Analisi multiscala ed equazioni integrabili discrete

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Introduction to multiscale analysis

- Multiscale analysis: perturbation technique for constructing (going ⇑) *uniformly* valid approximations to solutions of perturbation problems;
- *Nonuniformity* arises from *secularity*: *unphysical unbound growth* with time of amplitudes.
- **ODE example**: *damped harmonic oscillator*

$$\ddot{x} + x = -\varepsilon \dot{x}, \qquad x \in \mathcal{R}, \qquad x\left(0\right) = 1, \qquad \dot{x}\left(0\right) = 0,$$

$\varepsilon > 0$ damping parameter. Exact solution:

$$x\left(t;\varepsilon\right) = e^{-\varepsilon t/2}\left[\cos\left(t\sqrt{1-\varepsilon^2/4}\right) + \frac{\varepsilon/2}{\sqrt{1-\varepsilon^2/4}}\sin\left(t\sqrt{1-\varepsilon^2/4}\right)\right].$$

1) As $t \to +\infty$ *amplitude decreases* with time scale $1/\varepsilon$; 2) *frequency shift* with time scale $1/\varepsilon^2$: $\omega \doteq \sqrt{1-\varepsilon^2/4} \approx 1 - \varepsilon^2/8$;

- Need for a dependence of the solution on the *slow variables* $t_1 \doteq \varepsilon t$, $t_2 \doteq \varepsilon^2 t, \ldots$: $x\left(t;\varepsilon\right) = x\left(t, t_1, t_2, \ldots\right)$;

Introduction to multiscale analysis

- Time derivative expansion:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \mathcal{O}(\varepsilon^3).$$

The variables t, t_j are now supposed *independent*;

- Solution of the form

$$x(\mathbf{t};\varepsilon) = \sum_{n=0}^{+\infty} \varepsilon^n x_n(\mathbf{t}).$$

Plug into our equation;

- Order ε^0:

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0.$$

Solution:

$$x_0 = A(t_1, t_2, \ldots) e^{\mathrm{i}t} + B(t_1, t_2, \ldots) e^{-\mathrm{i}t};$$

Introduction

- Order ε:

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = -\mathrm{i}\left(A + 2\frac{\partial A}{\partial t_1}\right) e^{\mathrm{i}t} + \mathrm{i}\left(B + 2\frac{\partial B}{\partial t_1}\right) e^{-\mathrm{i}t}.$$

Solution:

$$x_1 = hom. - \frac{t}{2}\left[\left(A + 2\frac{\partial A}{\partial t_1}\right) e^{\mathrm{i}t} + \left(B + 2\frac{\partial B}{\partial t_1}\right) e^{-\mathrm{i}t}\right].$$

Unbound growth of x_1 with t: no *dumping* effect. $e^{\pm \mathrm{i}t}$ solve homogeneous equation $\Rightarrow$ *secular terms.* Then

$$A + 2\frac{\partial A}{\partial t_1} = 0, \qquad A = A_1(t_2)\, e^{-t_1/2},$$

$$B + 2\frac{\partial B}{\partial t_1} = 0, \qquad B = B_1(t_2)\, e^{-t_1/2}.$$

Now one can choose $x_1 = 0$;

- Order ε^2:

$$x(t;\varepsilon) = e^{-\varepsilon t/2}\left\{\cos\left[\left(1-\frac{\varepsilon^2}{8}\right)t\right]+\frac{\varepsilon}{2}\sin\left[\left(1-\frac{\varepsilon^2}{8}\right)t\right]+\mathcal{O}\left(\varepsilon^3\right)\right\},$$

in *agreement* with the exact solution to the desired order in ε.

Multiscale analysis and integrability

- Multiscale analysis: perturbation technique for testing *integrability* of a given nonlinear system [Calogero, Eckhaus '87, '88], [Calogero, Degasperis, Xiaoda '00, '01];
- *Integrability* is *maintained* in the reduction process [Zakharov, Kuznetsov '86: *PDE*], [Levi et al. '07, '08: $P\Delta E$] (method for exact solution going $\Downarrow$);
- Lax pair starting eq. $\Rightarrow$ Lax pairs *reduced* eqs. [Procesi '97], [Degasperis, Procesi '99];
- Integrability: characterization in terms of a *hyerarchy of reduced* systems [Degasperis, Manakov, Santini '97]!

