CHIRAL PERTURBATION THEORY AND $\eta \rightarrow 3\pi$: AN INTRODUCTION

Study of Strongly Interacting Matter
HadronPhysics

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Vetenskapsrådet

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Overview

Chiral Perturbation Theory

Exploring the consequences of
the chiral symmetry of QCD
and its spontaneous breaking
using effective field theory techniques

Derivation from QCD:
H. Leutwyler,
On The Foundations Of Chiral Perturbation Theory,
Ann. Phys. 235 (1994) 165 [hep-ph/9311274]

For references to lectures see:
http://www.thep.lu.se/∼bijnens/chpt.html

Chiral Perturbation Theory

A general Effective Field Theory:

- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:

- Degrees of freedom: Goldstone Bosons from spontaneous breaking of chiral symmetry
- Powercounting: Dimensional counting in momenta/masses
- Breakdown scale: Resonances, so about M_ρ.

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Chiral Symmetry

Chiral Symmetry

QCD: N_f light quarks: equal mass: interchange: $SU(N_f)_V$

But $$\mathcal{L}_{QCD} = \sum_{q=u,d,s} \left[i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L + \bar{q}_L q_R \right) \right]$$

So if $m_q = 0$ then $SU(3)_L \times SU(3)_R$.

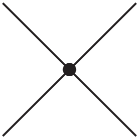
Spontaneous breakdown

- $\langle \bar{q} q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
- Mechanism: see talk by L. Giusti
- $SU(3)_L \times SU(3)_R$ broken spontaneously to $SU(3)_V$
- 8 generators broken $\Longrightarrow$ 8 massless degrees of freedom and interaction vanishes at zero momentum

Goldstone Bosons

Power counting in momenta: Meson loops, Weinberg powercounting

rules

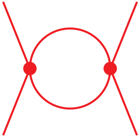


p^2

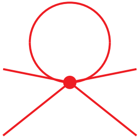
$1/p^2$

$\int d^4p$ $\quad p^4$

one loop example



$$(p^2)^2\,(1/p^2)^2\,p^4 = p^4$$



$$(p^2)\,(1/p^2)\,p^4 = p^4$$

Lagrangians: Lowest order

$U(\phi) = \exp(i\sqrt{2}\Phi/F_0)$ parametrizes Goldstone Bosons

$$\Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix}.$$

LO Lagrangian: $\mathcal{L}_2 = \frac{F_0^2}{4}\{\langle D_\mu U^\dagger D^\mu U\rangle + \langle \chi^\dagger U + \chi U^\dagger\rangle\}\,,$

$D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu\,,$
left and right external currents: $r(l)_\mu = v_\mu + (-)a_\mu$

Scalar and pseudoscalar external densities: $\chi = 2B_0(s + ip)$ quark masses via scalar density: $s = \mathcal{M} + \cdots$

$\langle A\rangle = Tr_F\,(A)$

Lagrangians: Lagrangian structure

	2 flavour		3 flavour		PQChPT/N_f flavour	
p^2	F, B	2	F_0, B_0	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
p^6	c_i^r	52+4	C_i^r	90+4	K_i^r	112+3

p^2: Weinberg 1966

p^4: Gasser, Leutwyler 84,85

p^6: JB, Colangelo, Ecker 99,00

- ➠ L_i LEC = Low Energy Constants = ChPT parameters
- ➠ H_i: contact terms: value depends on definition of currents/densities
- ➠ Finite volume: no new LECs
- ➠ Other effects: (many) new LECs

Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars
- Chiral logarithms
- includes Isospin and the eightfold way ($SU(3)_V$)
- Unitarity included perturbatively

$$m_\pi^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2}\log\frac{(2B\hat{m})}{\mu^2} + 2l_3^r(\mu)\right] + \cdots$$

$$M^2 = 2B\hat{m}$$

(Partial) History/References

- Original determination at p^4: Gasser, Leutwyler, Annals Phys.158 (1984) 142, Nucl. Phys. B250 (1985) 465
- p^6 2 flavour: several papers (see later)
- p^6 3 flavour: Amorós, JB, Talavera, Nucl. Phys. B602 (2001) 87 [hep-ph/0101127]
- Review article two-loops: JB, Prog. Part. Nucl. Phys. 58 (2007) 521 [hep-ph/0604043]
- Update of fits + new input: JB, Jemos, Nucl. Phys. B 854 (2012) 631 [arXiv:1103.5945]
- Recent review with more p^6 input: JB, Ecker, arXiv:1405.6488, Ann. Rev. Nucl. Part. Sc.(in press)
- Review Kaon physics: Cirigliano, Ecker, Neufeld, Pich, Portoles, Rev.Mod.Phys. 84 (2012) 399 [arXiv:1107.6001]
- Lattice: FLAG reports:, Colangelo et al., Eur.Phys.J. C71 (2011) 1695 [arXiv:1011.4408] Aoki et al., arXiv:1310.8555

Three flavour LECs: uncertainties

- $m_K^2, m_\eta^2 \gg m_\pi^2$
- Contributions from p^6 Lagrangian are larger
- Reliance on estimates of the C_i much larger
- Typically: C_i^r: (terms with)
  kinematical dependence $\equiv$ measurable
  quark mass dependence $\equiv$ impossible (without lattice)
  100% correlated with L_i^r
- How suppressed are the $1/N_c$-suppressed terms?
- Are we really testing ChPT or just doing a phenomenological fit?

