quad b_3 = -0.018, \quad b_4 = 0.302,$$

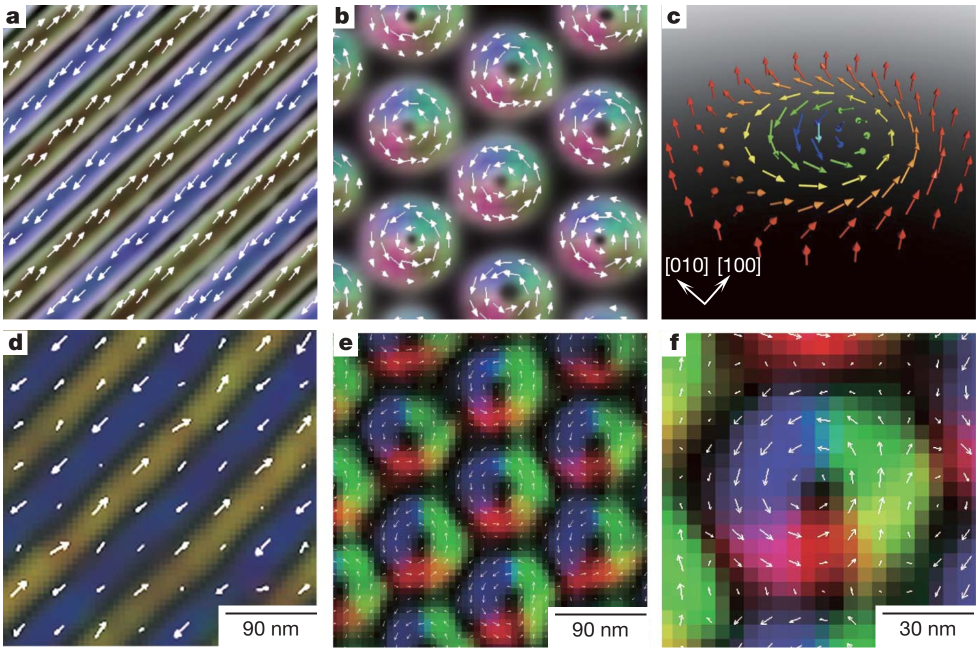


Figure 1 | Topological spin textures in the helical magnet $Fe_{0.5}Co_{0.5}Si$. **a**, **b**, Helical (**a**) and skyrmion (**b**) structures predicted by Monte Carlo simulation. **c**, Schematic of the spin configuration in a skyrmion. **d–f**, The experimentally observed real-space images of the spin texture, represented by the lateral magnetization distribution as obtained by TIE analysis of the Lorentz TEM data: helical structure at zero magnetic field (**d**), the crystal (SkX) structure for a weak magnetic field (50 mT) applied the thin plate (**e**) and a magnified view of **e** (**f**). The colour map arrows represent the magnetization direction at each point.

X. Z. Yu, Y. Onose, N. Kanazawa, J. H. Park, J. H. Han, Y. Matsui, N.

Relativistic Skyrme - Faddeev

$g_{\mu\nu} = \text{diag}\,(+,-,-,-)$

$$\mathcal{L} = \frac{1}{32\pi^2}\left(\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{\lambda}{4}\left(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi}\right)\cdot\left(\partial^\mu\vec{\phi}\times\partial^\nu\vec{\phi}\right)\right) - \kappa\left(1-\vec{\phi}\cdot\vec{\phi}\right), \tag{7}$$

Polar representation

$$\vec{\phi} = (\sin w\cos u, \sin w\sin u, \cos w)\,, \tag{8}$$

Euler - Lagrange Equations

$$\begin{aligned}\partial_\mu w^\mu &= \tfrac{1}{2}\sin(2w)u_\nu u^\nu + \tfrac{\lambda}{2}\sin w\;u_\nu\,\partial_\mu[\sin w(w^\mu u^\nu - w^\nu u^\mu)],\\ w_\mu u^\mu\sin(2w) &+ \sin^2 w[\partial_\mu u^\mu + \tfrac{\lambda}{2}w_\nu\partial_\mu(u^\mu w^\nu - u^\nu w^\mu)] = 0.\end{aligned} \tag{9}$$

