New and Old Results on the 3D Ising Model.

Michele Caselle

Universitá degli Studi di Torino

Genova 15/05/2014

Summary:

1. Introduction and motivation
2. Duality
3. Lorentz invariance and "universality".
4. Application: the boundary term of the effective action
5. Effective string theory description of the Ising spectrum.

The Ising Model.

Few ingredients:

- Cubic lattice (lattice spacing a)
- On each site i a spin $S_i \in \{1, -1\}$
- Hamiltonian:

$$\mathcal{H}\{s_i\} = -J \sum_{<i,j>} s_i s_j - H \sum_i s_i$$

  where $< i, j > =$ sum on n.n. sites (links of the lattice)

- Partition Function:

$$Z = \sum_{\{s_i\}} e^{-\frac{1}{kT}\mathcal{H}} = \sum_{\{s_i\}} e^{\beta \sum_{<i,j>} s_i s_j + h \sum_i s_i}$$

  with $\beta = J/kT$ and $h = H/kT$

The Ising Model.

Very rich behaviour:

- Z_2 global symmetry: $s_i \to -s_i \quad \forall i$
- Second order phase transition at a finite value $\beta_c = 0.2216543(2)$ of the coupling
- Z_2 symmetry spontaneously broken in the low T phase
- Order parameter: magnetization

$$M \equiv \frac{\partial}{\partial h} lnZ = < \sum_i s_i >$$

- Several experimental realizations:
  - Uniaxial antiferromagnets
  - Binary mixtures
  - liquid-vapour transition

Critical behaviour.

In the vicinity of the critical point we observe a non trivial (i.e. non mean field) critical behaviour:

Defining $t \equiv \frac{T-T_c}{T_c} \equiv \frac{\beta_c-\beta}{\beta}$

- Magnetization

$$M(-t) \sim (-t)^{-\beta}, \qquad \beta = 0.3270(6)$$

- Susceptibility

$$\chi(t) \sim t^{-\gamma}, \qquad \gamma = 1.2390(15)$$

- Specific heat

$$C(t) \sim t^{-\alpha}, \qquad \alpha = 0.107(4)$$

- Correlation length

$$\xi(t) \sim t^{-\nu}, \qquad \nu = 0.6310(15)$$

Renormalization group approach.

Scaling relations:

$$\alpha + 2\beta + \gamma = 2$$

$$d\nu = 2 - \alpha \qquad\qquad (d \leq 4)$$

Only two of the indices are really independent, they are related to the RG eigenvalues of the two relevant operators of the theory: energy (which is Z_2 even) and magnetization (which is Z_2 odd).

Irrelevant operators (for instance those related to the breaking of rotational invariance on the lattice) give corrections to scaling contributions. Their RG eigenvalues are related to the so called "correction to scaling exponents".

Universality.

- The cubic lattice Ising model belongs to the "3d Ising universality class", whose field theory realization is the 3d ϕ^4 (with $\phi \in \mathbf{R}$)
- Besides the critical indices other "universal ratios" of scaling amplitudes can be constructed (and measured in experiments). A few examples:

$$\Gamma_\chi \equiv \lim_{t \to 0^+} \frac{\chi(t)}{\chi(-t)}$$

$$\Gamma_\xi \equiv \lim_{t \to 0^+} \frac{\xi(t)}{\xi(-t)}$$

$$\Gamma_A \equiv \lim_{t \to 0^+} \frac{C(t)}{C(-t)}$$

The Ising Model.

These universal quantities can be estimated using

- Perturbative expansions in the 3d ϕ^4 theory (two options: ϵ expansion starting from $d = 4$ and then setting $\epsilon = 1$ or direct FT calculations in $d = 3$)
- Strong coupling expansions on the lattice
- Montecarlo simulations

Operator	Z_2	Δ
σ	$-$	0.5182(3)
σ'	$-$	> 4.5
ϵ	$+$	1.413(1)
ϵ'	$+$	3.84(4)
ϵ''	$+$	4.67(11)

Table : Critical dimensions of some low-lying operators of the 3D Ising model.

The Ising Model.

- The agreement between Montecarlo simulations and perturbative results seems to suggest that there is nothing really interesting in the model and that everything can be understood perturbatively.
- This is wrong. There are a few important observables for which perturbative results and simulations strongly disagree.
- They open a window to new unexpected physics (and maybe can be also observed in experiments).
- To understand them duality (and the corresponding mapping to the 3d gauge Ising model) and string theory is mandatory

The ξ/ξ_{2m} ratio.

The large distance behavior of connected two point function $G(\tau) \equiv \langle s_0 s_n \rangle_c$ is:

$$G(\tau) =\sim \exp(-\tau/\xi)$$

where ξ is the exponential correlation length and is given by:

$$\frac{1}{\xi} = - \lim_{|n|\to\infty} \frac{1}{|n|} \log \langle s_0 s_n \rangle_c \quad .$$

An useful estimator for ξ is the so called second moment correlation length defined as follows:

$$\xi_{2m}^2 = \frac{\mu_2}{2d\mu_0} \quad ,$$

where

$$\mu_0 = \lim_{L\to\infty} \frac{1}{V} \sum_{m,n} \langle s_m s_n \rangle_c$$

and

$$\mu_2 = \lim_{L\to\infty} \frac{1}{V} \sum_{m,n} (m-n)^2 \langle s_m s_n \rangle_c \ ,$$

The ξ/ξ_{2m} ratio.

The relation between ξ and ξ_{2m} becomes clear if we write:

$$\xi_{2m}^2 = \frac{\sum_{\tau=-\infty}^{\infty} \tau^2 \; G(\tau)}{2\sum_{\tau=-\infty}^{\infty} \; G(\tau)} \quad .$$

Assuming a simple exponential decay for $G(\tau)$,

$$\langle S_0 \; S_\tau \rangle_c \propto c \; \exp(-|\tau|/\xi) \quad ,$$

and replacing the summation by an integration over τ we get

$$\xi_{2m}^2 = \frac{1}{2} \; \frac{\int_{\tau=0}^{\infty} d\tau \; \tau^2 c \; \exp(-\tau/\xi)}{\int_{\tau=0}^{\infty} d\tau \; c \exp(-\tau/\xi)} = \xi^2,$$

The ξ/ξ_{2m} ratio.

