Methods in Amplitude Analysis: theory, models and implementation

Adam Szczepaniak
Indiana University/JLab

•Constraints and Examples
•1-to-3 decay amplitude models
•KT Equations
•Collaboration strategy
Why Amplitude Analysis

Experimental Measurement

QCD Measurement

Physics quantities: form factors, resonance parameters, masses, etc.

Reaction amplitudes

\[ d\sigma_{\text{Measured}} = \text{Detector Acceptance} \otimes dPS |A|^2 \]
Amplitude construction

Axiomatic S-matrix principles:

• Crossing relations: \( A(s, t) \) describes all processes related by line reversal.

• Analyticity: Cuts determined by unitarity (i.e. in the physical region, continuation is complicated, Mandelstam representation known only for 4-point function).

  • Asymptotic behavior (\( A(s_i) < s_i \log s_i \))

  • Bound state poles

• Regge behavior: Analyticity of “the second kind”

• Global symmetries: EM, chiral, …
• “All constraints are equal but some may be more equal then other”

\[ A(s, t) = \sum_{l}^{\infty} (2l + 1) f_l(s) P_l(z_s) \rightarrow \sum_{l} (2l + 1) f_l(s) P_l(z_s) \]

(s-channel) Isobar model

\[ (s - 4) R^2 \sim \frac{s - 4}{m_e^2} \ll 1 \]

alternatively use t/u channel isobars
COMPASS

$\pi^- (190 \text{GeV}^2) p \rightarrow \eta(\prime) \pi^- p$

$\pi^- t \sim \cos(\theta_{GJ})$

Pomeron

$\eta(\text{backward})$

$\eta(\text{forward})$

$s \ t$

$m(\eta \pi^-) [\text{GeV}/c^2]$

$m(\eta' \pi^-) [\text{GeV}/c^2]$

$\pi^- \ a_0, a_2, \cdots \ \pi^-$

$\pi^- \ \eta$

$\pi^- \ P \ \eta$

$\pi^- \ \pi^-$
• When cross-channel channel singularities are all nearby, there are no known amplitudes that satisfy all S-matrix constraints (except perturbatively, e.g. chiral p.t.)

\[ M(\eta, \omega, \phi, \cdots) \rightarrow m + m + m \ (\pi^+, \pi^-, \pi^0, \cdots) \]

\[ M \gtrsim 3m \]

Two general class of models

• Two-body unitarization, of low partial waves

  violate analyticity of the 2nd kind

• Resonance/Regge Duality

  violate analyticity of individual partial waves
- General properties:

  \[
  M(1) + \bar{m}(2) \to m(3) + m(4) \\
  s = (p_3 + p_4) \\
  t = (p_4 - p_2)^2 = (p_4 + p_2)^2 \\
  u = (p_3 - p_2)^2 = (p_3 + p_2)^2 
  \]

  in decay-channel, s,t,u become the Dalitz variables

- Partial waves:

  \[
  A(s, t) = \sum_{l} (2l + 1) A_l(s) P_l(z_s) \\
  A_l(s) = A_l^R(s) + A_l^L(s) 
  \]

- Two-body unitarity:

  \[
  \Delta A_l^R(s) = \rho(s) \frac{N_l(s)}{D_l^*(s)} A_l(s) \\
  \frac{N_l(s)}{D_l(s)} = m + m \to m + m \\
  \text{p.w} \\
  \]

- Solution: (Frazer-Fulco/Omnes/Mandelstam)

  \[
  A_l(s) = A_l^L(s) + \frac{1}{D(s)} \int_{4m^2} ds' \frac{\rho(s') N_l(s') A_l^L(s')}{\pi s' - s} 
  \]
The solution depends on and generalized to inelastic case

\[ A_l(s) = A^L_l(s) + \frac{1}{D(s)} \int_{4m^2} ds' \frac{\rho(s') N_l(s') A^L_l(s')}{\pi} \frac{1}{s' - s} \]

can also be represented in the standard form

\[ A_l(s) = \frac{1}{D_l(s)} \left( G^L_l(s) = \int_{-\infty} \frac{ds'}{\pi} \frac{D_l(s') A^L_l(s')}{s' - s} \right) \]

The solution depends on

exact representation:

\[ A_l(s) = \frac{1}{D^e_l(s)} \left( \sum_n a^n_l \omega^n_L(s) + \sum_m b^l_m \omega^m_{in}(s) \right) \]
Model examples (implementation of crossing defines the l.h.c)

• "product form"

\[ A(s, t, u) = \frac{f(s, t, u)}{D_0(s)D_0(t)D_0(u)} \]  

\[ G^L_0(s) = f(s, t, u) \int_{-1}^{1} \frac{dz_s}{D_0(t)D_0(u)} \]

• "sum form" = Khuri-Treiman (equation)

\[ A(s, t, u) = f_0(s) \cdot \frac{1}{D_0(s)} + f_0(t) \cdot \frac{1}{D_0(t)} + f_0(u) \cdot \frac{1}{D_0(u)} \]

\[ A_l(s) = \frac{1}{D_l(s)} \int_{4m^2} ds' \frac{\rho(s') N_l(s') A^L_l(s')}{s' - s} + A^L_l(s) \]
General remarks