The d'Alembert-homogeneous Eikonal reduction: $w = const$

$$\partial_\mu u^\mu = 0, \qquad u_\nu u^\nu = 0,$$

$$G\left(u, A_\mu\left(u\right)x^\mu, B_\mu\left(u\right)x^\mu\right) = 0, \qquad A_\mu A^\mu = B_\mu B^\mu = A_\mu B^\mu = 0,$$

with G, A_μ and B_μ arbitrary real regular functions.

Orthogonality reduction

By imposing

$$w_\mu u^\mu = 0, \qquad u_\nu u^\nu = \alpha \quad (\alpha = \text{ constant } \in \mathbb{R})\,.$$

the system reduces to the equations

$$\partial_\mu u^\mu = 0, \qquad u_\nu u^\nu = \alpha,$$
$$w_\mu u^\mu = 0, \qquad \partial_\mu w^\mu = \frac{\alpha}{2}\frac{\sin(2w)}{1 - \frac{\lambda\,\alpha}{2}\sin^2 w}(1 + \frac{\lambda}{2} w^\mu w_\mu),$$

which are highly nonlinear for the w field.

General solution of the d'Alembert-Eikonal system

1. Fushchich V I, Zhdanov R Z and Revenko I V 1991 General solutions of the nonlinear wave equation and of the eikonal equation *Ukr. Mat. Z.* **43** 1471-1487;
2. Zhdanov R Z, Revenko I V and Fushchich V I 1995 On the general solution of the d'Alembert equation with a nonlinear eikonal constraint and its applications

$$
\begin{aligned}
u &= A_\mu(\tau)\,x^\mu + R_1(\tau)\,,\\
& \quad B_\mu(\tau)\,x^\mu + R_2(\tau) = 0,\\
A_\mu A^\mu &= \alpha, \quad A_\mu B^\mu = A'_\mu B^\mu = B_\mu B^\mu = 0,
\end{aligned}
$$

Then, for $\alpha = -\eta^2$, the general solution is

$$u = x_k A_k(\tau) + A_0(\tau), \quad t = x_k B_k(\tau) + B_0(\tau),$$

$$A_1 = \eta\cos(f(\tau))\sin(g(\tau)), A_2 = \eta\sin(f(\tau))\sin(g(\tau)), A_3 = \eta\cos(g(\tau)),$$

being $f(\tau)$ and $g(\tau)$ arbitrary functions.

The reduced Skyrme–Faddeev system

By setting to zero the coefficients of all functions of w

$$w_\mu u^\mu = 0, \quad w_\mu w^\mu = -\frac{2}{\lambda}, \quad \partial_\mu w^\mu = 0,$$

$$\partial_\mu u^\mu + \frac{\lambda}{2} w_\nu \partial_\mu (u^\mu w^\nu - u^\nu w^\mu) = 0, \quad u_\nu \partial_\mu (w^\mu u^\nu - w^\nu u^\mu) = 0,$$

quasilinear system in (u^μ, w^μ)

$$\partial_\mu w^\mu = 0, \qquad w_\mu w^\mu = -\epsilon^2, \qquad u_\mu w^\mu = 0, \tag{10}$$

d'Alembert-Eikonal $\quad (u \to w, \alpha \to -\epsilon^2) \quad$ orthogonality condition

$$u_\nu \partial_\mu (w^\mu u^\nu - w^\nu u^\mu) = 0, \quad \epsilon^2 \partial_\mu u^\mu + w_\nu \partial_\mu (u^\mu w^\nu - u^\nu w^\mu) = 0,$$