The ratio ξ/ξ_{2m} is universal and can be evaluated perturbatively in the ϕ^4 theory. One finds (at one loop)

$$\xi/\xi_{2m} \sim 1.0065$$

why a result greater than one? The point is that the single exponential ansatz is a too trivial representation of the connected correlator. A more refined description assumes a multiple exponential ansatz:

$$\langle S_0 \; S_\tau \rangle_c \propto \sum_i \; c_i \; \exp(-|\tau|/\xi_i) \;\; ,$$

The ξ/ξ_{2m} ratio.

Then we obtain

$$\xi_{2m}^2 = \frac{1}{2}\,\frac{\int_{\tau=0}^{\infty} d\tau\ \tau^2 \sum_i c_i\ \exp(-\tau/\xi_i)}{\int_{\tau=0}^{\infty} d\tau\ \sum_i c_i \exp(-\tau/\xi_i)} = \frac{\sum_i c_i \xi_i^3}{\sum_i c_i \xi_i}\ ,$$

The ξ/ξ_{2m} ratio is a tool to evaluate the spectrum of the theory!

$$\frac{\xi}{\xi_{2m}} \equiv \frac{\xi_1}{\xi_{2m}} = 1 + \frac{c_2}{2c_1}\frac{\xi_2}{\xi_1}\left(1 - \frac{\xi_2^2}{\xi_1^2}\right)$$

The ξ/ξ_{2m} ratio.

In the perturbative approach to ϕ^4 theory we have a cut located at twice the mass ($m = 1/\xi_1$) of the theory, this is the origin of the correction in the ratio:

$$
\begin{aligned}
G(\tau)_{pert} &= \frac{1}{2m_{ph}L^2} e^{-m_{ph}\tau} \left[1 + \frac{1}{32}\frac{u_R}{4\pi}\right] \\
&\quad + \frac{3u_R}{16\pi L^2 m_{ph}} \int_{2m_{ph}}^{\infty} d\mu \frac{e^{-\mu\tau}}{\mu\left(1 - \frac{\mu^2}{m_{ph}^2}\right)^2} \, .
\end{aligned}
$$

where u_R denotes the dimensionless renormalized coupling, $m_{ph} \equiv 1/\xi$ is the physical mass.

The ξ/ξ_{2m} ratio.

From this expression we find: in the low temperature phase

$$\frac{\xi}{\xi_{2m}} = 1 + 0.00573... \frac{u_R}{4\pi} \sim 1.0065$$

From Montecarlo simulation[1] instead we find

$$\frac{\xi}{\xi_{2m}} = 1.030(5)$$

From Strong coupling expansion[2] we find

$$\frac{\xi}{\xi_{2m}} = 1.032(6)$$

[1] M.C., M.Hasenbusch, J.P.**A30** (1997) 4963
[2] Arisue, Tabata N.P.**B435** (1995) 555

The ξ/ξ_{2m} ratio.

- Note: no discrepancy is observed in the high T phase!
- Question: Is there some problem with universality (i.e. $\phi^4 \neq$Ising) ?
- No! direct lattice simulation of a discretized version of ϕ^4 gives again $\frac{\xi}{\xi_{2m}} \sim 1.030$
- The only possible explanation is that at least one state (presumably more) must exist in the spectrum besides the lowest mass and its cuts

The ξ/ξ_{2m} ratio.

Test: direct evaluation of the spectrum via MC simulation: we found a new state with mass $m_2 = 1.85m$
(i.e. below threshold), most probably a bound state of the lowest mass. We also found[1] [2] :

- indications of a rich spectrum of non perturbative states with higher masses
- exactly the same pattern of states in Ising and ϕ^4

To understand what's going on we need

Duality and **String Theory**

[1] M.C., M.Hasenbusch and P.Provero, N.P.**B556** (1999) 575

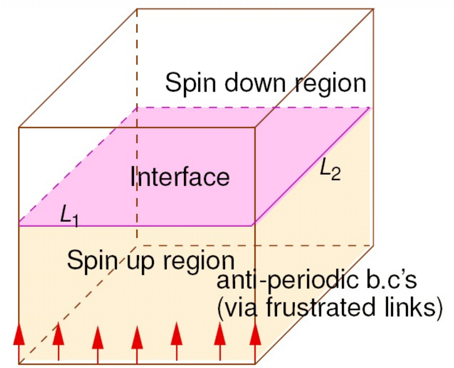
[2] M.C., M.Hasenbusch, P.Provero and K.Zarembo, P.R.**D62** (2000) 17901

Fluid interfaces

- Interfaces play an important role in several physical systems ranging from soft condensed matter to high energy physics.
- 3D spin models (and in particular the Ising model) offer a simple context where interfaces appear and can be studied, e.g. by using numerical simulations to check theoretical predictions.
- Between the roughening and the critical temperature of the 3D Ising model, interfaces are dominated by long wavelength fluctuations (i.e. they behave as *fluid* interfaces).

Fluid interfaces

The simplest way to force the presence of an interface in the Ising model is by fixing antiperiodic boundary condition in one direction.



The interface free energy is given by $F = -\log \frac{Z_{ap}}{Z_p}$ where (Z_{ap}) Z_p is the partition function of the Ising model with (anti)periodic b.c.

The capillary wave model

An effective model widely used to describe a rough interface is the *capillary wave model* (CWM) [1]

$$S_0 = \frac{1}{2}\int_0^{L_1} dx \int_0^{L_2} dy \ \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 \quad .$$

where $\phi(x,y)$ describes the transverse displacement of the minimal surface: a flat torus of area $L_1 L_2$ if periodic b.c in the longitudinal directions are chosen. Main assumption: no foldings nor self-intersections nor overhangs $\rightarrow$ $\phi(x,y)$ is a single–valued function of x and y.