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Testing if ChPT works: relations

Yes: JB, Jemos, Eur.Phys.J. C64 (2009) 273-282 [arXiv:0906.3118]

Systematic search for relations between observables that do not depend on the C_i^r

Included:

- m_M^2 and F_M for π, K, η.
- 11 $\pi\pi$ threshold parameters
- 14 πK threshold parameters
- 6 $\eta \to 3\pi$ decay parameters,
- 10 observables in $K_{\ell 4}$
- 18 in the scalar formfactors
- 11 in the vectorformfactors
- Total: 76

We found 35 relations

Relations at NNLO: summary

- We did numerics for $\pi\pi$ (7), πK (5) and $K_{\ell 4}$ (1)
  13 relations
- $\pi\pi$: similar quality in two and three flavour ChPT
  The two involving a_3^- significantly did not work well
- πK: relation involving a_3^- not OK
  one more has very large NNLO corrections
- The relation with $K_{\ell 4}$ also did not work: related to that ChPT has trouble with curvature in $K_{\ell 4}$
- Conclusion: Three flavour ChPT "sort of" works

Fits: inputs

Amorós, JB, Talavera, Nucl. Phys. B602 (2001) 87 [hep-ph/0101127] (ABC01)

JB, Jemos, Nucl. Phys. B 854 (2012) 631 [arXiv:1103.5945] (BJ12)

JB, Ecker, arXiv:1405.6488, Ann. Rev. Nucl. Part. Sc.(in press) (BE14)

- $M_\pi, M_K, M_\eta, F_\pi, F_K/F_\pi$
- $\langle r^2\rangle^\pi_S$, c^π_S slope and curvature of F_S
- $\pi\pi$ and πK scattering lengths a^0_0, a^2_0, $a_0^{1/2}$ and $a_0^{3/2}$.
- Value and slope of F and G in $K_{\ell 4}$
- $\frac{m_s}{\hat{m}} = 27.5$ (lattice)
- $\bar{l}_1, \dots, \bar{l}_4$
- more variation with C^r_i, a penalty for a large p^6 contribution to the masses
- 17+3 inputs and 8 L^r_i+34 C^r_i to fit

Main fit

	ABC01	BJ12	L_4^r free	BE14
	old data			
$10^3 L_1^r$	0.39(12)	0.88(09)	0.64(06)	0.53(06)
$10^3 L_2^r$	0.73(12)	0.61(20)	0.59(04)	0.81(04)
$10^3 L_3^r$	$-2.34(37)$	$-3.04(43)$	$-2.80(20)$	$-3.07(20)$
$10^3 L_4^r$	$\equiv 0$	0.75(75)	0.76(18)	$\equiv 0.3$
$10^3 L_5^r$	0.97(11)	0.58(13)	0.50(07)	1.01(06)
$10^3 L_6^r$	$\equiv 0$	0.29(8)	0.49(25)	0.14(05)
$10^3 L_7^r$	$-0.30(15$	$-0.11(15)$	$-0.19(08)$	$-0.34(09)$
$10^3 L_8^r$	0.60(20)	0.18(18)	0.17(11)	0.47(10)
χ^2	0.26	1.28	0.48	1.04
dof	1	4	?	?
F_0 [MeV]	87	65	64	71

$? = (17 + 3) - (8 + 34)$

Main fit: Comments

- All values of the C_i^r we settled on are "reasonable"
- Leaving L_4^r free ends up with $L_4^r \approx 0.76$
- keeping L_4^r small: also L_6^r and $2L_1^r - L_2^r$ small (large N_c relations)
- Compatible with lattice determinations
- Not too bad with resonance saturation both for L_i^r and C_i^r
- decent convergence (but enforced for masses)
- Many prejudices went in: large N_c, resonance model, quark model estimates,...

Some results of this fit

Mass:

$$
\begin{aligned}
m_\pi^2/m_{\pi phys}^2 &= 1.055(p^2) - 0.005(p^4) - 0.050(p^6)\,,\\
m_K^2/m_{Kphys}^2 &= 1.112(p^2) - 0.069(p^4) - 0.043(p^6)\,,\\
m_\eta^2/m_{\eta phys}^2 &= 1.197(p^2) - 0.214(p^4) + 0.017(p^6)\,,
\end{aligned}
$$

Decay constants:

$$
\begin{aligned}
F_\pi/F_0 &= 1.000(p^2) + 0.208(p^4) + 0.088(p^6)\,,\\
F_K/F_\pi &= 1.000(p^2) + 0.176(p^4) + 0.023(p^6)\,.
\end{aligned}
$$

Scattering:

$$
\begin{aligned}
a_0^0 &= 0.160(p^2) + 0.044(p^4) + 0.012(p^6)\,,\\
a_0^{1/2} &= 0.142(p^2) + 0.031(p^4) + 0.051(p^6)\,.
\end{aligned}
$$