$$\Updownarrow \; a_\mu w^\mu = 0 \quad \text{with} \quad a = u^\nu u_\nu \qquad \text{identity} \tag{11}$$

where $\epsilon^2 = \frac{2}{\lambda}$. Compatibility condition is the Monge-Ampére equation

$$\mathrm{Det}\,[w_{ij}] = 0, \tag{12}$$

Compatibility

(1+2)-dim

$$\Rightarrow u = F\left[w_1, w_2\right], \quad a \equiv 0 \quad \forall w$$

(1+3)-dim

$$\Rightarrow u = F\left[w_1, w_2, w_3\right]$$

$$(x_m B'_m(\tau) + B'_0(\tau))d\tau = dt - B_k(\tau)dx_k, \quad X_k = x_k,$$
$$[X_m(B'_m(\tau)A_p(\tau) - A'_m(\tau)B_p(\tau)) + B'_0(\tau)A_p(\tau) - A'_0(\tau)B_p(\tau)]u_{X_p} = 0$$

Phase - pseudo-phase Solutions

$$w = \Theta\left[\theta\right], \; u = \Phi\left[\theta\right] + \tilde{\theta}, \; \text{where} \;\; \theta = \alpha_\mu x^\mu, \tilde{\theta} = \beta_\mu x^\mu$$

A 3-parametric family of equations

$$\left[2B_3 - \frac{\lambda}{4}\mathcal{B}\sin^2\Theta\right]\Theta_{\theta\theta} = \sin 2\Theta\left(\frac{\lambda}{8}\mathcal{B}\,\Theta_\theta^2 + B_3\Phi_\theta^2 + B_2\Phi_\theta + B_1\right) \quad (13)$$

$$2B_3\sin^2\Theta\;\Phi_{\theta\theta} + \Theta_\theta\sin 2\Theta\left(2B_3\Phi_\theta + B_2\right) = 0, \quad (14)$$

where $B_1 = -\beta_\mu\beta^\mu$, $B_2 = -2\alpha_\mu\beta^\mu$ $B_3 = -\alpha_\mu\alpha^\mu$ and $\mathcal{B} = B_2^2 - 4B_1B_3$.

