$\eta \to \pi^+\pi^-\pi^0$ theory:
dispersive construction of the two-loop amplitude

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based on work done in collaboration with K. Kampf, J. Novotný, M. Zdrahal

[K. Kampf, M. K., J. Novotný, M. Zdrahal, Phys. Rev. D 84, 114015 (2011) and in preparation]

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OUTLINE

- Introduction
- Theory
- Applications
- Summary

Introduction

$\eta \to \pi\pi\pi$ is a $\Delta I = 1$ transition and hence requires isospin breaking (IB)

Two sources of IB:

$$H^{\Delta I=1} = H^{\Delta I=1}_{\rm QCD} + H^{\Delta I=1}_{QED}$$

$$\begin{aligned} H^{\Delta I=1}_{\rm QCD} &= \frac{1}{2}(m_u - m_d) \int d^4x \left(\bar{u}u - \bar{d}d\right)(x) \\ H^{\Delta I=1}_{\rm QED} &= \frac{e^2}{2} \int d^4x d^4y \, D_F(x-y)\eta_{\mu\nu} T\{j^\mu_{\rm em}(x) j^\nu_{\rm em}(y)\}^{\Delta I=1} \end{aligned}$$

$$A(s,t,u) = A_{\rm QCD}(s,t,u) + A_{\rm QED}(s,t,u)$$

$$\begin{aligned} A_{\rm QCD}(s,t,u) &= (m_u - m_d)\left[f_0(s,t,u,m_{u,d,s},...) + f_1(s,t,u,m_{u,d,s},...) + \cdots\right] \\ A_{\rm QED}(s,t,u) &= e^2\left[g_0(s,t,u,m_{u,d,s},...) + g_1(s,t,u,m_{u,d,s},...) + \cdots\right] \end{aligned}$$

Electromagnetic contributions were found to be very small:

Tree level $\mathcal{O}(e^2 E^0)$

[D. G. Sutherland, Phys. Lett. 23, 384 (1966)]

[J. S. Bell, D. G. Sutherland, Nucl. Phys. B 4, 315 (1968)]

One loop $\mathcal{O}(e^2 \hat{m}, e^2 m_s)$

[R. Baur, J. Kambor, D. Wyler, Nucl. Phys. B 460, 127 (1996)]

One loop $\mathcal{O}(e^2 (m_u - m_d))$

[C. Ditsche, B. Kubis, U.-G. Meißner, Eur. Phys. J. C 60, 83 (2009)]

$\longrightarrow A_{\mathrm{QCD}}(s,t,u) \propto (m_u - m_d)$ dominates

$\longrightarrow$ to a very good approximation

$$\Gamma^{\eta\to\pi\pi\pi} \propto (m_u - m_d)^2$$

$$\Gamma_{\mathrm{exp}}^{\eta\to\pi^0\pi^+\pi^-} = 300(12)\ \mathrm{eV}$$

What is known about $A_{\mathrm{QCD}}(s,t,u)$?

Strong interaction contributions computed up to NNLO:

Tree level $\mathcal{O}(E^2)$

[J. A. Cronin, Phys. Rev. 161, 1483 (1967)]

[J. S. Bell, D. G. Sutherland, Nucl. Phys. B 4, 315 (1968)]

One loop $\mathcal{O}(E^4)$

[J. Gasser, H. Leutwyler, Nucl. Phys. B 250, 539 (1985)]

Two loops $\mathcal{O}(E^6)$

[J. Bijnens, K. Ghorbani, Eur. Phys. JHEP 0711, 030 (2007)]

for details, cf. talk by J. Bijnens

Limitations:

- slow convergence $\longrightarrow$ attributed to $\pi\pi$ final-state rescattering effects
- many $\mathcal{O}(E^6)$ low-energy constants contribute at NNLO
- analytical expression very long, not published (FORTRAN codes available)

$\longrightarrow$ other approaches have been considered:

- Dispersive analysis, based on a factorized representation, shared by the ChPT amplitude up to and including NNLO (numerical treatment, iterative resummation of $\pi\pi$ final-state interactions to all orders, no inclusion of $M_{\pi^\pm} \neq M_{\pi^0}$ effects)

[J. Kambor, C. Wiesendanger, D. Wyler, Nucl. Phys. B 465, 215 (1996)]

[A. V. Anisovich, H. Leutwyler, Phys. Lett. B 375, 335 (1996)]

$\longrightarrow$ cf. talk by E. Passemar

- NREFT (analytic representation, worked out up to two loops, limited to the physical region, can include $M_{\pi^\pm} \neq M_{\pi^0}$ effects)

[S. P. Schneider, B. Kubis, C. Ditsche, JHEP 1102, 028 (2011)]

Normalization

$$A_{\mathrm{QCD}}(s,t,u) = \frac{\sqrt{3}}{4R}\mathcal{M}(s,t,u)$$

$s + t + u = M_\eta^2 + 3M_{\pi^\pm}^2 \equiv 3s^c$ ($M_{\pi^0} = M_{\pi^\pm}$)

$$R \equiv \frac{m_s - m_{ud}}{m_d - m_u}$$

Simple relation to

$$Q^2 \equiv \frac{m_s^2 - m_{ud}^2}{m_d^2 - m_u^2} = \frac{1}{2}\left(1 + \frac{m_s}{m_{ud}}\right) R$$

- Dispersive representation of amplitude at NNLO (this talk)

  - Based on very general properties

    [J. Stern, H. Sazdjian, N. H. Fuchs, Phys. Rev. D 47, 3814 (1993)]

  - Compact analytical expression (up to a few loop functions)

  - Only $\pi\pi$ intermediate states included explicitly
  - $M_{\pi^\pm} \neq M_{\pi^0}$ effects can be included

    [K. Kampf, M. K., J. Novotný, M. Zdrahal, unpublished]

    [S. Descotes-Genon, M. K., Eur. Phys. J. C 72, 1962 (2012)]

$\eta \rightarrow \pi\pi\pi$: theory

Input:

- Relativistic invariance, analyticity, unitarity, crossing

$\longrightarrow$ Dispersive representation of $P\pi \to \pi\pi$ (and $\pi\pi \to \pi\pi$) scattering amplitudes

$$\begin{aligned}\mathcal{M}(s,t,u) &= a(t) + (s-u)b(t) + (s-u)^2 c(t) \\ &+ \frac{s^3}{\pi}\int_{s_{\rm thr}}^{\infty} \frac{dx}{x^3} \frac{\operatorname{Im}\mathcal{M}_s(x,t,3s^c-x-t)}{x-s} + \frac{u^3}{\pi}\int_{u_{\rm thr}}^{\infty} \frac{dx}{x^3} \frac{\operatorname{Im}\mathcal{M}_u(x,t,3s^c-x-t)}{x-u}\end{aligned}$$

- Chiral counting for partial-wave projections

$$\mathcal{M}(s,t,u) = 16\pi \sum_{\ell\geq 0}(2\ell+1)P_\ell(\cos\theta)t_\ell(s) \qquad \mathcal{A}(s,t,u) = 16\pi \sum_{\ell\geq 0}(2\ell+1)P_\ell(\cos\theta)f_\ell(s)$$

$$\begin{aligned}&\text{Re}\, t_\ell(s),\ \text{Re}\, f_\ell(s) \sim \mathcal{O}(E^2) \qquad \text{Im}\, t_\ell(s),\ \text{Im}\, f_\ell(s) \sim \mathcal{O}(E^4)\ \ell = 0,1, \\ &\text{Re}\, t_\ell(s),\ \text{Re}\, f_\ell(s) \sim \mathcal{O}(E^4) \qquad \text{Im}\, t_\ell(s),\ \text{Im}\, f_\ell(s) \sim \mathcal{O}(E^8),\ \ell \geq 2\end{aligned}$$