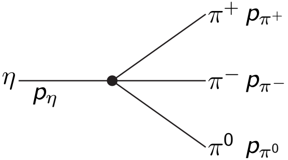
ChPT aspects of $\eta \to 3\pi$

1. Chiral Perturbation Theory
2. Determination of LECs in the continuum
3. $\eta \to 3\pi$: Some model independent comments/results
   - Definitions
   - Experiment
   - Why?
4. $\eta \to 3\pi$ in ChPT
   - LO
   - LO and NLO
   - NNLO
5. Conclusions

Definitions: $\eta \to 3\pi$

Reviews: JB, Gasser, Phys.Scripta T99(2002)34 [hep-ph/0202242]

JB, Acta Phys. Slov. 56(2005)305 [hep-ph/0511076]



$$
\begin{aligned}
s &= (p_{\pi^+} + p_{\pi^-})^2 = (p_\eta - p_{\pi^0})^2 \\
t &= (p_{\pi^-} + p_{\pi^0})^2 = (p_\eta - p_{\pi^+})^2 \\
u &= (p_{\pi^+} + p_{\pi^0})^2 = (p_\eta - p_{\pi^-})^2
\end{aligned}
$$

$$s + t + u = m_\eta^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2 \equiv 3s_0\,.$$

$$\langle \pi^0\pi^+\pi^- \text{out}|\eta\rangle = i\,(2\pi)^4\,\delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right)\,A(s,t,u)\,.$$

$$\langle \pi^0\pi^0\pi^0 \text{out}|\eta\rangle = i\,(2\pi)^4\;\delta^4\left(p_\eta - p_1 - p_2 - p_3\right)\,\overline{A}(s_1,s_2,s_3)$$

$$\overline{A}(s_1,s_2,s_3) = A(s_1,s_2,s_3) + A(s_2,s_3,s_1) + A(s_3,s_1,s_2)$$

Obervables: $\Gamma(\eta \to \pi^+\pi^-\pi^0)$ and $r = \frac{\Gamma(\eta\to\pi^0\pi^0\pi^0)}{\Gamma(\eta\to\pi^+\pi^-\pi^0)}$

Definitions: Dalitz plot

$$x = \sqrt{3}\frac{T_+ - T_-}{Q_\eta} = \frac{\sqrt{3}}{2m_\eta Q_\eta}(u-t)$$

$$y = \frac{3T_0}{Q_\eta} - 1 = \frac{3\left((m_\eta - m_{\pi^o})^2 - s\right)}{2m_\eta Q_\eta} - 1 \overset{\text{iso}}{=} \frac{3}{2m_\eta Q_\eta}(s_0 - s)$$

$$Q_\eta = m_\eta - 2m_{\pi^+} - m_{\pi^0}$$

T^i is the kinetic energy of pion π^i

$$z = \frac{2}{3}\sum_{i=1,3}\left(\frac{3E_i - m_\eta}{m_\eta - 3m_{\pi^0}}\right)^2$$ E_i is the energy of pion π^i

$$|M|^2 = A_0^2\left(1 + ay + by^2 + dx^2 + fy^3 + gx^2y + \cdots\right)$$

$$|\overline{M}|^2 = \overline{A}_0^2\left(1 + 2\alpha z + \cdots\right)$$

Note: neutral, next order: x and y appear separately

Relations

Expand amplitudes and use isospin: JB, Ghorbani, arXiv:0709.0230

$$M(s,t,u) = A\left(1 + \tilde{a}(s-s_0) + \tilde{b}(s-s_0)^2 + \tilde{d}(u-t)^2 + \cdots\right)$$

$$\overline{M}(s,t,u) = A\left(3 + (\tilde{b}+3\tilde{d})\left((s-s_0)^2 + (t-s_0)^2 + (u-s_0)^2\right)\right)$$

Gives relations ($R_\eta = (2m_\eta Q_\eta)/3$)

$$a = -2R_\eta\,\mathrm{Re}(\tilde{a})\,, \quad b = R_\eta^2\left(|\tilde{a}|^2 + 2\mathrm{Re}(\tilde{b})\right)\,, \quad d = 6R_\eta^2\,\mathrm{Re}(\tilde{d})\,.$$

$$\alpha = \frac{1}{2}R_\eta^2\,\mathrm{Re}\left(\tilde{b}+3\tilde{d}\right) = \frac{1}{4}\left(d + b - R_\eta^2|\tilde{a}|^2\right) \le \frac{1}{4}\left(d + b - \frac{1}{4}a^2\right)$$

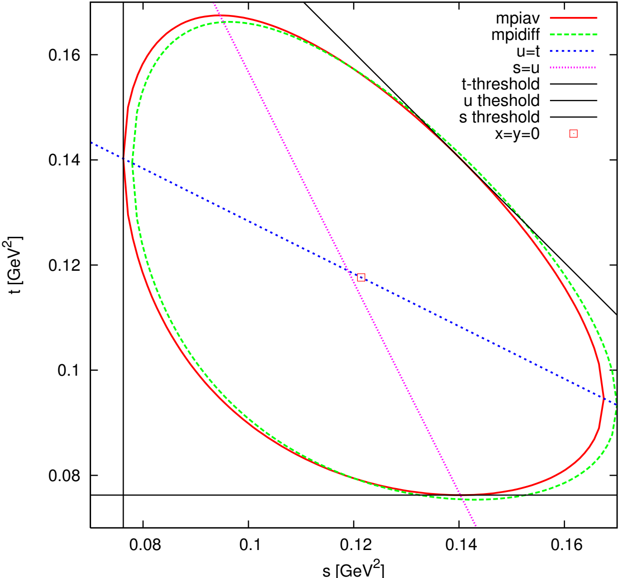
equality if $\mathrm{Im}(\tilde{a}) = 0$