Conservation laws

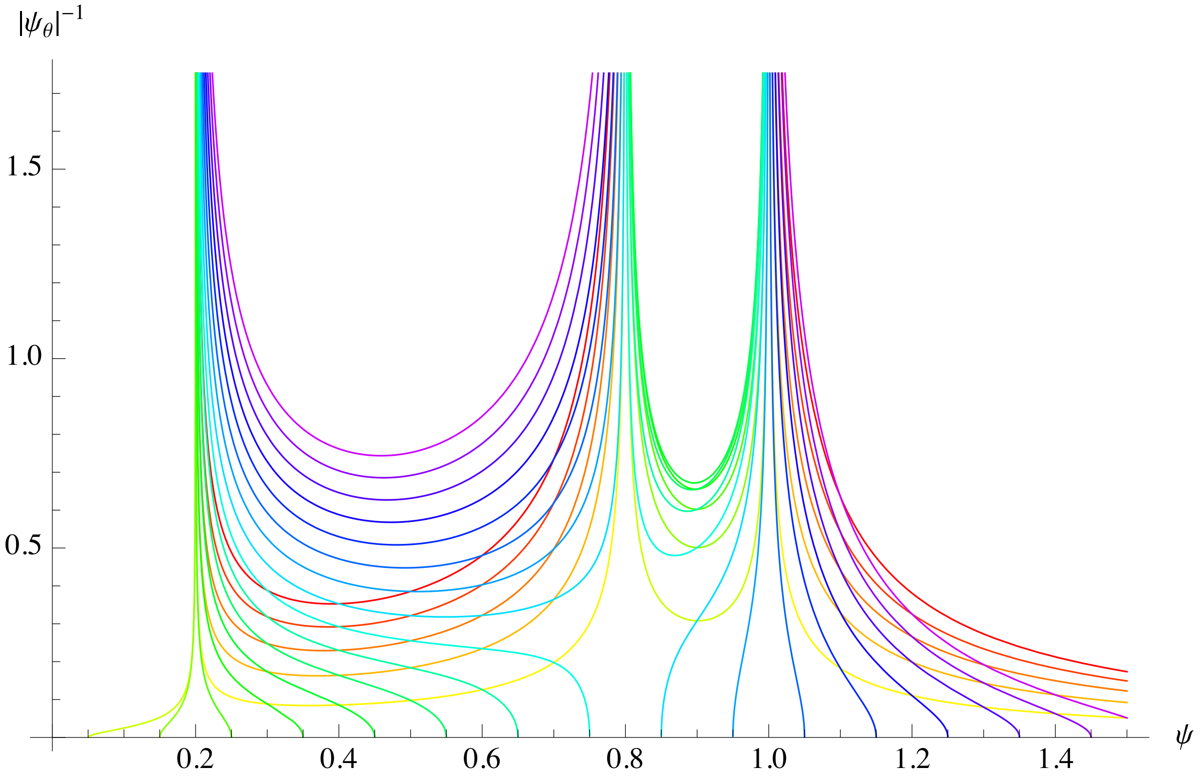
$$\mathcal{E}^0 = \frac{-1}{32\pi^2}\left\{B_3\alpha_0\Theta_\theta^2 + \sin^2\Theta\left[2\vec{\alpha}\cdot\vec{\beta}\,\beta_0 + \left(B_1 - 2\vec{\beta}^2\right)\alpha_0 + B_3\left(2\beta_0 + \alpha_0\Phi_\theta\right)\Phi_\theta - \frac{\lambda\mathcal{B}}{8}\alpha_0\Theta_\theta^2\right]\right\},$$

$$\mathcal{E}^i = \frac{-1}{32\pi^2}\left\{B_3\alpha_i\Theta_\theta^2 + \sin^2\Theta\left[B_2\beta_i - B_1\alpha_i + B_3\left(2\beta_i + \alpha_i\Phi_\theta\right)\Phi_\theta - \frac{\lambda\mathcal{B}}{8}c\right.\right.$$

$B_3 \neq 0$ and substitution

$$\Theta = \arcsin\sqrt{\psi},$$

$$\psi_\theta^2 = \frac{64(\psi-1)(\psi-A_1)(\psi-A_2)}{\lambda^2\mathcal{B}\psi_1(\psi_1-\psi)}.$$


$|\psi_\theta|^{-1}$
1.5
1.0
0.5
0.2
0.4
0.6
0.8
1.0
1.2
1.4
ψ

Parametric form solutions

$$\theta(\psi) = \theta_0 + \frac{1}{4}\sqrt{\frac{\mathcal{B}\lambda^2\psi_1(\psi_1 - A_1)^2}{(A_1-1)(A_2-\psi_1)}}\Pi\left[\frac{A_1-A_2}{\psi_1-A_2}; Z|\frac{(\psi_1-1)(A_1-A_2)}{(A_1-1)(\psi_1-A_2)}\right],$$

$$\psi = -\frac{A_2\psi_1\sin^2 Z + A_1\left(\psi_1\cos^2 Z - A_2\right)}{A_1\sin^2 Z + A_2\cos^2 Z + \psi_1}$$

$$\begin{aligned}\Phi &= -\frac{B_2U_2}{2B_3}\left[\int\frac{d\theta}{\psi(\theta)} + \theta\right] + \Phi_0 = \\ &- \frac{s_1}{2\psi_1}\left[\sqrt{\frac{2\psi_1(A_1-\psi_1)^2(B_1\lambda\psi_1+2)}{(A_1-1)(A_2-\psi_1)}}\Pi\left(\frac{A_2-A_1}{A_2-\psi_1}; Z\left|\frac{(A_1-A_2)(\psi_1-1)}{(A_1-1)(\psi_1-A_2)}\right.\right)\right. \\ &\left.+2s_2\sqrt{\frac{A_2\psi_1(A_1-\psi_1)^2}{A_1(A_1-1)(A_2-\psi_1)}}\Pi\left(\frac{(A_1-A_2)\psi_1}{A_1(\psi_1-A_2)}; Z\left|\frac{(A_1-A_2)(\psi_1-1)}{(A_1-1)(\psi_1-A_2)}\right.\right)\right],\end{aligned}$$