[1] F. P. Buff, R. A. Lovett and F. H. Stillinger Jr., Phys. Rev. Lett. 15 (1965) 621

The capillary wave model

Within this approximation, the interface partition function takes the form

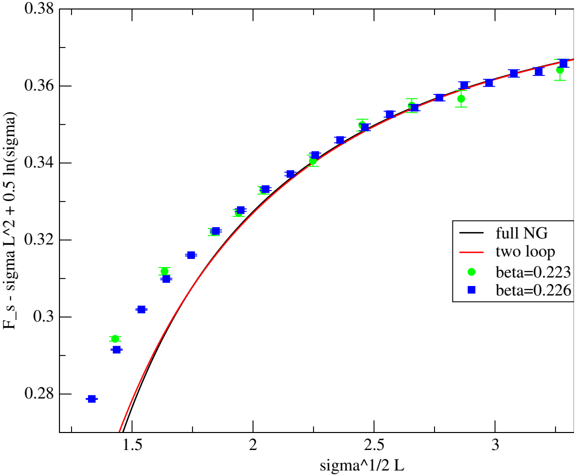
$$Z \equiv e^{-F} = \lambda \, e^{-\sigma L_1 L_2} Z_q^{(g)}(u) \quad .$$

where $u = \frac{L_2}{L_1}$, λ is an undetermined constant and $Z_q^{(g)}(u)$ is the result of the (gaussian) functional integration over the ϕ (i.e. the contribution to the free energy of the sum over all the possible "shapes" of the interface).
For $u = 1$, i.e. $L_1 = L_2 \equiv L$, $Z_q^{(g)}(u) = 1$ and one finds

$$Z = \lambda \, e^{-\sigma L^2} \quad \rightarrow \quad F = \sigma L^2 + const$$

This result however is largely unsatisfactory. Montecarlo simulations show large deviations which behave as $\delta F = c/\sigma L^2$. Fitting the data one finds $c \sim 0.25$

square interfaces



Duality: Z_2 gauge theory

The building blocks of the Z_2 gauge model are the link variables $g_{n;\mu} \in \{-1, 1\}$, which play the role of gauge fields. Denoting by μ the direction of the link, the action is

$$S_{gauge} = -\beta \sum_{n,\mu<\nu} g_{n;\mu\nu}$$

where $g_{n;\mu\nu}$ are the plaquette variables, defined by

$$g_{n;\mu\nu} = g_{n;\mu}\, g_{n+\mu;\nu}\, g_{n+\nu;\mu}\, g_{n;\nu} \quad .$$

This action is invariant under local Z_2 gauge transformations. The plaquettes in the action, like any other product of links along a closed path, are invariant under this gauge transformation.

Duality and the Z_2 gauge model

The $3D$ Ising model and Z_2 gauge theory are related by an exact duality transformation known as Kramers–Wannier duality. Main properties:

- It relates the partition functions of the two models evaluated at two different values of the coupling constants:

$$Z_{gauge}(\beta) \propto Z_{spin}(\tilde{\beta}) \quad \tilde{\beta} = -\frac{1}{2} \log\left[\tanh(\beta)\right]$$

  where $\tilde{\beta}$ is the "dual coupling"

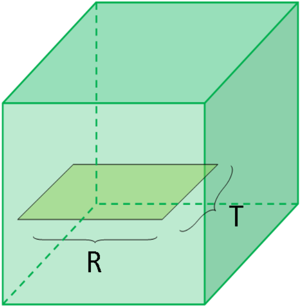
- Low values of β are mapped into high values of $\tilde{\beta}$ and vice versa.
- The broken symmetry phase of the spin model is mapped into the confining phase of the gauge theory.
- the end points of these two phases, the deconfinement transition and the magnetization transition, are mapped into each other.

Duality: Interfaces

- Duality relates the interface to two important non local, gauge invariant, observables in gauge theories: The Wilson loop and the correlator of two Polyakov loops.
- A Wilson loop is the (trace of the) ordered product of links variables along a closed path, usually a rectangle of size RxT
- A Polyakov loop is the (trace of the) ordered product of link variables along a line which winds in one of the directions in which the have chosen periodic boundary conditions for the model. It forms a closed path thanks to the periodic boundary conditions.
- They are very important in LGTs, since they allow to evaluate the "interquark potential".
- Interfaces are obtained imposing antiperiodic boundary conditions, i.e. "frustrating" a layer of links in the 3d Ising model. Wilson loops and Polyakov loop correlators are the dual transformation of the observables obtained frustrating only the links inside the (dual) of the wilson loop or the (dual) of the region bordered by two Polyakov loops.

Wilson Loop.

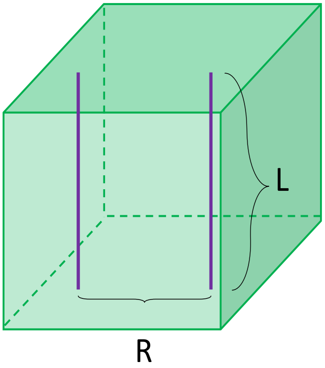
A Wilson loop of size $R \times T$



$$V(R) = \lim_{T \to \infty} -\frac{1}{T} \log < W(R,T) >$$

Polyakov loop correlator.

Expectation value of two Polyakov loops at distance R and Temperature $T = 1/L$



$$V(R) = -\lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^{\dagger} >$$

Lattice determination of the interquark potential.

In pure lattice gauge theories the interquark potential is usually extracted from two (almost) equivalent observables

- Wilson loop expectation values $< W(R,T) >$ ("zero temperature potential")

$$V(R) = \lim_{T\to\infty} -\frac{1}{T} \log < W(R,T) >$$

- Polyakov loop correlators $< P(0)P(R)^\dagger >$ ("finite temperature potential")

$$< P(0)P(R)^\dagger > \sim \sum_{n=0}^{\infty} c_n \; e^{-LE_n}$$

where L is the inverse temperature, i.e. the length of the lattice in the compactified imaginary time direction

$$E_0 = V(R) = -\lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^\dagger >$$

Wilson Loops.