Relations

Consequences:

- Relations between the charged and neutral decay
- Relations between r and Dalitz plot
  (see also Gasser, Leutwyler, Nucl. Phys. B **250** (1985) 539)
- If you can calculate $\mathrm{Im}(\tilde{a})$ then relation:
  nonrelativistic pion EFT
  Schneider, Kubis and Ditsche, JHEP **1102** (2011) 028 [1010.3946].

Definitions: Dalitz plot



x variation: vertical

y variation: parallel to $t = u$

Experiment: Decay rates

Width: determined from $\Gamma(\eta \to \gamma\gamma)$ and Branching ratios
Using the PDG12 partial update 2013 numbers

$\Gamma(\eta \to \pi^+\pi^-\pi^0) = 300 \pm 12$ eV (in JB,Ghorbani 295 ± 17 eV)

r: 1.426 ± 0.026 (our fit)
1.48 ± 0.05 (our average)

Experiment: charged

Exp.	a	b	d	f
WASA (prel)	$-1.104(3)$	$0.144(3)$	$0.073(3)$	$0.153(6)$
KLOE (prel)	$-1.074(23)(3)$	$0.179(27)(8)$	$0.059(25)(10)$	$0.089(58)$
KLOE	$-1.090(5)(^{+8}_{-19})$	$0.124(6)(10)$	$0.057(6)(^{+7}_{-16})$	$0.14(1)(2)$
Crystal Barrel	$-1.22(7)$	$0.22(11)$	$0.06(4)$ (input)	
Layter et al.	$-1.08(14)$	$0.034(27)$	$0.046(31)$	
Gormley et al.	$-1.17(2)(21)$	$0.21(3)$	$0.06(4)$	

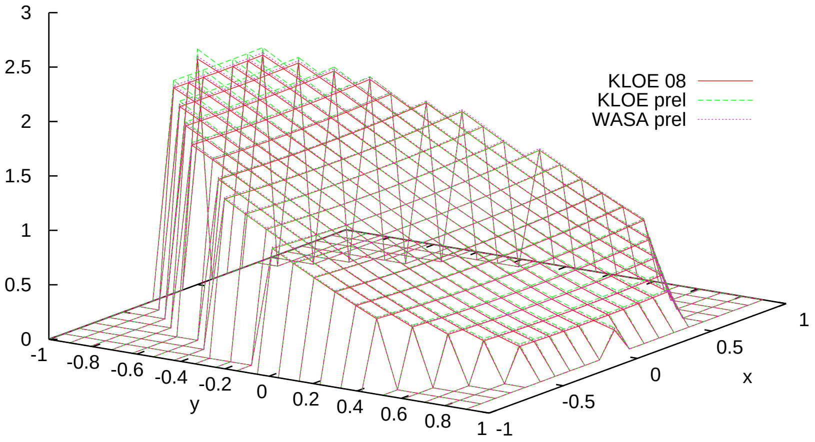
Crystal Barrel: d input, but a and b insensitive to d

Large correlations: KLOE:

	a	b	d	f
a	1	-0.226	-0.405	-0.795
b		1	0.358	0.261
d			1	0.113
f				1

Experiment: charged

But very good agreement:



Experiment: neutral

Exp.	α
GAMS2000	-0.022 ± 0.023
SND	$-0.010 \pm 0.021 \pm 0.010$
Crystal Barrel	$-0.052 \pm 0.017 \pm 0.010$
Crystal Ball (BNL)	-0.031 ± 0.004
WASA/CELSIUS	$-0.026 \pm 0.010 \pm 0.010$
KLOE	$-0.0301 \pm 0.0035^{+0.0022}_{-0.0035}$
WASA@COSY	$-0.027 \pm 0.008 \pm 0.005$
Crystal Ball (MAMI-B)	$-0.032 \pm 0.002 \pm 0.002$
Crystal Ball (MAMI-C)	-0.032 ± 0.003

All experiments in good agreement

Why is $\eta \to 3\pi$ interesting?