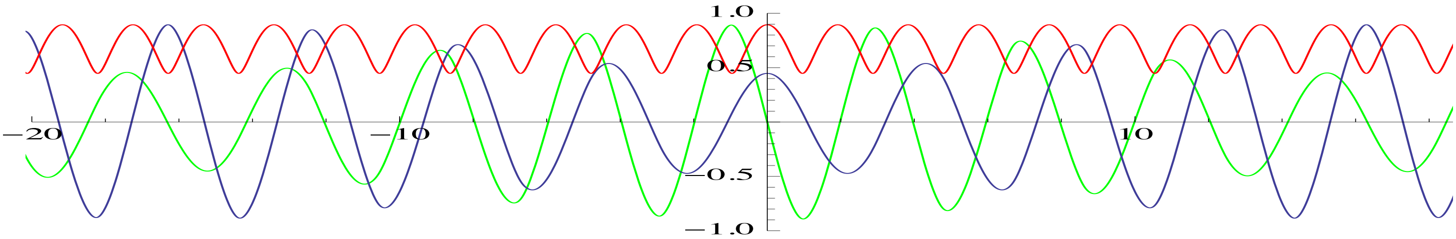


Figura : The graphic for the ϕ_1 (green), ϕ_2 (blue) and ϕ_3 () as function of x^3 for a choice of the parameters
$A_1 = 0.2, A_2 = 0.8, \psi_1 = 0.9, \mathcal{B} = 1, \lambda = 1, B_1 = 1, s_1 = -1, s_2 = -1$.
Accordingly, the wave vectors for the phase and pseudo-phase have been chosen to be $\alpha_\mu = (0, 0, 0, 0.33541)$ and $\beta_\mu = (1.49638, 1, 0, -1.49638)$, respectively