In the Wilson loop framework confinement is equivalent to the well known area-perimeter-constant law:

$$< W(R,T) >= e^{-(\sigma RL + c(R+T) + k)}$$

which implies $V(R) = \sigma R + c$.
Confinement is usually associated to the creation (via a mechanism which still has to be understood) of a thin flux tube joining the quark antiquark pair. (Nielsen-Olesen, 't Hooft, Wilson, Polyakov, Nambu ....) However if we accept this picture we cannot neglect the quantum fluctuations of this flux tube. The area law is thus only the classical contribution to the interquark potential and we should expect quantum corrections to its form. The theory which describes these quantum fluctuations is known as "effective string theory".

Effective string action

The simplest choice for the effective string action is to describe the quantum fluctuations of the flux tube as free massless bosonic degrees of freedom

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \, [\partial_\alpha X \cdot \partial^\alpha X] \; ,$$

where:

- S_{cl} describes the usual ("classical") area-perimeter term.
- $X_i(\xi_0, \xi_1)$ $(i = 1, \ldots, d-2)$ parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand
- ξ_0, ξ_1 are the world-sheet coordinates.

This is nothing else than the old Capillary Wave Model !

CWM for the Wilson loop

- The functional integration is a trivial gaussian integral,

$$< W(R,T) >= \int e^{-\sigma RT - \frac{\sigma}{2}\int d^2\xi X^i(-\partial^2)X^i}$$

- In the effective string language this is equivalent to sum over all the possible string configuration compatible with the Wilson loop (i.e. with Dirichlet boundary conditions along the Wilson loop).
- The result is

$$< W(R,T) > = e^{-\sigma RT} \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}} \quad .$$

where $\eta(\tau)$ is the Dedekind function

$$\eta(\tau) = q^{1/24}\Pi_{n=1}^{\infty}(1-q^n)$$

with $q \equiv e^{2\pi i\tau}$ and $\tau = iT/R$.

Evaluation of the CWM correction for the Wilson loop.

- The gaussain integration gives:

$$\int e^{-\frac{\sigma}{2}\int d^2\xi X^i(-\partial^2)X^i} \propto \left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} \quad .$$

- The determinant must be evaluated with Dirichlet boundary conditions. The spectrum of $-\partial^2$ with Dirichlet boundary conditions is:

$$\lambda_{mn} = \pi^2 \left(\frac{m^2}{T^2} + \frac{n^2}{R^2}\right)$$

corresponding to the normalized eigenfunctions

$$\psi_{mn}(\xi) = \frac{2}{\sqrt{RT}} \sin\frac{m\pi\tau}{T} \sin\frac{n\pi\varsigma}{R} \quad .$$

Evaluation of the CWM correction for the Wilson loop.

- The determinant can be regularized with the ζ-function technique: defining

$$\zeta_{-\partial^2}(s) \equiv \sum_{mn=1}^{\infty} \lambda_{mn}^{-s}$$

the regularized determinant is defined through the analytic continuation of $\zeta'_{-\partial^2}(s)$ to $s = 0$:

$$\det(-\partial^2) = \exp\left[-\zeta'_{-\partial^2}(0)\right] \quad .$$

- The result is

$$\left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} = \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}} \quad .$$

where $\eta(\tau)$ is the Dedekind function

$$\eta(\tau) = q^{1/24}\Pi_{n=1}^{\infty}(1-q^n)$$

with $q \equiv e^{2\pi i\tau}$ and $\tau = iT/R$.

The Interface case

- In the interface case the only change is in the boundary conditions. As a result we find
-

$$Z_q^{(g)}(u) = \frac{1}{\sqrt{u}} \left| \eta(iu) / \eta(i) \right|^{-2} \quad ,$$

  with $u = \frac{L_2}{L_1}$
- Setting $u = 1$ we find, as anticipated, $Z_q^{(g)} = 1$

Implications for the Interquark potential: the "Lüscher term"

- Using the definition of the interquark potential

$$V(R) = \lim_{T\to\infty} -\frac{1}{T} \log < W(R,T) >$$

one finds

$$V(R) = \sigma R - \frac{(d-2)\pi}{24R} + c$$

- This quantum correction is known as "Lüscher term" and is universal i.e. it does not depend on the ultraviolet details of the gauge theory but only on the geometric properties of the flux tube.

The Lüscher term.

This correction is in remarkable agreement with numerical simulations. First high precision test in d=4 SU(3) LGT more than ten years ago. [1]

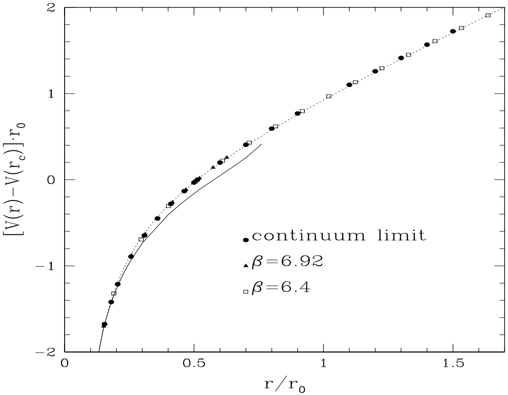


Figure : The static potential. The dashed line represents the bosonic string model and the solid line the prediction of perturbation theory.

[1] S. Necco and R. Sommer, Nucl.Phys. B622 (2002) 328

The Lüscher term.

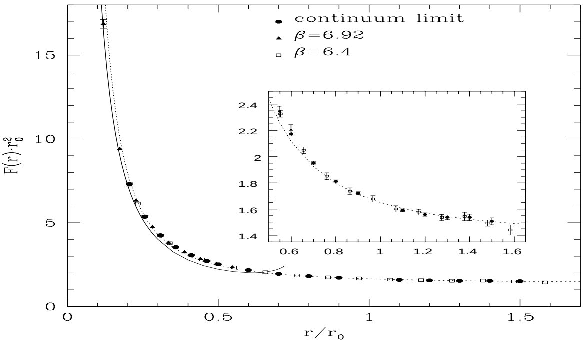


Figure : The force in the continuum limit and for finite resolution, where the discretization errors are estimated to be smaller than the statistical errors. The full line is the perturbative prediction. The dashed curve corresponds to the bosonic string model normalized by $r_0^2 F(r_0) = 1.65$.