$$\begin{aligned}\mathcal{M}(s,t,u) &= 16\pi[t_0(s) + 3t_1(s)\cos\theta] + \mathcal{M}_{\ell\geq 2}(s,t,u) \\ \mathcal{A}(s,t,u) &= 16\pi[f_0(s) + 3f_1(s)\cos\theta] + \mathcal{A}_{\ell\geq 2}(s,t,u)\end{aligned}$$

$$\text{Re}\,\mathcal{M}_{\ell\geq 2},\ \text{Re}\,\mathcal{A}_{\ell\geq 2} \sim \mathcal{O}(E^4) \qquad \text{Im}\,\mathcal{M}_{\ell\geq 2},\ \text{Im}\,\mathcal{A}_{\ell\geq 2} \sim \mathcal{O}(E^8)$$

- Chiral counting (cont'd)

$$\text{Re}\, f_\ell(s) = \underbrace{\varphi_\ell(s)}_{\sim\mathcal{O}(E^2)} + \underbrace{\psi_\ell(s)}_{\sim\mathcal{O}(E^4)} + \mathcal{O}(E^6) \qquad \text{Re}\, t_\ell(s) = \underbrace{\tilde{\varphi}_\ell(s)}_{\sim\mathcal{O}(E^2)} + \underbrace{\tilde{\psi}_\ell(s)}_{\sim\mathcal{O}(E^4)} + \mathcal{O}(E^6)$$

$$|f_\ell(s)|^2 = [\text{Re}\, f_\ell(s)]^2 + \mathcal{O}(E^8) = [\varphi_\ell(s)]^2 + 2\varphi_\ell(s)\psi_\ell(s) + \mathcal{O}(E^8),\ \ell = 0,1$$
$$|t_\ell(s)|^2 = [\text{Re}\, t_\ell(s)]^2 + \mathcal{O}(E^8) = [\tilde{\varphi}_\ell(s)]^2 + 2\tilde{\varphi}_\ell(s)\tilde{\psi}_\ell(s) + \mathcal{O}(E^8),\ \ell = 0,1$$

- Unitarity

provides the absorptive parts

$$\text{Im}\, t_\ell^{i\to f}(s) = \sum_k \frac{1}{S_k} \frac{\lambda_k^{1/2}(s)}{s} t_\ell^{i\to k}(s) \Big[f_\ell^{f\to k}(s) \Big]^\star \times \theta(s - s_k^{\text{thr}})$$

$$\text{Im}\, f_\ell^{i\to f}(s) = \sum_k \frac{1}{S_k} \frac{\lambda_k^{1/2}(s)}{s} f_\ell^{i\to k}(s) \Big[f_\ell^{f\to k}(s) \Big]^\star \times \theta(s - s_k^{\text{thr}})$$

sum goes over all the possible two-pion intermediate states k

$\longrightarrow$ All other two-pseudoscalar-meson intermediate-states thresholds are far enough from the decay region of $\eta \to \pi\pi\pi$ to be described by a Taylor expansion in s, t, u
(additional thresholds are beyond NNLO, or beyond ChPT)

Output:

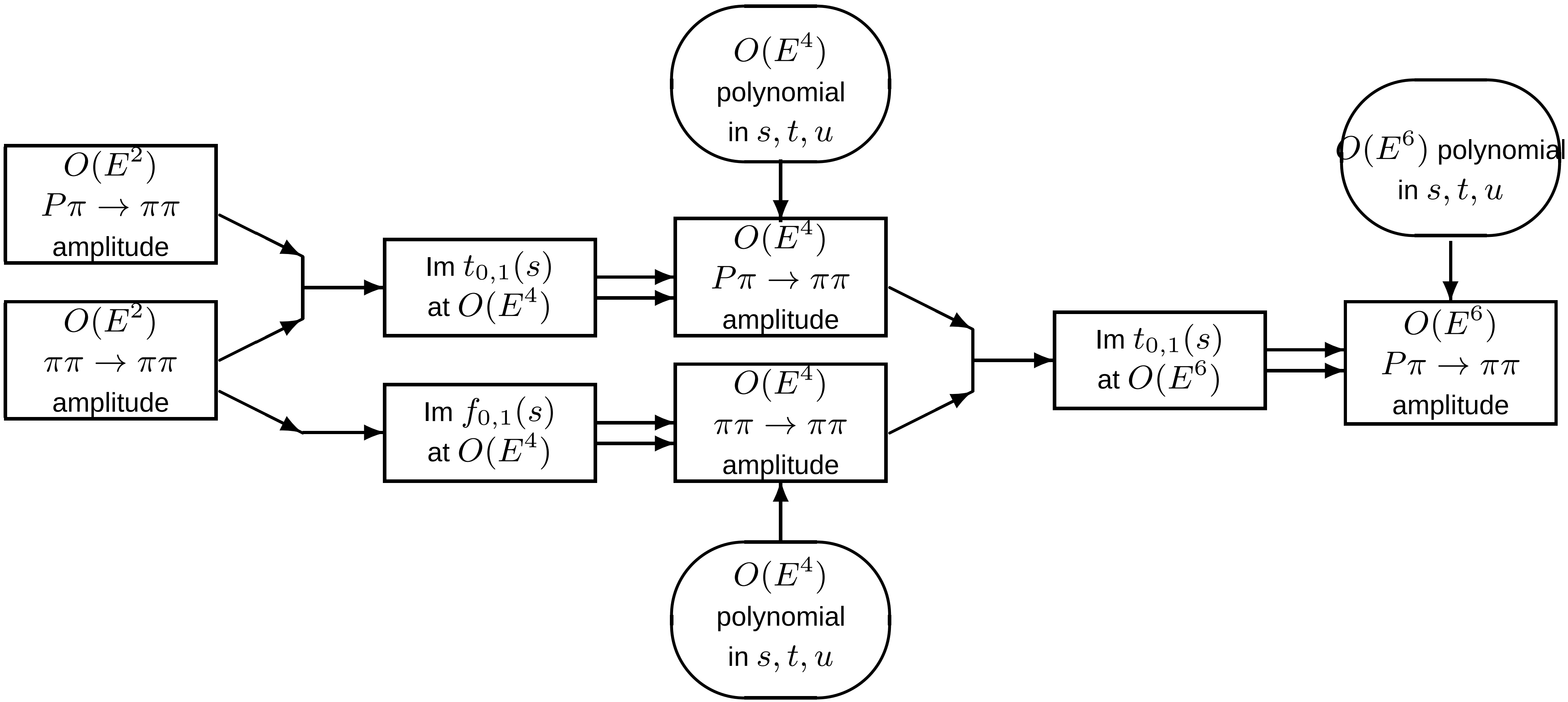

Figure 1: Schematic representation of the iterative two-steps reconstruction procedure for the $P\pi \to \pi\pi$ amplitudes.

Output:

$$\mathcal{M}(s,t,u) = \mathcal{P}(s,t,u) + \mathcal{U}(s,t,u) + \mathcal{O}(E^8)$$

• $\mathcal{P}(s,t,u)$: third order polynomial expressed in terms of six free parameters corresponding to the $t-u$ symmetric expansion at the center of the Dalitz plot

$$\mathcal{P}(s,t,u) = A_x M_\eta^2 + B_x(s-s^c) + C_x(s-s^c)^2 + D_x\left((t-s^c)^2 + (u-s^c)^2\right) \\ +E_x(s-s^c)^3 + F_x\left((t-s^c)^3 + (u-s^c)^3\right)$$

• Description of higher two-meson thresholds ($\eta\pi$, $K\bar{K}$, . . .) included in this polynomial