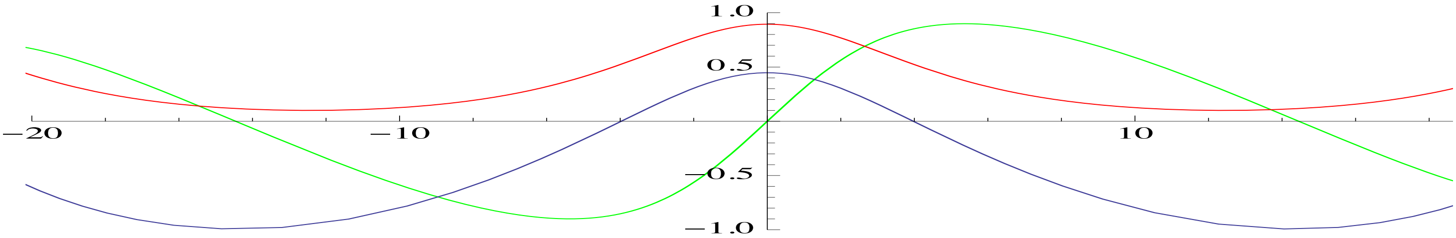
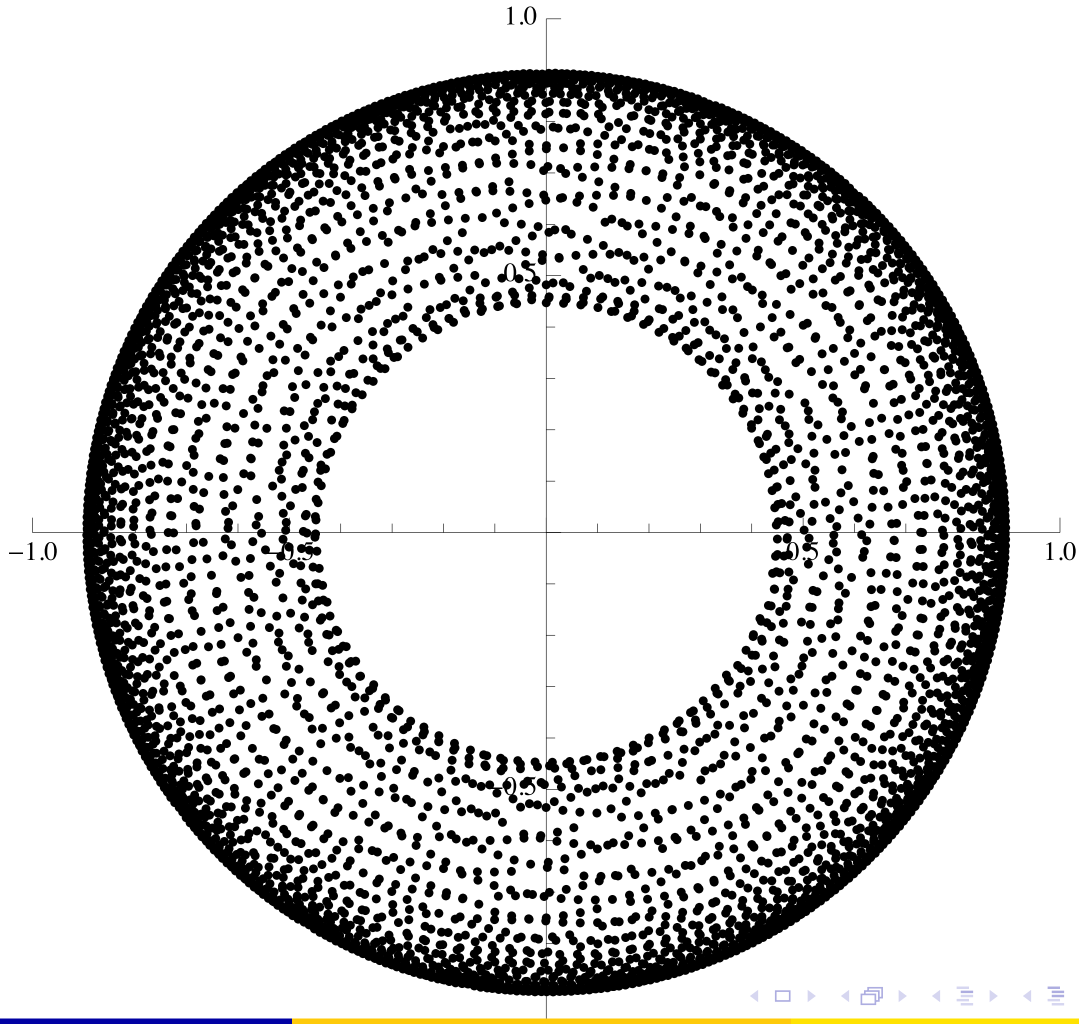
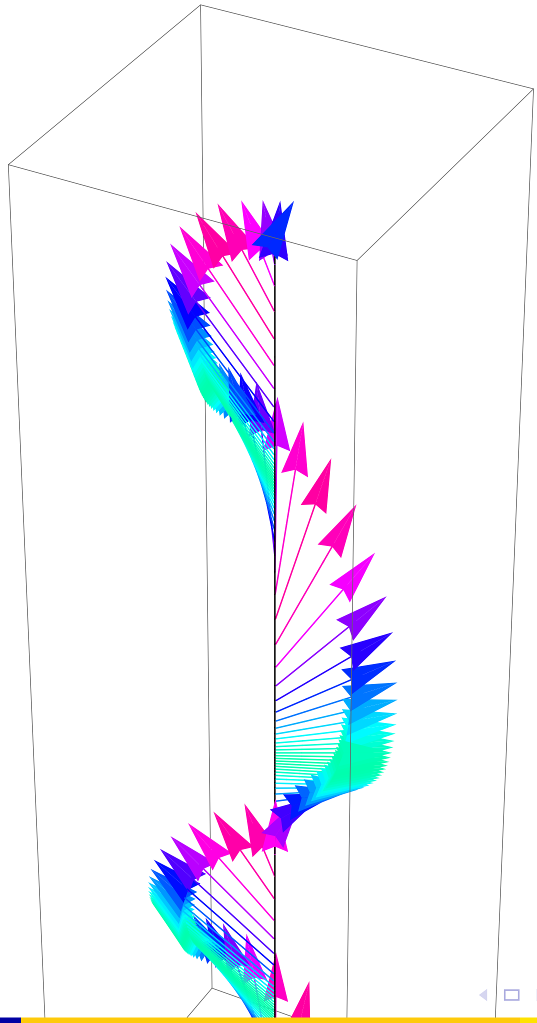