The Nambu-Goto action.

- Evaluation of higher order quantum corrections requires further hypothesis on the nature of the interface (or of the flux tube in the language of LGTs). The simplest choice is the Nambu-Goto string in which quantum corrections are evaluated summing over all the possible surfaces bordered by the Wilson loop with a weight proportional to their area.

$$S = \sigma \int d^2\xi \sqrt{det(\eta_{\alpha\beta} + \partial_\alpha X \cdot \partial_\beta X)}$$

$$\sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8}(\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4}(\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right],$$

Derivation of the Nambu-Goto action.

- The Nambu-Goto action is given by the area of the world–sheet:

$$S = \sigma \int_0^T d\tau \int_0^R d\varsigma \sqrt{g} \quad ,$$

where g is the determinant of the two–dimensional metric induced on the world–sheet by the embedding in R^d:

$$g = \det(g_{\alpha\beta}) = \det\ \partial_\alpha X^\mu \partial_\beta X^\mu \quad (\alpha, \beta = \tau, \varsigma,\ \mu = 1, \ldots, d)$$

- Choosing the "physical gauge"

$$X^1 = \tau \quad X^2 = \varsigma$$

g may be expressed as a function of the transverse degrees of freedom only:

$$\begin{aligned} g &= 1 + \partial_\tau X^i \partial_\tau X^i + \partial_\varsigma X^i \partial_\varsigma X^i \\ &\quad + \partial_\tau X^i \partial_\tau X^i \partial_\varsigma X^j \partial_\varsigma X^j - (\partial_\tau X^i \partial_\varsigma X^i)^2 \ (i = 3, \ldots, d) \ . \end{aligned}$$

- Expanding we find:

$$S \sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8} (\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4} (\partial_\alpha X \cdot \partial_\beta X)^2 + \ldots \right] ,$$

Interface free energy from the Nambu-Goto action.

The N-G action can be evaluated and resummed to all orders using standard covariant quantization techniques for the bosonic string[1].

$$\mathcal{Z}^{(d)} = 2\left(\frac{\sigma}{2\pi}\right)^{\frac{d-2}{2}} V_T \sqrt{\sigma \mathcal{A} u} \sum_{m=0}^{\infty} \sum_{k=0}^{m} c_k c_{m-k} \left(\frac{\mathcal{E}}{u}\right)^{\frac{d-1}{2}} K_{\frac{d-1}{2}}\left(\sigma \mathcal{A} \mathcal{E}\right) ,$$

where $\mathcal{A} = L_1 L_2$, V_T is the product of the system sizes in the transverse directions, K_l denotes the Bessel function of order l and $\mathcal{E}$ denotes the spectrum levels:

$$\mathcal{E} = \mathcal{E}_{k,m} = \sqrt{1 + \frac{4\pi}{\sigma L_1^2}\left(m - \frac{d-2}{12}\right) + \frac{4\pi^2}{\sigma^2 L_1^4}(2k-m)^2} , \qquad (1)$$

[1] M. Billo', M. Caselle, L. Ferro, JHEP 0602 (2006) 070

Interface free energy from the Nambu-Goto action.

- This result allows also to fix the scaling behaviour of the λ constant which in three dimensions turns out to be $\lambda \sim \sqrt{\sigma}\ V_T$
- Defining $F_s \equiv -logZ/V_T$, we find:

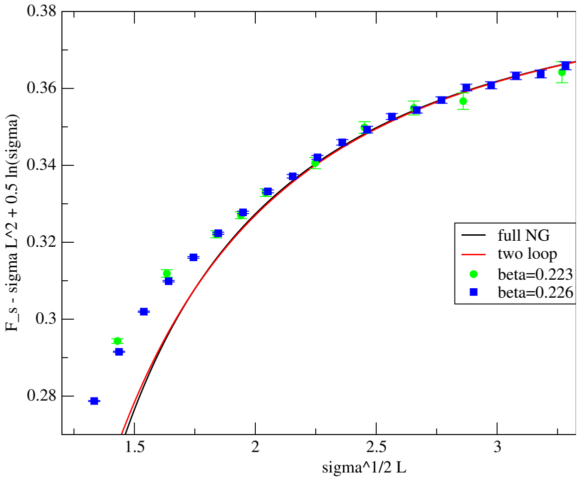
$$F_s = \sigma L^2 - \frac{1}{2}\ln\sigma - \frac{1}{4\sigma L^2} + O\left(\frac{1}{(\sigma L^2)^2}\right) \quad . \tag{2}$$

In remarkable agreement with MC simulation[1]!

- Notice two nontrivial features of this agreement: even if the theory is not renormalizable it predicts exactly that
  - there is no $1/L$ correction.
  - the coefficient of the $1/L^2$ term is $1/4$

[1] M.C., M. Hasenbusch and M.Panero JHEP 0603 (2006) 084, JHEP 0709:117,2007.

square interfaces



Interquark potential from the Nambu-Goto action.

- In the framework of the Nambu-Goto action one can evaluate exactly the energy of all the excited states of the flux tube:

$$E_n(R) = \sqrt{\sigma^2 R^2 + 2\pi\sigma \left(n - \frac{D-2}{24}\right)}$$

- In particular $E_0(R)$ corresponds to the interquark potential

$$V(R) = E_0(R) = \sqrt{\sigma^2 R^2 - 2\pi\sigma \frac{D-2}{24}} ,$$

$$V(R) \quad \sim \quad \sigma R - \frac{\pi(D-2)}{24R} - \frac{1}{2\sigma R^3} \left(\frac{\pi(D-2)}{24}\right)^2 + O(1/R^5) ,$$

The Nambu-Goto action.