- Pions are in $I = 1$ state $\Longrightarrow A \sim (m_u - m_d)$ or α_{em}
- α_{em} effect is small
  - but is there via $(m_{\pi^+} - m_{\pi^0})$ in kinematics
  - Lowest order vanishes (current algebra)
  - $\alpha \hat{m}$ and αm_s small

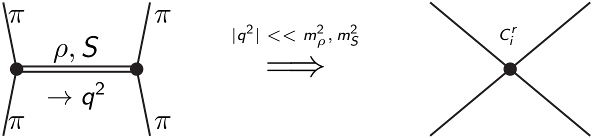
    Baur, Kambor, Wyler, Nucl. Phys. B **460** (1996) 127
  - $\eta \to \pi^+\pi^-\pi^0\gamma$ needs to be included directly

    Ditsche, Kubis, Meissner, Eur. Phys. J. C **60** (2009) 83 [0812.0344]

    Estimates the corrections of $\alpha(m_u - m_d)$ as well
  - Conclusion: at the precision I will discuss not relevant
  - Exception: Cusps and Coulomb at $\pi^+\pi^-$ thresholds
- So $\eta \to 3\pi$ gives a handle on $m_u - m_d$

C_i^r

Most analysis use (i.e. almost all of mine):
C_i^r from (single) resonance approximation



Motivated by large N_c: large effort goes in this

Ananthanarayan, JB, Cirigliano, Donoghue, Ecker, Gamiz, Golterman, Kaiser, Knecht, Peris, Pich, Prades, Portoles, de Rafael,. . .

C_i^r

$$\begin{aligned}
\mathcal{L}_V &= -\frac{1}{4}\langle V_{\mu\nu}V^{\mu\nu}\rangle + \frac{1}{2}m_V^2\langle V_\mu V^\mu\rangle - \frac{f_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle \\
&\quad -\frac{ig_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu,u^\nu]\rangle + f_\chi\langle V_\mu[u^\mu,\chi_-]\rangle \\
\mathcal{L}_A &= -\frac{1}{4}\langle A_{\mu\nu}A^{\mu\nu}\rangle + \frac{1}{2}m_A^2\langle A_\mu A^\mu\rangle - \frac{f_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle \\
\mathcal{L}_S &= \frac{1}{2}\langle\nabla^\mu S\nabla_\mu S - M_S^2S^2\rangle + c_d\langle Su^\mu u_\mu\rangle + c_m\langle S\chi_+\rangle \\
\mathcal{L}_{\eta'} &= \frac{1}{2}\partial_\mu P_1\partial^\mu P_1 - \frac{1}{2}M_{\eta'}^2P_1^2 + i\tilde{d}_m P_1\langle\chi_-\rangle\,.
\end{aligned}$$

$f_V = 0.20,\ f_\chi = -0.025,\ g_V = 0.09,\ c_m = 42\text{ MeV},\ c_d = 32\text{ MeV},$
$\tilde{d}_m = 20\text{ MeV},\ m_V = m_\rho = 0.77\text{ GeV}, m_A = m_{a_1} = 1.23\text{ GeV},$
$m_S = 0.98\text{ GeV},\ m_{P_1} = 0.958\text{ GeV}$

f_V, g_V, f_χ, f_A: experiment

c_m and c_d from resonance saturation at $\mathcal{O}(p^4)$

LUND UNIVERSITY

Chiral perturbation theory and $\eta\to 3\pi$: an introduction

Johan Bijnens

C_i^r

Problems:

- Weakest point in the numerics
- However not all results presented depend on this
- Unknown so far: C_i^r in the masses/decay constants and how these effects correlate into the rest
- No μ dependence: obviously only estimate

What we do/did about it:

- Vary resonance estimate by factor of two
- Vary the scale μ at which it applies: 600-900 MeV
- Check the estimates for the measured ones
- Again: kinematic can be had, quark-mass dependence difficult

L_i^r and C_i^r

Full NNLO fits of the L_i^r

- Amorós,JB,Talavera, 2000, 2001 (fit 10)
  simple C_i^r
- JB, Jemos, 2011 (BJ12)
  simple C_i^r
- JB,Ecker, 2014, (BE14)
  Continuum fit with more input for C_i^r
- Numerics presented for $\eta \to 3\pi$ is mostly with fit 10
  JB,Ghorbani, 2007

Lowest order

ChPT:Cronin 67: $A(s,t,u) = \frac{B_0(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left\{1 + \frac{3(s-s_0)}{m_\eta^2 - m_\pi^2}\right\}$

with $Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$ or $R \equiv \frac{m_s - \hat{m}}{m_d - m_u}$ $\hat{m} = \frac{1}{2}(m_u + m_d)$

$$A(s,t,u) = \frac{1}{Q^2}\frac{m_K^2}{m_\pi^2}(m_\pi^2 - m_K^2)\frac{\mathcal{M}(s,t,u)}{3\sqrt{3}F_\pi^2},$$

$$A(s,t,u) = \frac{\sqrt{3}}{4R}M(s,t,u)$$

LO: $\mathcal{M}(s,t,u) = \frac{3s - 4m_\pi^2}{m_\eta^2 - m_\pi^2}$ $M(s,t,u) = \frac{1}{F_\pi^2}\left(\frac{4}{3}m_\pi^2 - s\right)$

Lowest order

ChPT:Cronin 67: $A(s,t,u) = \frac{B_0(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left\{1 + \frac{3(s-s_0)}{m_\eta^2 - m_\pi^2}\right\}$

with $Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$ or $R \equiv \frac{m_s - \hat{m}}{m_d - m_u}$ $\hat{m} = \frac{1}{2}(m_u + m_d)$

$$A(s,t,u) = \frac{1}{Q^2}\frac{m_K^2}{m_\pi^2}(m_\pi^2 - m_K^2)\frac{\mathcal{M}(s,t,u)}{3\sqrt{3}F_\pi^2},$$

$$A(s,t,u) = \frac{\sqrt{3}}{4R}M(s,t,u)$$

LO: $\mathcal{M}(s,t,u) = \frac{3s - 4m_\pi^2}{m_\eta^2 - m_\pi^2}$ $M(s,t,u) = \frac{1}{F_\pi^2}\left(\frac{4}{3}m_\pi^2 - s\right)$