• $\mathcal{U}(s,t,u) \to$ non-analytic unitarity part:

$$\mathcal{U}(s,t,u) \quad = \quad 16\pi\left[\mathcal{W}_0(s) + 3(t-u)\mathcal{W}_1(s) + \mathcal{W}_0^t(t) + 3(u-s)\mathcal{W}_1^t(t) + \mathcal{W}_0^u(u) + 3(t-s)\mathcal{W}_1^u(u)\right]$$

$\mathcal{W}_\ell(s)$, $\mathcal{W}_\ell^t(s)$, $\mathcal{W}_\ell^u(s)$ ($\ell = 0, 1$)
$\to$ analytical in the complex s-plane, except for a right-hand cut, with discontinuities

$$\mathrm{Im}\,\mathcal{W}_0(s) = \mathrm{Im}\,t_0(s) + 3\,\frac{\Delta_{P1}\Delta_{23}}{s}\,\frac{\mathrm{Im}\,t_1(s)}{2K(s)} \qquad \mathrm{Im}\,\mathcal{W}_1(s) = \frac{\mathrm{Im}\,t_1(s)}{2K(s)} \qquad \text{etc}\ldots$$

depend on A_x, B_x, C_x, D_x and on the $\pi\pi$ scattering lengths

Applications

Connection with ChPT

$$\begin{aligned} \mathcal{M}_{\text{ChPT}}(s,t,u) &= \mathcal{M}^{(\text{LO})}_{\text{ChPT}}(s,t,u) + \mathcal{M}^{(\text{NLO})}_{\text{ChPT}}(s,t,u) + \mathcal{M}^{(\text{NNLO})}_{\text{ChPT}}(s,t,u) + \mathcal{O}(E^8) \\ &= \mathcal{P}_{\text{ChPT}}(s,t,u) + \mathcal{U}_{\text{ChPT}}(s,t,u) + \mathcal{O}(E^8) \end{aligned}$$

- In $\mathcal{M}_{\text{ChPT}}(s,t,u)$, expand (in principle) contributions from two-meson states other than $\pi\pi$ ($\eta\pi$, $K\bar{K}$, . . .) as a polynomial of at most third order in s, t, u

- In $\mathcal{M}(s,t,u)$, formally split the parameters as

$$A_x = A_x^{(\text{LO})} + \Delta A_x^{(\text{NLO})} + \Delta A_x^{(\text{NNLO})} \qquad B_x = B_x^{(\text{LO})} + \Delta B_x^{(\text{NLO})} + \Delta B_x^{(\text{NNLO})}$$
$$C_x = C_x^{(\text{NLO})} + \Delta C_x^{(\text{NNLO})} \quad \text{etc}\ldots$$

- Express $A_x^{(\text{LO})}$, $\Delta A_x^{(\text{NLO})}$, etc in terms of chiral logarithms and low-energy constants L_i and C_i (matching partly done numerically to NNLO).

Connection with ChPT

- Use the resonance determination $C_i = C_i^{\rm Res}$ as in [Bijnens and Ghorbani]

[V. Cirigliano et al., Nucl. Phys. B 753, 139 (2006)]

[K. Kampf et al., Eur. Phys. J. C 50, 385 (2007)]

- Reproduce the result of [Bijnens and Ghorbani]

$$R = 40.9 \quad [\rm ChPT(NNLO),\, C_i = C_i^{\rm Res}]$$

- Test approximations made

Dalitz-plot parameters

	a	b	d	f
Gormley et al.	-1.17 $\pm$ 0.02	0.21 $\pm$ 0.03	0.06 $\pm$ 0.04	
Layter et al.	-1.08 $\pm$ 0.014	0.034 $\pm$ 0.027	0.046 $\pm$ 0.031	
Crystal Barrel	-1.22 $\pm$ 0.07	0.22 $\pm$ 0.11	0.06 $\pm$ 0.04	
KLOE	-1.090 $\pm$ 0.020	0.124 $\pm$ 0.012	0.057 $\pm$ 0.017	0.14 $\pm$ 0.02
ChPT NNLO	-1.271 $\pm$ 0.075	0.394 $\pm$ 0.102	0.055 $\pm$ 0.057	0.025 $\pm$ 0.160

- What corrections ΔA_x, ΔB_x, etc are necessary in order to reproduce the KLOE data?