High precision fit in the SU(2) case in 2+1 dimensions (A. Athenodorou, B. Bringoltz, M. Teper JHEP 1105:042 (2011))

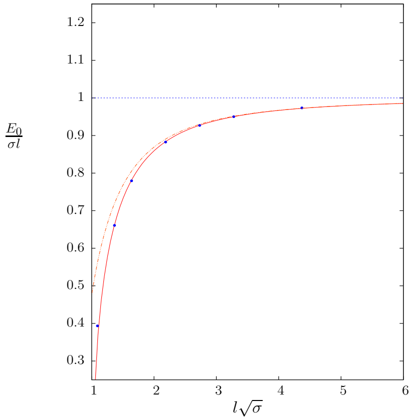


Figure 6: Energy of absolute ground state for SU(2) at $\beta = 5.6$. Compared to full Nambu-Goto (solid curve) and just the Lüscher correction (dashed curve).

Interquark potential via Polyakov Loop correlators.

- In this case we have different boundary conditions in the two directions (space R and inverse temperature L).
- The novel feature of this observable is that by exchanging R and L (the so called "open-closed string transformation") we can study the finite temperature behaviour of the string tension.

$$V(R) = \sigma(T)R, \quad \sigma(T) = \sigma_0 \sqrt{1 - \frac{(d-2)\pi T^2}{3\sigma_0}}$$

where T is now the temperature and σ_0 the zero temperature string tension

- From this expression we may deduce a "Nambu-Goto" prediction for the critical temperature:

$$\frac{T_c}{\sqrt{\sigma_0}} = \sqrt{\frac{3}{(d-2)\pi}}$$

which turns out to be in remarkable agreement with LGT results both in d=3 and d=4.

Can we really trust these results?

- These results look nice, but they depend on a set of ad hoc assumptions on the behaviour of the flux tube. Why should we prefer the Nambu-Goto action to other possible choices for the flux tube action?
- They are "too universal" and show no dependence on the gauge group.
- It is somehow surprising that the Nambu-Goto model which looks so complex can be solved exactly at the quantum level (to all orders!!). How is it possible?
- Is there a "boundary" contribution due to the quarks at the flux tube boundaries?

In the past few years two important results changed our understanding of effective string theories and allowed us to answer to the above questions

Universality of effective string corrections.

- The Effective String action is strongly constrained by Lorentz invariance. The first few orders of the action are universal and coincide with those of the Nambu-Goto action. This explains why N.-G. describes so well the infrared regime of Wilson loops or Polyakov Loop correlators.[1] [2] [3]
- The Nambu-Goto effective theory can be described as a free 2d bosonic theory perturbed by the irrelevant operator $T\bar{T}$ (where T and $\bar{T}$ are the two chiral components of the energy momentum tensor). This perturbation turns out to be quantum integrable and yields, using the Thermodynamic Bethe Ansatz (TBA), a spectrum which, in a suitable limit, coincides with the Nambu-Goto one. [4]

[1] M. Luscher and P. Weisz JHEP07(2004)014
[2] H. B. Meyer JHEP05(2006)066
[3] O. Aharony and M. Field JHEP01(2011)065
[4] M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo JHEP07(2013)071

Effective string action

The most general action for the effective string can be written as a low energy expansion in the number of derivatives of the transverse fields ("physical gauge").

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + c_2 (\partial_\alpha X \cdot \partial^\alpha X)^2 + c_3 (\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right] + S_b \, ,$$

where:

- S_{cl} describes the usual ("classical") perimeter-area term.
- S_b is the boundary contribution characterizing the open string
- $X_i(\xi_0, \xi_1)$ $(i = 1, \dots, d-2)$ parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand
- ξ_0, ξ_1 are the world-sheet coordinates.

Effective string and spacetime symmetries.

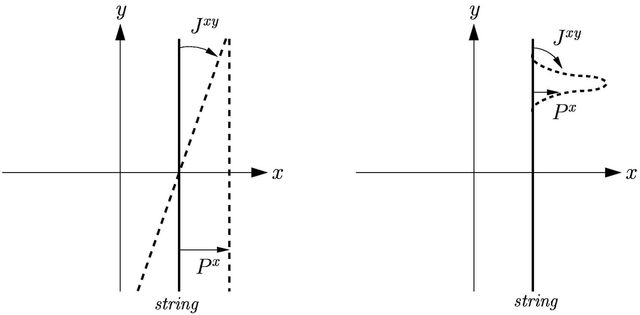
- Symmetries of the action must hold in the low energy regime.
- String vacuum is not Poincaré invariant.

$\Longrightarrow$ Poincaré symmetry is broken spontaneously.

$ISO(D-1,1) \to SO(D-2) \otimes ISO(1,1)$. $\Longrightarrow$ $3(D-2)$ Goldstone bosons?

Just $D-2$ tranverse fluctuations of the string.

The remaining $2(D-2)$ Lorentz transformations are realized non-linearly



Non-linear realization and long-string expansion.

An internal transformation of the fields realizes the Poincaré group:

- Broken translations:
  $X^i \to X^i + a^i$. $\Longrightarrow$ Only field derivatives in the effective action.
- Broken rotation in the plane $(1, 2)$:

$$\delta_{\epsilon}^{bj} X_i = \epsilon \left(-\delta_{ij}\xi_b - X_j \partial_b X_i \right)$$

Number of derivatives minus number of fields (scaling) preserved.

Fields and coordinates rescaling $\Longrightarrow$ Derivative expansion:

$$\partial_a X^i \longrightarrow \frac{1}{\sqrt{\sigma} R} \partial_a X^i.$$

Variations by broken rotation mix orders $\Longrightarrow$ Recurrence relations.

$ISO(1, 1)$ and $SO(D-2)$ invariance $\Longrightarrow$ Contraction of indices.

Effective string action is strongly constrained! [1] [2] [3]

- the terms with only first derivatives coincide with the Nambu-Goto action to all orders in the derivative expansion.
- The first allowed correction to the Nambu-Goto action turns out to be the the six derivative term

$$c_4 \left(\partial_\alpha \partial_\beta X \cdot \partial^\alpha \partial^\beta X\right) \left(\partial_\gamma X \cdot \partial^\gamma X\right)$$

  with arbitrary coefficient c_4
- however this term is non-trivial only when $d > 3$. For $d = 3$ the first non-trivial deviation of the Nambu-Goto action is an eight-derivative term
- The fact that the first deviations from the Nambu-Goto string are of high order, especially in $d = 3$, explains why in early Monte Carlo calculations a good agreement with the Nambu-Goto string was observed.