$\eta \to 3\pi$: p^2 and p^4

- $\Gamma\left(\eta \to 3\pi\right) \propto |A|^2 \propto Q^{-4}$ allows a PRECISE measurement
- Q^2 form lowest order mass relation: $\boxed{Q \approx 24}$
  $\Longrightarrow \Gamma(\eta \to \pi^+\pi^-\pi^0)_{\rm LO} \approx 66$ eV
- $m^2_{K^+} - m^2_{K^0} \sim Q^{-2}$ at NNLO: $\boxed{Q = 20.0 \pm 1.5}$
  $\Longrightarrow \Gamma(\eta \to \pi^+\pi^-\pi^0)_{\rm LO} \approx 140$ eV
- At order p^4 Gasser-Leutwyler 1985: $\dfrac{\int dLIPS|A_2 + A_4|^2}{\int dLIPS|A_2|^2} = 2.4$,

  ($LIPS$=Lorentz invariant phase-space)

  Major source: large S-wave final state rescattering
- Experiment: 300 ± 12 eV (PDG 2012/13)

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$\eta \to 3\pi$: LO, NLO, NNLO, NNNLO,...

- IN Gasser,Leutwyler, 1985 ($\sqrt{2.4} = 1.55$):
  about half: $\pi\pi$-rescattering
  other half: everything else
- $\pi\pi$-rescattering important Roiesnel, Truong, 1981
- Dispersive approach (talks: Passemar, Knecht, Szczepaniak): resum all $\pi\pi$
- assume rescattering + rest separable:

↑ Other effects

⋯	⋯	⋯	⋯
NNLO	⋯	⋯	⋯
NLO	NNLO	⋯	⋯
LO	NLO	NNLO	⋯

→ $\pi\pi$-rescattering

dispersive does this all the way

Why look at it this way?

$\uparrow$ Other effects

$\cdots$	$\cdots$	$\cdots$	$\cdots$
NNLO	$\cdots$	$\cdots$	$\cdots$
NLO	NNLO	$\cdots$	$\cdots$
LO	NLO	NNLO	$\cdots$

$\rightarrow$ $\pi\pi$-rescattering

dispersive does this all the way

- $\delta_\pi = 0.3$, $\delta_O = 0.3$
- LO $= 1$
- NLO $= \delta_\pi + \delta_O = 0.6$
- NNLO $= \delta_\pi^2 + \delta_\pi \delta_O + \delta_O^2 = 0.27$
- Squared: $1 \rightarrow 2.6 \rightarrow 3.5$
- Underlying other is: $1 + 0.3 + 0.09$
- Goal: remove dispersive from ChPT, then add again via dispersion relations (but now all boxes)
- Problem: Separation is not trivial

Why look at it this way?

↑ Other effects

$\cdots$	$\cdots$	$\cdots$	$\cdots$
NNLO	$\cdots$	$\cdots$	$\cdots$
NLO	NNLO	$\cdots$	$\cdots$
LO	NLO	NNLO	$\cdots$

$\rightarrow$ $\pi\pi$-rescattering

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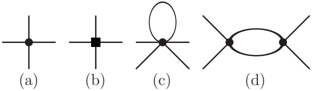
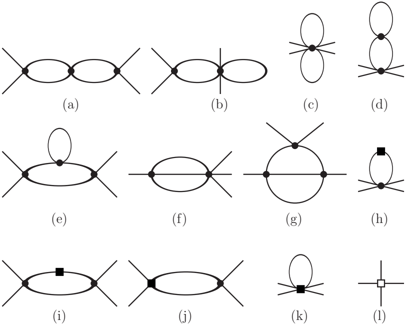
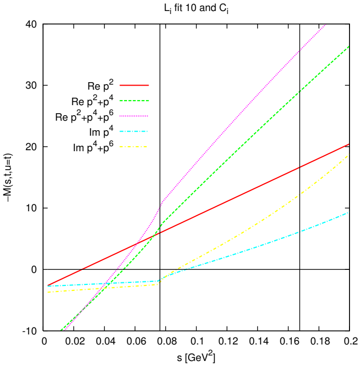
Diagrams



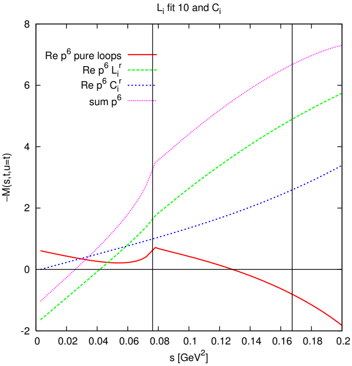


- Include mixing, renormalize, pull out factor $\frac{\sqrt{3}}{4R}$, ...
- Two independent calculations (comparison lots of work)
- You have to carefully define which LO ($\mathcal{M}$ or M)
- You have to carefully define which NLO
- Integrals only in numerical form: (g) is the hardest one