	# 174	# 2500
ΔA_x	-0.05 ± 0.3	-0.029 ± 0.003
ΔB_x	-0.5 ± 1	-0.46 ± 0.01
ΔC_x	-7 ± 2	-6.97 ± 0.07
ΔD_x	-0.7 ± 0.8	-0.64 ± 0.02
ΔE_x	-37 ± 18	-36 ± 3
ΔF_x	24 ± 5	24 ± 1

	cor.set	fit to KLOE
A_x	0.575 ± 0.006	0.575 ± 0.001
B_x	1.99 ± 0.04	2.15 ± 0.02
C_x	-6.8 ± 0.3	-5.8 ± 0.2
D_x	0.94 ± 0.03	0.87 ± 0.08
E_x	-31 ± 3	-19 ± 9
F_x	20 ± 1	21 ± 5

[units: appropriate powers of GeV]

$$R = 37.4(2.8) \quad [\mathrm{ChPT(NNLO) + KLOE}]$$

Using the full two-loop dispersive representation

- the parameters in the polynomial and the unitarity parts are the same

- Dalitz-plot data alone do not allow to fix the overall normalization

- Use ChPT at NNLO to find region where there is stability when going from NLO to NNLO, independently from the values of the C_is

- $\longrightarrow$ look at the absorptive part of the ChPT amplitude

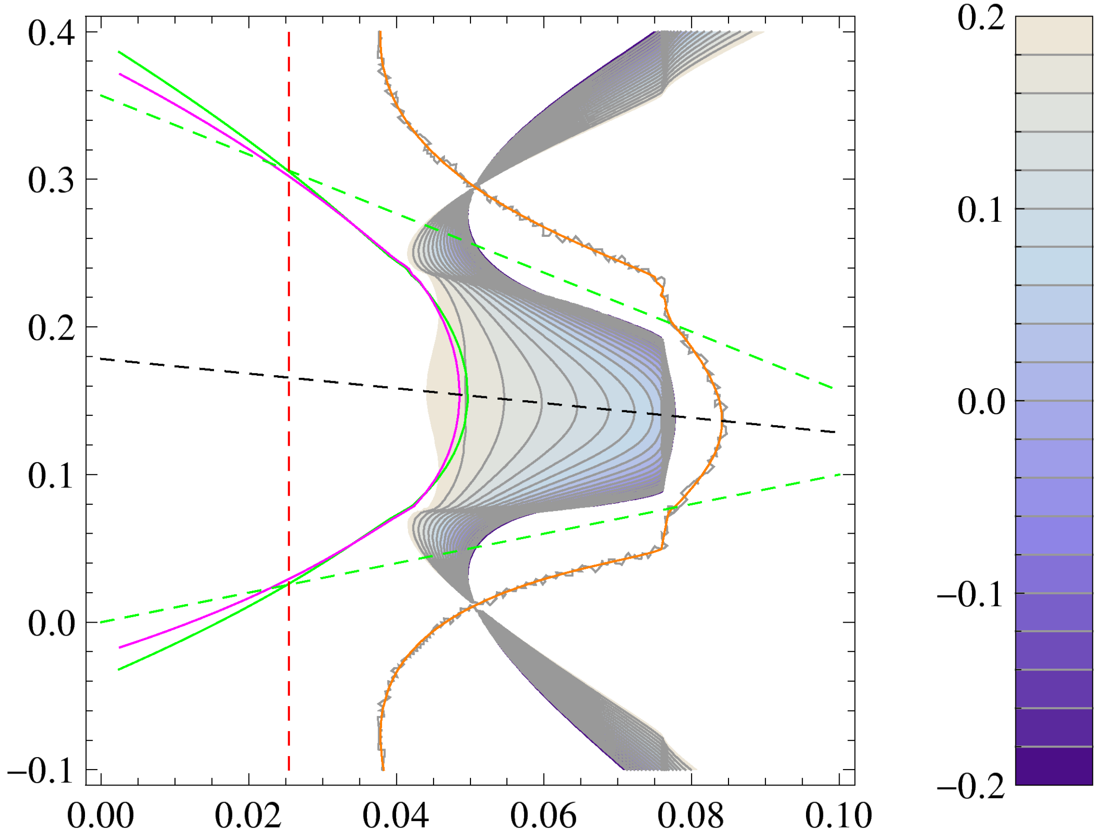

Figure 2: Chiral convergence of the imaginary part of the amplitude stemming from NNLO ChPT computation. The figure shows the variation of $\eta_{46} \equiv 1 - \mathrm{Im}\mathcal{A}^{\mathrm{ChPT}}(s,t,u)|_{\mathrm{NLO}}/\mathcal{A}^{\mathrm{ChPT}}(s,t,u)|_{\mathrm{NNLO}}$ in the (s,t,u) plane. Only the region where $|\eta_{46}|$ is less then 20% have been coloured. Abscissa: $s\ [GeV^2]$, Ordinate: $t\ [GeV^2]$