[1] M. Luscher and P. Weisz JHEP07(2004)014
[2] H. B. Meyer JHEP05(2006)066
[3] O. Aharony and M. Field JHEP01(2011)065

Application: the boundary term of the effective action: Constraints imposed by the Lorentz invariance

If the boundary is a Polyakov line in the ξ_0 direction placed at $\xi_1 = 0$, on which we assume Dirichlet boundary conditions $X_i(\xi_0, 0) = 0$, the most general boundary action should be of this type

$$S_b = \int d\xi_0 \left[b_1 \partial_1 X \cdot \partial_1 X + b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X + b_3 (\partial_1 X \cdot \partial_1 X)^2 + \dots \right] .$$

Imposing Lorentz invariance one finds that $b_1 = 0$ and that the b_2 term is only the first term of a Lorentz invariant expression[1] :

$$b_2 \int d\xi_0 \left[\frac{\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} - \frac{(\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} \right] .$$

which is the analogous in the case of the boundary action of the Nambu-Goto action for the "bulk" effective action.

[1] M. Billo, M. Caselle, F. Gliozzi, M. Meineri, R. Pellegrini JHEP05(2012)130

The boundary contribution to the interquark potential

Following the above discussion, the leading correction coming from the boundary turns out to be:

$$S_{b,2}^{(1)} = \int d\xi_0 \left[b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X \right] .$$

Its contribution to the interquark potential can be evaluated performing a simple gaussian functional integration[1]

$$\langle S_{b,2}^{(1)} \rangle \; = \; -b_2 \frac{\pi^3 L}{60 R^4} E_4(\mathrm{i}\frac{L}{2R}) \; .$$

where the Eisenstein function E_4, is defined as

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \; ,$$

where $q = e\ 2\pi i \tau$ and $\sigma_p(n)$ is the sum of the p-th powers of the divisors of n:

$$\sigma_p(n) = \sum_{m|n} m^p \; .$$

[1] O. Aharony and M. Field JHEP01(2011)065

The boundary contribution to the interquark potential

- We end up with the folllowing expression for the interquark potential

$$V(R) = \sigma R - \frac{\pi(D-2)}{24R} - \frac{1}{2\sigma R^3}\left(\frac{\pi(D-2)}{24}\right)^2 - b_2 \frac{\pi^3(D-2)}{60R^4} + O(1/R^5)\ ,$$

  where b_2 is a new physical parameter, similar to the string tension σ, which depends on the theory that we study and should be determined by simulations and comparison with experiments.

- To test this picture we performed a set of high precision simulations in the case of the 3d gauge Ising model, which is the simplest possible confining gauge theory.

Simulation I: Polyakov loops

- In order to eliminate the non-universal perimeter and constant terms from the expectation value of Polyakov loop correlators $P(R,L)$ (where L is the length of the two loops and R their distance) we measured the following ratio:

$$R_P(R,L) = \frac{P(R+1,L)}{P(R,L)} \ .$$

- Due to the peculiar nature of our algorithm, based on the dual transformation to the 3d spin Ising model, this ratio can be evaluated for large values of R and L with very high precision.

Simulation settings

- We performed our simulations in the 3d gauge Ising model, using a dual algorithm

data set	β	L	σ	$1/T_c$
1	0.743543	68	0.0228068(15)	5
2	0.751805	100	0.0105255(11)	8
3	0.754700	125	0.0067269(17)	10

Table : Some information on the data sample

Results

- The values of b_2 extracted from the data show the expected scaling behaviour $b_2 \sim \frac{1}{\sqrt{\sigma}^3}$

data set	b_2	$b_2\sqrt{\sigma}^3$	χ^2
1	7.25(15)	0.0250(5)	1.2
2	26.8(8)	0.0289(9)	1.8
3	57.9(12)	0.0319(7)	1.3

Table : Values of b_2 as a function of β

Simulation II: Wilson loops

As a check of our analysis we performed the same simulation for the Wilson loops fixing the value of b_2 obtained above. In this case there is no more parameter to fit and we can directly compare our predictions with the results of the simulations. To eliminate all the non-universal parameters we constructed the following combination:

$$R_W^{'}(L, Lu) = \frac{W(L,R)}{W(L+1,R-1)} - \exp\{-\sigma(1+L(1-u))\}\ , \qquad u = R/L$$

Simulation II: Wilson loops

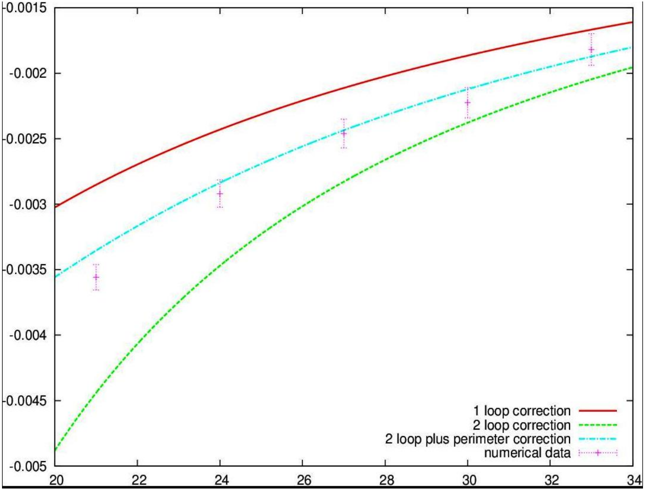


Figure : $R'_W(L, L^{\frac{4}{3}})$ at $\beta = 0.754700$.

Effective string theory description of glueballs: The Isgur-Paton model

- In the Isgur-Paton model[1] glueballs are considered as "closed flux tubes" kept together by the same string tension of the interquark potential.
- The model predicts glueball masses as adimensional ratios $\frac{m_i}{\sqrt{\sigma}}$ and gives results in good agreement with lattice simulations.