Classification of multilin., disp., quad-graph equations

- Multilinear, real, quad-graph equation:

$$
\begin{aligned}
& a_1 u_{n,m} + a_2 u_{n+1,m} + a_3 u_{n,m+1} + a_4 u_{n+1,m+1} + \\
& \quad +(\alpha_1 - \alpha_2)\, u_{n,m} u_{n+1,m} + (\alpha_1 + \alpha_2)\, u_{n,m+1} u_{n+1,m+1} + \\
& \quad + (\beta_1 - \beta_2)\, u_{n,m} u_{n,m+1} + (\beta_1 + \beta_2)\, u_{n+1,m} u_{n+1,m+1} + \\
& \quad + \gamma_1 u_{n,m} u_{n+1,m+1} + \gamma_2 u_{n+1,m} u_{n,m+1} + \\
& \quad + (\xi_1 - \xi_3)\, u_{n,m} u_{n+1,m} u_{n,m+1} + (\xi_1 + \xi_3)\, u_{n,m} u_{n+1,m} u_{n+1,m+1} + \\
& \quad + (\xi_2 - \xi_4)\, u_{n+1,m} u_{n,m+1} u_{n+1,m+1} + (\xi_2 + \xi_4)\, u_{n,m} u_{n,m+1} u_{n+1,m+1} + \\
& \quad + \zeta u_{n,m} u_{n+1,m} u_{n,m+1} u_{n+1,m+1} + \theta = 0,
\end{aligned}
$$

  a_1, a_2, a_3, a_4, α_1, α_2, β_1, β_2, γ_1, γ_2, ξ_1, ξ_2, ξ_3, ξ_4, ζ, θ real parameters; $u_{n,m}$ real function;
- Perturbation around pl. wave sol. of linear part $\Rightarrow \theta = 0$;
- Linear part dispersive iff:

$$
\begin{aligned}
a_4 = a_1, \quad a_3 = a_2, \quad |a_1| \neq |a_2|, \quad & \text{Class } Q_+ \text{ or} \\
a_4 = -a_1, \quad a_3 = -a_2, \quad |a_1| \neq |a_2|, \quad & \text{Class } Q_-.
\end{aligned}
$$

Classification of multilin., disp., quad-graph equations

- Class Q_+:

$$
\begin{aligned}
a_1 (u_{n,m} + u_{n+1,m+1}) &+ a_2 (u_{n+1,m} + u_{n,m+1}) + \\
&+(\alpha_1 - \alpha_2) u_{n,m} u_{n+1,m} + (\alpha_1 + \alpha_2) u_{n,m+1} u_{n+1,m+1} + \\
&+ (\beta_1 - \beta_2) u_{n,m} u_{n,m+1} + (\beta_1 + \beta_2) u_{n+1,m} u_{n+1,m+1} + \\
&+ \gamma_1 u_{n,m} u_{n+1,m+1} + \gamma_2 u_{n+1,m} u_{n,m+1} + \\
&+ (\xi_1 - \xi_3) u_{n,m} u_{n+1,m} u_{n,m+1} + (\xi_1 + \xi_3) u_{n,m} u_{n+1,m} u_{n+1,m+1} + \\
&+ (\xi_2 - \xi_4) u_{n+1,m} u_{n,m+1} u_{n+1,m+1} + (\xi_2 + \xi_4) u_{n,m} u_{n,m+1} u_{n+1,m+1} + \\
&+ \zeta u_{n,m} u_{n+1,m} u_{n,m+1} u_{n+1,m+1} = 0, \quad |a_1| \neq |a_2|;
\end{aligned}
$$

- Q_+ linear dispersion relation:

$$\omega(\kappa) = \arctan\left[\frac{(a_1^2 - a_2^2)\sin(\kappa)}{(a_1^2 + a_2^2)\cos(\kappa) + 2a_1 a_2}\right];$$

- Equivalence class: Subgroup of real Möbious transformations

$$u_{n,m} \Rightarrow \frac{u_{n,m}}{A u_{n,m} + B}.$$

Classification of multilin., disp., quad-graph equations

- Dispersive system: perturbation around a plane wave solution of Q_+ linear part $\Rightarrow$ *NLS* hierarchy;
- Seek for a solution:

$$u_{n,m}(\varepsilon) = \sum_{j=1}^{+\infty} \sum_{\alpha=-j}^{j} \varepsilon^j u_j^{(\alpha)}(n_1, m_1, m_2, \dots)\, e^{\mathrm{i}\alpha(\kappa n - \omega m)};$$

- Reality conditions: $u_n^{(-\alpha)} = \bar{u}_n^{(\alpha)}$, $\forall\alpha$.