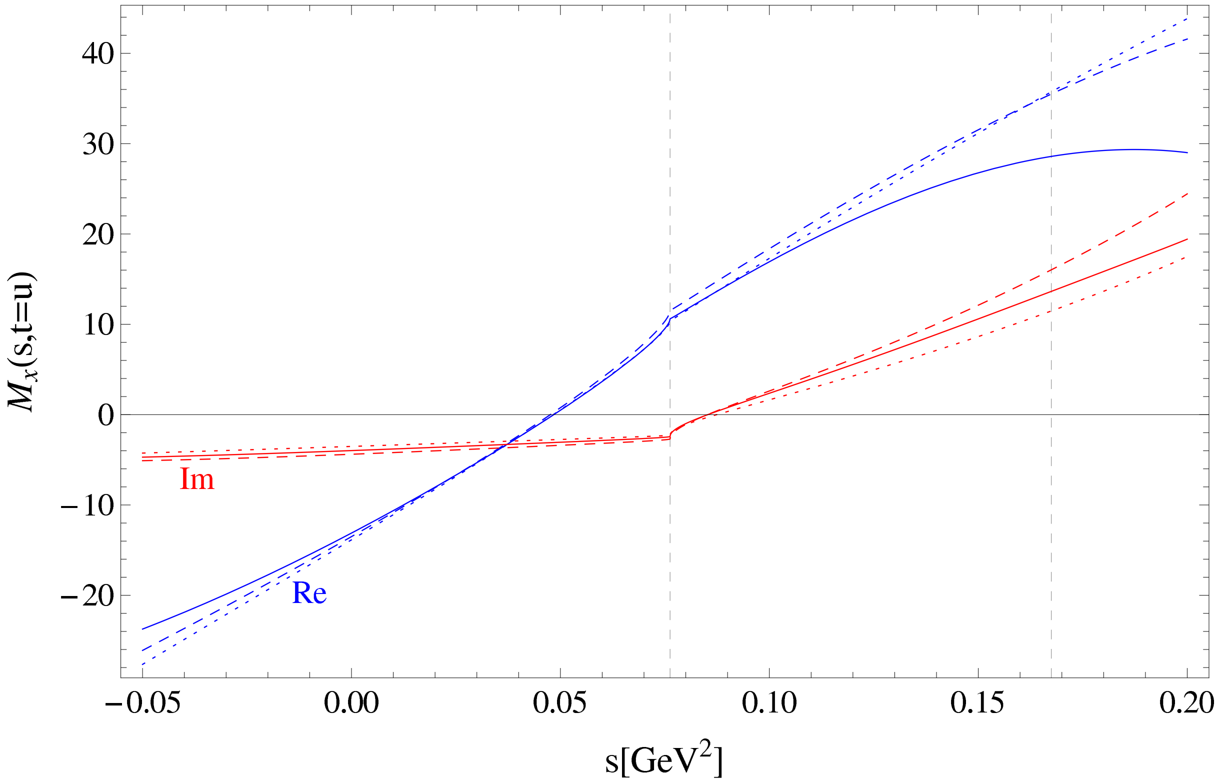

Figure 3: The real (blue) and the imaginary (red) part of the amplitude along the $t = u$ line. The dotted lines represent an order-by-order fit, the dashed ones stand for the resummed fit constructed from it (with the values of all parameters equal to their $O(p^6)$ values), and finally the solid lines reflect an overall fit corresponding to KLOE values, with normalization set to interpolate between the dotted and the dashed lines for the imaginary part

$$R = 37.5(3.3) \quad [\text{disp. rep.} + \text{KLOE} + \text{ChPT(NNLO)}]$$

Summary

- Construction of a (quasi-) analytical dispersive representation of $\eta \to \pi\pi\pi$ amplitudes that is a valid description whenever:

- order $\mathcal{O}(E^8)$ corrections are small
- two-meson contributions other than $\pi\pi$ intermediate states can be described by a third-order polynomial in s, t, u

- Reproduces the structure of the NNLO ChPT calculation

- NNLO ChPT + $C_i = C_i^{\mathrm{Res}}$ does not provide a good description of the Dalitz-plot parameters measured by KLOE

- Construction of a (quasi-) analytical dispersive representation of $\eta \to \pi\pi\pi$ amplitudes that is a valid description whenever:

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- Reproduces the structure of the NNLO ChPT calculation

- NNLO ChPT + $C_i = C_i^{\text{Res}}$ does not provide a good description of the Dalitz-plot parameters measured by KLOE $\longrightarrow$ however

	a	b	d	f