$\eta \to 3\pi$: $M(s, t = u)$

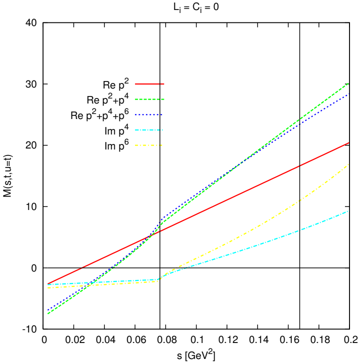


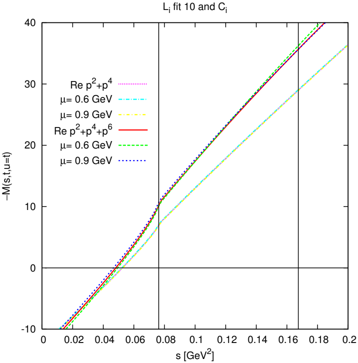
Along $t = u$



Along $t = u$ parts

$\eta \to 3\pi$: $M(s, t = u)$





Along $t = u$
$L_i^r = C_i^r = 0$

Along $t = u$: μ dependence
I.e. where $C_i^r(\mu)$ estimated

Neutral decay

	$\overline{A}_0^2$	α
LO	1090	0.000
NLO	2810	0.013
NLO ($L_i^r = 0$)	2100	0.016
NNLO	4790	0.013
NNLOq	4790	0.014
NNLO ($C_i^r = 0$)	4140	0.011
NNLO ($L_i^r = C_i^r = 0$)	2220	0.016
dispersive (KWW)	—	$-$(0.007—0.014)
tree dispersive	—	-0.0065
absolute dispersive	—	-0.007
Borasoy	—	-0.031
error	160	0.032

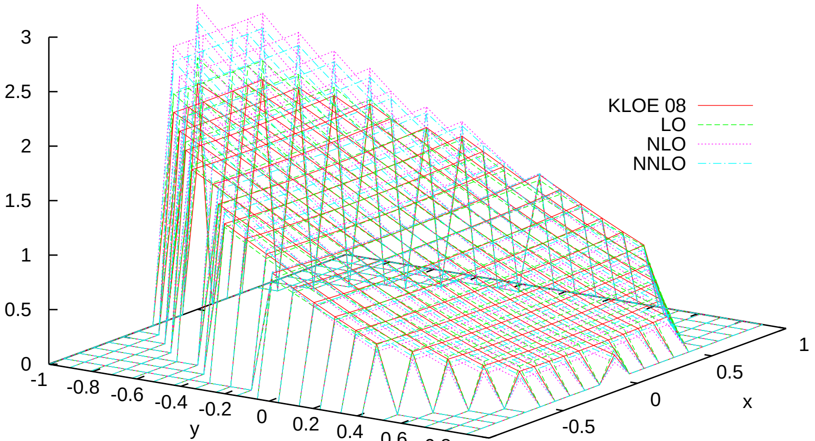
- experiment: $\alpha = -0.032$ with small error
- NNLO ChPT gets a_0^0 in $\pi\pi$ correct

Theory: charged

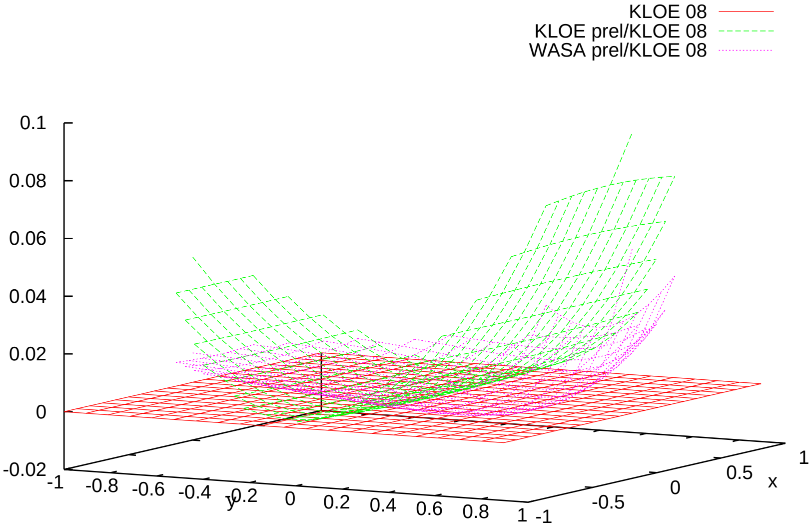
	A_0^2	a	b	d	f
LO	120	-1.039	0.270	0.000	0.000
NLO	314	-1.371	0.452	0.053	0.027
NLO ($L_i^r = 0$)	235	-1.263	0.407	0.050	0.015
NNLO	538	-1.271	0.394	0.055	0.025
NNLOp (y from T^0)	574	-1.229	0.366	0.052	0.023
NNLOq (incl $(x, y)^4$)	535	-1.257	0.397	0.076	0.004
NNLO ($\mu = 0.6$ GeV)	543	-1.300	0.415	0.055	0.024
NNLO ($\mu = 0.9$ GeV)	548	-1.241	0.374	0.054	0.025
NNLO ($C_i^r = 0$)	465	-1.297	0.404	0.058	0.032
NNLO ($L_i^r = C_i^r = 0$)	251	-1.241	0.424	0.050	0.007
BJ12	451	-1.303	0.406	0.060	0.031
BE14	614	-1.356	0.430	0.063	0.038
BE14free	552	-1.339	0.421	0.062	0.036
error	18	0.075	0.102	0.057	0.160
KLOE 08		-1.090	0.124	0.057	0.14