• Various forms satisfy: 2-body unitarity and crossing symmetry
• Can be analytically continued to the decay region
  minor complications from anomalous thresholds
  (Mandelstam/Kacser/Aitchison/Brehm)
• Can be extended to higher partial waves
• Analytical continuation of unitarity from 2-to-2 to 1-to-3 is not the same as imposing unitarity on 3-to-3 amplitude at the M-particle pole

Any model which truncate partial waves will

• Violate Mandelstam analyticity
• Have incorrect asymptotic behavior
• Violate analyticity in the complex-1 plane
• Dual models (Veneziano)  \[ A(s, t) = \frac{\Gamma(1 - \alpha(s))\Gamma(1 - \alpha(t))}{\Gamma(2 - \alpha(s) + \alpha(t))} \]

\[
A(s, t) = \sum_k \frac{\beta_k(t)}{k - \alpha(s)} = \sum_i \frac{\beta_k(s)}{k - \alpha(t)}
\]

\[
A_n(s, t; N) = \frac{2n - \alpha_s - \alpha_t}{(n - \alpha_s)(n - \alpha_t)} \sum_{i=1}^{n} a_{n,i}(-\alpha_s - \alpha_t)^{i-1}
\]

\[
\times \frac{\Gamma(N + 1 - \alpha_s)\Gamma(N + 1 - \alpha_t)}{\Gamma(N + 1 - n)\Gamma(N + n + 1 - \alpha_s - \alpha_t)}.
\]

Regge/Resonance duality

Can be generalized to any number of external particles

Can be extend to satisfy Mandelstam duality, but not known extensions to several trajectories
Regge

dual model $J/\psi$

standard isobar

dual model $\psi'$

PRELIMINARY
Dispersive analysis of $\omega/\phi \rightarrow 3\pi$

Solution: (e.g. P-waves only)

\[
A^R(s) = \frac{1}{D(s)} \int_{4m^2}^{\infty} \frac{d(s') \rho(s') N(s') A^L(s')}{\pi s' - s}
\]

Easily generalized to inelastic case

\[
a^R(s) = \frac{1}{D^{el}(s)} \left( \int_{4m^2}^{s_i} \frac{d(s') \rho(s') N(s)(s') A^L(s')}{\pi s' - s} + A^{in}(s) \right)
\]

el = only elastic cut \quad in = only inelastic cut
Dispersive analysis of $\omega/\phi \to 3\pi$

$w(s)$ is the conformal map of inelastic contributions:
Coefficients $a_i$ play the role of improved subtraction constants

\[ a^R(s) = \frac{1}{D_{el}(s)} \left( \int_{4m^2}^{s_i} ds' \frac{\rho(s') N(s)(s') A^L(s')}{\pi (s'-s)} + \sum_{i=0}^{N} a_i \omega^i(s) \right) \]

Different from
Niecknig et. al. 2012
Anisovich et. al. 1998

All details in: I. Danilkin et al., arXiv1076363
Dalitz plots

- Only one parameter (overall normalization) $\rightarrow$ fixed from $\Gamma_{\text{exp}}(\omega/\phi \rightarrow 3\pi)$
- $\phi \rightarrow 3\pi$: distribution clearly shows $\rho$-meson resonances
- $\omega \rightarrow 3\pi$: distribution is relatively flat;
- upcoming high-statistic data from CLAS, KLOE, WASA, etc.

I. Danilkin et al.
Transition form factors: $\omega/\phi \rightarrow \pi\gamma$

$$f_{V\pi}(s) = \int_{4m^2}^{s_i} \frac{ds'}{\pi} \frac{\Delta f_{V\pi}(s')}{s' - s} + \sum_{i=0}^{N} C_i \omega^i(s)$$

![Graph showing results for $\omega \rightarrow \pi^0\gamma^*$ and $\omega \rightarrow \pi^0\mu^+\mu^-$](image)

- **Black**: standard VMD (fails to describe the data)
- **Blue**: $N=0$ ($C_0$ from $\Gamma_{\exp}(V \rightarrow \pi\gamma)$)
- **Red**: $N=1$
- **Green**: $N=2$ (fit to the data)

**Nature of the steep rise?**
Exp. analysis of $\phi \rightarrow \pi\gamma$ is very important

I. Danilkin et al.
Transition form factors: $\omega/\phi \rightarrow \pi \gamma$

$$f_{V\pi}(s) = \int_{4m^2}^{s_i} \frac{ds'}{\pi} \frac{\Delta f_{V\pi}(s')}{s' - s} + \sum_{i=0}^{N} C_i \omega^i(s)$$

- **Black**: standard VMD (fails to describe the data)
- **Blue**: $N=0$ ($C_0$ from $\Gamma_{exp}(V \rightarrow \pi \gamma)$)
- **Grey**: $N=0$ (no 3b effects)
Joint Physics Analysis Center (JPAC)

theory

experiment

working on CLAS data now

η and ω codes will be available online

“generic” parametrizations will be available online