ChPT NNLO	$-1.271(75)$	$+0.394(102)$	$+0.055(57)$	$+0.025(160)$
KLOE	$-1.090(20)$	$+0.124(12)$	$+0.057(17)$	$+0.14(2)$
WASA	$-1.144(18)$	$+0.219(19)(37)$	$+0.086(18)(18)$	$+0.115(37)$
Δ	$-0.054(23)$ (-2.2σ)	$+0.095(44)$ $(+2.3\sigma)$	$+0.029(28)$ $(+1.0\sigma)$	$-0.025(43)$ (-0.6σ)

ChPT NNLO: [J. Bijnens, K. Ghorbani, J. High Energy Phys. 11, 030 (2007)]

KLOE: [F. Ambrosino et al [KLOE], J. High Energy Phys. 05, 006 (2008)]

WASA: [P. Adlarson et al. [WASA at COSY], arXiv:1406.2505 [hep-ex] (2014)]

cf. talks by L. Caldeira-Balkestahl and A. Somov

- Construction of a (quasi-) analytical dispersive representation of $\eta \to \pi\pi\pi$ amplitudes that is a valid description whenever:

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- two-meson contributions other than $\pi\pi$ intermediate states can be described by a third-order polynomial in s, t, u

- Reproduces the structure of the NNLO ChPT calculation

- NNLO ChPT + $C_i = C_i^{\text{Res}}$ does not provide a good description of the Dalitz-plot parameters measured by KLOE

- Fit of the two-loop dispersive representation to the KLOE data and to the absorptive part of the two-loop ChPT amplitude (does not depend on the C_is in the stability region along the line $t = u$, below the physical region)

$$R = 37.4(2.2)$$

to be compared to

$$R = \begin{cases} 35.8(1.9)(1.8) & [2+1] \\ 40.7(2.7)(2.2) & [2] \end{cases} \qquad Q = \begin{cases} 22.6(7)(6) & [2+1] \\ 24.3(1.4)(0.6) & [2] \end{cases}$$

[FLAG review of lattice results, arXiv:1310.8555v3 [hep-lat]]