Figura : The graphic for the ϕ_1 (green), ϕ_2(blue) and ϕ_3 () for a choice of the parameters
$A_1 = 0.2, A_2 = 0.99, \psi_1 = 20.01, \mathcal{B} = 1, \lambda = 1, B_1 = 1, s_1 = -1, s_2 = -1$. The wave vectors are $\alpha_\mu = (0, 0, 0, -1.58153)$ and $\beta_\mu = (-1.04879, 1, 0, 1.04879)$, respectively


1.0
0.5
−1.0
−0.5
0.5
1.0
−0.5



The Whitham averaging method

$$\hat{\mathcal{L}}_p = \sin^2(\Theta)\left(-\frac{1}{2}\lambda\left(\frac{B_2^2}{4} - B_1 B_3\right)\Theta_\theta^2 + B_3\Phi_\theta^2 + B_2\Phi_\theta + B_1\right) + B_3\Theta_\theta^2,$$

Averaged constrained Lagrangian on a period

$$L \equiv \frac{1}{2\pi}\oint \hat{\mathcal{L}}_p\, d\theta, \quad \oint d\theta = 2\pi, \quad <\Phi_\theta> = \oint \Phi d\theta = 2\pi m,$$

$$L = \left(B_1 - \frac{B_2^2}{4B_3}\right)\left(A_1 + A_2 + W\sqrt{\frac{\lambda}{2}B_3}\right) + \frac{B_2 + 2mB_3}{2B_3}\sqrt{A_1 A_2 (B_2^2 - 4B_1 B_3)},$$

where

$$W = \frac{1}{2\pi}\oint\sqrt{\frac{(\psi - A_1)(\psi - A_2)(\psi - \psi_1)}{1-\psi}}\frac{d\psi}{\psi}.$$

$L_{A_1} = 0$ and $L_{A_2} = 0$
$\omega = -\theta_{X^0}, k_i = \theta_{X^i}$ and $\gamma = -\tilde{\theta}_{X^0}, \beta_i = \tilde{\theta}_{X^i}$, where X^0, X^1, X^2, X^3 are the so called "slow" variables in comparison with "fast" variables x^0, x^1, x^2, x^3

$$\partial_0 L_\omega = \partial_i L_{k^i}, \quad \partial_0 L_\gamma = \partial_i L_{\beta^i}, \tag{15}$$

with the compatibility conditions

$$\begin{aligned} \partial_0 k^1 + \partial_i \omega &= 0, \quad , \quad \partial_j k^i = \partial_i k^j \quad i \neq j, \\ \partial_0 \beta^i + \partial_i \gamma &= 0, \quad , \quad \partial_j \beta^i = \partial_i \beta^j \quad i \neq j. \end{aligned} \tag{16}$$