- NLO to NNLO changes, but no large ones
- Error: $\Delta|M(s,t,u)|^2 = |M^{(6)}\, M(s,t,u)|$

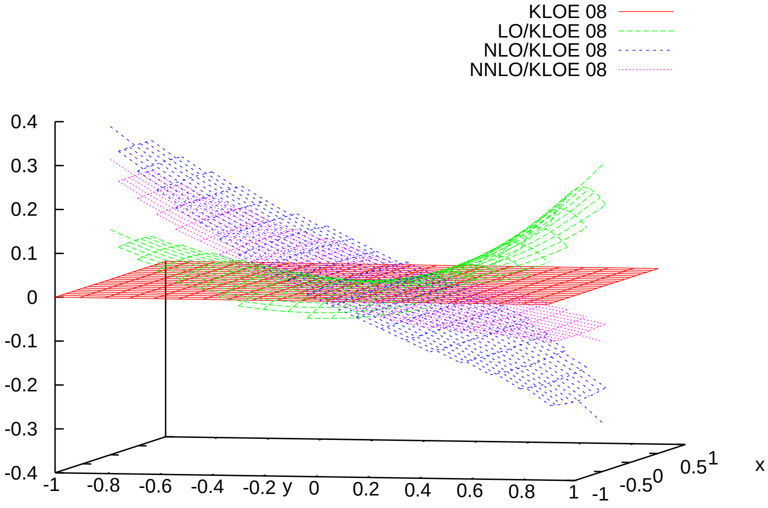
Experiment vs Theory: charged



Experiment : relative



Experiment vs Theory: relative



r and decay rates

$\sin\epsilon = \frac{\sqrt{3}}{4R} + \mathcal{O}(\epsilon^2)$

$$
\begin{aligned}
\Gamma(\eta \to \pi^+\pi^-\pi^0) &= \sin^2\epsilon \cdot 0.572\ \mathrm{MeV} && \mathrm{LO}\,,\\
&\quad \sin^2\epsilon \cdot 1.59\ \mathrm{MeV} && \mathrm{NLO}\,,\\
&\quad \sin^2\epsilon \cdot 2.68\ \mathrm{MeV} && \mathrm{NNLO}\,,\\
&\quad \sin^2\epsilon \cdot 2.33\ \mathrm{MeV} && \mathrm{NNLO}\ C_i^r = 0\,,\\
\Gamma(\eta \to \pi^0\pi^0\pi^0) &= \sin^2\epsilon \cdot 0.884\ \mathrm{MeV} && \mathrm{LO}\,,\\
&\quad \sin^2\epsilon \cdot 2.31\ \mathrm{MeV} && \mathrm{NLO}\,,\\
&\quad \sin^2\epsilon \cdot 3.94\ \mathrm{MeV} && \mathrm{NNLO}\,,\\
&\quad \sin^2\epsilon \cdot 3.40\ \mathrm{MeV} && \mathrm{NNLO}\ C_i^r = 0\,.
\end{aligned}
$$

r and decay rates

$$r \equiv \frac{\Gamma(\eta \to \pi^0\pi^0\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)}$$

$$\begin{aligned} r_{\mathrm{LO}} &= 1.54 \\ r_{\mathrm{NLO}} &= 1.46 \\ r_{\mathrm{NNLO}} &= 1.47 \\ r_{\mathrm{NNLO}\, C_i^r=0} &= 1.46 \end{aligned}$$

PDG 2013

$$\begin{aligned} r &= 1.48 \pm 0.05 \quad \text{our average}\,. \\ r &= 1.426 \pm 0.026 \quad \text{our fit}\,, \end{aligned}$$

Reasonable agreement

R and Q from $eta \to 3\pi$

	LO	NLO	NNLO	NNLO ($C_i^r = 0$)
R (η)	18.9	31.5	40.9	38.2
R (Dashen)	44	44	37	—
R (Dashen-violation)	36	37	32	—
Q (η)	16.5	21.3	24.3	23.4
Q (Dashen)	24	24	22	—
Q (Dashen-violation)	22	22	20	—

LO from $R = \dfrac{m_{K^0}^2 + m_{K^+}^2 - 2m_{\pi^0}^2}{2\left(m_{K^0}^2 - m_{K^+}^2\right)}$ (QCD part only)

NLO and NNLO from masses: Amorós, JB, Talavera 2001

$$Q^2 = \frac{m_s + \hat{m}}{2\hat{m}} R = 14.4R$$

($m_s/\hat{m} = 27.8$ used for $\eta \to 3\pi$)

Conclusions

- Short introduction to ChPT
- New best fit for the L_i^r: BE14
- Overview of $\eta \to 3\pi$
- $\eta \to 3\pi$ at NNLO in ChPT plus some preliminary updates