Expansion Parameters

1. $0 \le \varepsilon \ll 1$: perturbative parameter;
2. $n_1 \doteq \varepsilon n$: slow "space" variable;
3. $m_j \doteq \varepsilon^j m$, $j \ge 1$ slow "times" variables;

Classification results at order ε^6.

Theorem of A_4 integrability: *Up to restric. Möbius, exch. $n \leftrightarrow m$ and inv. $n \to -n$, the only A_4-asypt. S-integr. eqs. in $\mathcal{Q}_+$ class are: (1 free param.)*

- *Case T1*:

$$v + v_{1,2} + 2(v_1 + v_2) + v_1 v_2 (1 + \tau) + \\ + (v_1 v_{1,2} + v v_2)\tau + v_2 v_{1,2} + v v_1 + v_1 v_2 (v + v_{1,2})\tau = 0;$$

- *Case T2*:

$$-1 < \varepsilon < 1, \quad \varepsilon \neq 0,\ 1/2, \quad v + v_{1,2} + \varepsilon(v_1 + v_2) + v v_{1,2} - v_1 v_2 + \\ + \left(1 - \frac{1}{\varepsilon}\right)[v_1 v_2 (v + v_{1,2}) - v v_{1,2}(v_1 + v_2)] + \left(1 - \frac{1}{\varepsilon^2}\right) v v_1 v_2 v_{1,2} = 0;$$

Classification results at order ε^6.

Theorem of A_4 integrability (cont.): *(3 free param.: $-1 < \varepsilon < 1$, $\varepsilon \neq 0$, $\delta = \pm 1$, $\tau \geq 0$)*

- *Case T3*:

$$v + v_{1,2} + \varepsilon (v_1 + v_2) + \delta [\varepsilon v_1 v_2 (v + v_{1,2}) + v v_{1,2} (v_1 + v_2)] + \\ + \tau v v_1 v_2 v_{1,2} = 0;$$

- *Case T4*:

$$v + v_{1,2} + \varepsilon (v_1 + v_2) + \delta [v_1 v_2 (v + v_{1,2}) + \varepsilon v v_{1,2} (v_1 + v_2)] + \\ + \tau v v_1 v_2 v_{1,2} = 0.$$

Analysis of the effective integrable (sub) cases.

Take in *Case T3* $w \doteq \delta sgn(\varepsilon)/v$ and in *Case T4* $w \doteq \delta/v$ (not allow.)

- *Case T3′*:

$$w + w_{1,2} + \frac{1}{\varepsilon}(w_1 + w_2) + \delta\left[\frac{1}{\varepsilon} w_1 w_2 (w + w_{1,2}) + w w_{1,2}(w_1 + w_2)\right] + \frac{\tau}{|\varepsilon|} = 0;$$

- *Case T4′*:

$$\tilde{w} + \tilde{w}_{1,2} + \varepsilon(\tilde{w}_1 + \tilde{w}_2) + \delta\left[\tilde{w}_1 \tilde{w}_2 (\tilde{w} + \tilde{w}_{1,2}) + \varepsilon \tilde{w} \tilde{w}_{1,2}(\tilde{w}_1 + \tilde{w}_2)\right] + \tau = 0.$$

- Each just a perturbation by a constant term of an integrable system: $\tau = 0 \rightarrow$ Hirota s-G/pot. s-G resp. (in T3′ $\varepsilon \rightarrow 1/\varepsilon$, in T4′ $\delta \rightarrow s\delta$, $s \doteq sgn(\varepsilon)$)

$$w = |\varepsilon|^{1/2} \frac{\tilde{w}_1 + \tilde{w}_2}{1 + s\delta \tilde{w}_1 \tilde{w}_2}.$$

Analysis of the effective integrable (sub) cases.

$\tau = 0$: three-points generalized symmetries in n direction

- *Case T3′*:

$$w_{,t} = \frac{\left(\delta w^2 - \varepsilon\right)\left(\delta\varepsilon w^2 - 1\right)\left(w_1 - w_{-1}\right)}{\left(1 + \delta w w_1\right)\left(1 + \delta w w_{-1}\right)};$$

- *Case T4′*:

$$\tilde{w}_{,\tilde{t}} = \left[Y \frac{\left(\tilde{w}_1 - \tilde{w}_{-1}\right)}{\delta \tilde{w}_1 \tilde{w}_{-1} - 1} + (-1)^n \kappa + (-1)^m \theta\right]\left(\delta \tilde{w}^2 - 1\right).$$

- t, $\tilde{t}$ group parameters;
- add three-points gen. symm. in m direct. $\Rightarrow$ five-points gen. symm.!
- $\tau \neq 0$: no three-points gen. symm.: algebraic entropy $\Rightarrow$ not integrable!