[1] N. Isgur and J. E. Paton, Phys. Rev. D 31 (1985) 2910

Glueballs in SU(N) LGT in d=4.

$m_G/\sqrt{\sigma}$		
J^{PC}	SU(∞)	IP model
0^{++}	4.065(55)	3.12
0^{++*}	6.18(13)	6.46
0^{++**}	7.99(22)	8.72
$2^{\pm+}$	6.88(16)	6.79
$2^{\pm+*}$	8.62(38)	9.06
0^{-+}	9.02(30)	13.86
$4^{\pm+}$	–	9.64
$1^{\pm+}$	10.00(25)	10.84
$3^{\pm+}$	–	8.30

Table : Glueball masses in units of the string tension. Predictions of the simple no-parameter Isgur-Paton flux tube model compared to the actual spectrum of the SU($N=\infty$) theory.[1]

[1] R.W. Johnson, M. Teper, Phys.Rev. D66 (2002) 036006

[1] R.W. Johnson, M. Teper, Phys.Rev. D66 (2002) 036006

Glueballs in Z_2 LGT in d=3.

J^P	Ising	IP	$R_{max}\sqrt{\sigma}$
0^+	3.08(3)	2.00	0.50
$(0^+)'$	5.80(4)	5.94	
$(0^+)''$	7.97(11)	8.35	
$2^\pm$	7.98(8)	6.36	0.65
$(2^\pm)'$	9.95(20)	8.76	
0^-	10.0(5)	13.82	1.20
$(0^-)'$	13.8(6)	15.05	
$(1/3)^\pm$	12.7(5)	8.04	0.75

Table : Glueball masses in units of the string tension. Predictions of the simple no-parameter Isgur-Paton flux tube model compared to the actual spectrum of the Z_2 gauge theory in three dimensions.In the last column we report the IP predictions for the glueball radii.

Glueballs in Z_2 LGT in d=3.

- Using Duality it can be proved that: the spectrum of the Z_2 gauge theory in the confining phase with periodic boundary conditions coincides with the symmetric sector of the Ising spin model spectrum in the low temperature phase with periodic boundary conditions. The proof relies on the existence of a non-zero interface tension and therefore applies exclusively to the broken symmetry phase of the spin model[1].
- A few important consequences:
  - The lowest mass of the Ising model coincides with the 0^+ glueball of the dual gauge Ising model and the first glueball excitation (which is again in the 0^+ channel) is mapped onto the new state observed in MC simulation.
  - In general the glueball spectrum of a gauge theory is much richer than the particle spectrum of a QFT. All these states are mapped by duality into non-perturbative states of the 3d Ising model.
  - String theory (via the IP model) allows to predict the masses and the angular momentum of all these bound states .

[1] M.C., M.Hasenbusch, P.Provero and K.Zarembo, N.P.**B 623** (2002) 474

Conclusions

- The Effective String action is strongly constrained by Lorentz invariance. The first few orders of the bulk and of the boundary action are universal This explains why the Nambu-Goto effective theory describes so well the infrared regime of the interquark potential and of the interface free energy.
- In the 3d gauge Ising model also the first universal boundary correction can be reliably estimated and agrees with predictions
- The Nambu-Goto action can be described as a free 2d bosonic theory perturbed by the irrelevant operator $T\bar{T}$. This perturbation is quantum integrable and yields, via TBA, a spectrum which, in a suitable limit, coincides with the Nambu-Goto one
- An effective string description exist also for the glueball masses. All these states are mapped by duality into non-perturbative states of the 3d Ising model.

Acknowledgements

Collaborators:

Marco Billó[1],
Ferdinando Gliozzi[1],
Martin Hasenbusch [2]
Marco Meineri[3],
Meineri Panero[4],
Roberto Pellegrini[1],
Roberto Tateo[1]

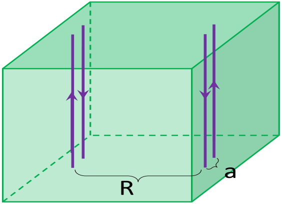
[1] Dipartimento di Fisica, Università di Torino
[2] Institut fur Physik, Humboldt-Universitaat zu Berlin
[3] Scuola Normale Superiore, Pisa
[4] Instituto de Fisica Teorica, Universidad Autonoma de Madrid

The Isgur-Paton model at Finite Temperature

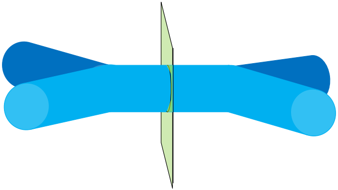
- The Isgur-Paton model can be extended also to finite temperature. The only modification is that one must change the string tension σ with its finite temperature counterpart $\sigma(T)$
- As a consequence the model predicts that the glueball masses should vanish at the deconfinement transition keeping the ratio $m(T)/\sqrt{\sigma(T)}$ constant.
- We tested this prediction[3] (again in the 3d Ising model)
- Evaluating the glueball mass on the lattice at finite temperature is not trivial. It requires looking at the connected correlator of four Polyakov loops

[3] M. Caselle, R. Pellegrini Phys. Rev. Lett. 111 (2013) 132001

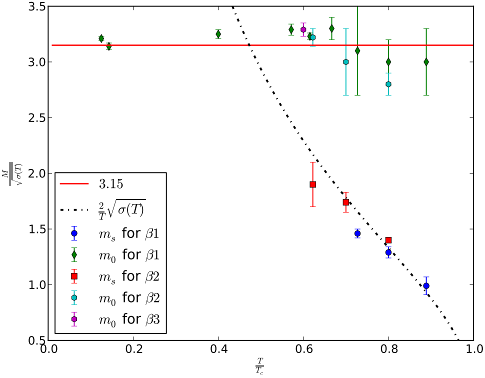
Extracting the glueball mass from the connected correlator of four Polyakov loops.



Effective string interpretation of the correlator. The external legs correspond to the four Polyakov loops, while the glueballs are the excitations of the closed string joining together the four legs.



Results.



The lowest glueball mass scales as $\sigma(T)$