Take in *Case T2* $v \doteq \frac{|\rho|^{1/2}w+1}{|\rho|^{1/2}w-1/\varepsilon}$, $\rho \doteq \frac{-1+2\varepsilon}{\varepsilon(\varepsilon-2)} \neq 0$ (not allow.)

- *Case T2'*:

$$w + w_{1,2} + w w_{1,2}\left[s\left(w_1 + w_2\right) + c^{-1} w_1 w_2\right] + sc = 0,$$

$$s \doteq sgn\left(\rho\right) = sgn\left(1/\varepsilon - 2\right), \quad c \doteq \frac{1}{\varepsilon|\rho|^{3/2}}.$$

- Real, discr. Tzitzeica [Adler '12]: 3×3 Lax pair, no three-points but five-points gen. symm.: integrable!

Analysis of the effective integrable (sub) cases.

Take in *Case T1* $\tau=0$: $v \doteq \sqrt{3}w-1$; $\tau=1$: $v \doteq 2^{1/3}w-1$; $\tau\neq 0,1$: $v \doteq \frac{1-\tau}{\tau}w-1$ (not allow.) resp.

- *Case T1′*:

$$w w_1 + w_1 w_2 + w_2 w_{1,2} - 1 = 0;$$

- *Case T1″*:

$$w_1 w_2 \left(w + w_{1,2}\right) - 1 = 0;$$

- *Case T1‴*:

$$w w_1 + w_1 w_2 + w_2 w_{1,2} + w_1 w_2 \left(w + w_{1,2}\right) + \chi = 0, \quad \chi \doteq \frac{(\tau-3)\,\tau^2}{(1-\tau)^3}.$$

- *Case T1″*: deg. Tzitzeica [Mikhailov, Xenitidis '13]: 3×3 Lax pair, no three-points but five-points gen. symm.: integrable!
- *Case T1‴*: no three-points gen. symm.: algebraic entropy $\Rightarrow$ integrable!

Analysis of the effective integrable (sub) cases.

Case T1′: five-points gen. symm. (z, $\tilde{z}$ group param.): new integr.!

- n direction (no three-points gen. symm.):

$$w_{,z} = w\left(w w_1 - 1\right)\left(w w_{-1} - 1\right)\left(w_{11} w_1 - w_{-1} w_{-11}\right);$$

- m direction (no three-points gen. symm.):

$$w_{,\tilde{z}} = \frac{w\left(w_2 + w\right)\left(w + w_{-2}\right)\left(w_{22} + w_2 - w_{-2} - w_{-22}\right)}{\left(w_{22} + w_2 + w\right)\left(w_2 + w + w_{-2}\right)\left(w + w_{-2} + w_{-22}\right)}.$$

relation with Bogoyavlensky-type lattices

- n direction: potentiation $t \doteq w w_1 - 1$

$$t_{,z} = t\left(t + 1\right)\left(t_{11} t_1 - t_{-1} t_{-11}\right);$$

- m direction: Miura $\tilde{t} \doteq -\frac{w_1 + w_2}{w + w_1 + w_2}$

$$\tilde{t}_{,\tilde{z}} = \tilde{t}\left(\tilde{t} + 1\right)\left(\tilde{t}_{22} \tilde{t}_2 - \tilde{t}_{-2} \tilde{t}_{-22}\right).$$

- T1$'$ $\Rightarrow$T1$'''$, $\chi = 0$:

$$\frac{1}{\widehat{w}} \doteq w w_1 - 1 \qquad (\text{Invertible})\,;$$

- T1$'$ $\Rightarrow$T1$'''$, $\chi = 0$, $n \leftrightarrow m$:

$$\widehat{w} \doteq -\frac{w}{w_1 + w_2} - 1;$$

- T1$''$ $\Rightarrow$T1$'''$, $\chi = 0$:

$$\widehat{w} \doteq \tilde{w}_{-1} \tilde{w} \tilde{w}_1 - 1 \qquad (\text{Invertible})\,;$$

- T1$'$ Lax pair:

$$\Phi_{3,0} = Y_{0,0}\Phi_{0,0}, \quad \Phi_{0,3} = X_{0,0}\Phi_{0,0} \qquad (\text{Non standard})\,.$$