Beam loading and betatron radiation from a bubble in a deep plasma channel

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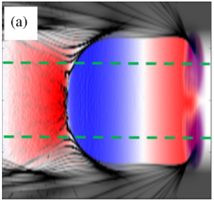
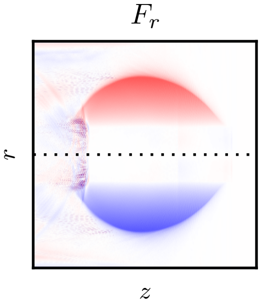
Suggestion to use a deep channel in plasma

Pukhov et al., *Field-Reversed Bubble in Deep Plasma Channels for High-Quality Electron Acceleration*, PRL **113**, 2014

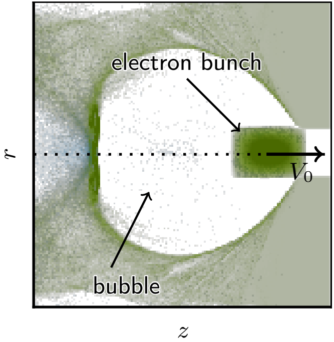


Advantages over homogeneous plasma:

- Reduction of the transversal force acting on accelerated electrons to almost zero.
  Allows to improve accelerated beam quality.
- Increase of the phase velocity of the wake.
  Allows to increase the energy gain of the accelerated particles.



Bubble theory in a deep plasma channel



Electron density distribution (green) in a bubble obtained from 3D PIC simulations.
The driving electron bunch pushes plasma electrons away and generates the bubble.

Lu et al., *Nonlinear Theory for Relativistic Plasma Wakefields in the Blowout Regime*, PRL **96**, 2006

Assumptions:

- Cylindrical coordinates: $\mathbf{r} = (r, \phi, z)$.
- Radially-inhomogeneous plasma: $n(\mathbf{r}) = n(r)$.
- Axial symmetry (no dependence on ϕ).
- Ions are immobile.
- Electrons are initially at rest (their thermal velocity can be neglected).

Equations for the EM field potentials

Quasistationary approximation:

$$f(r,z,t) = f(r,\xi), \quad \xi = t - z,$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial z} = -\frac{\partial}{\partial \xi}.$$

EM field is described by a vector-potential (A_z, A_r) and a wakefield potential:

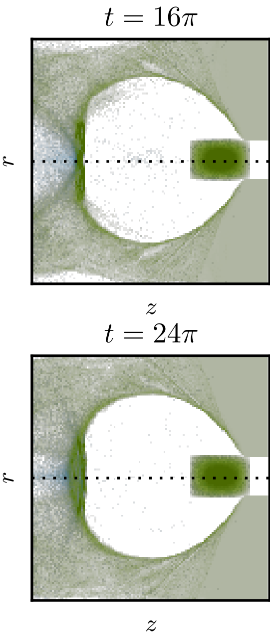
$$\Psi = \varphi - A_z.$$

Lorentz gauge:

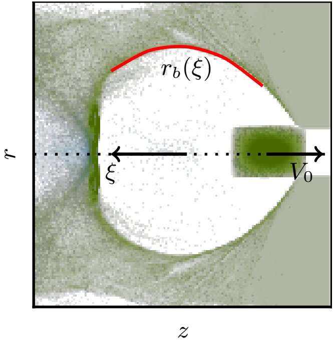
$$\frac{1}{r}\frac{\partial}{\partial r}(rA_r) = -\frac{\partial \Psi}{\partial \xi}.$$

Equations for the potentials:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = -J_z, \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) = J_z - \rho.$$



Model of bubble



$J_z - \rho = S(\xi, r)$ — source for Ψ.
Model for $S(\xi, r)$:

$$S(\xi, r) = \begin{cases} s_i(r), & r < r_b(\xi), \\ s_0(\xi), & r_b(\xi) < r < r_b(\xi) + \Delta, \\ 0, & r > r_b(\xi) + \Delta, \end{cases}$$

No plasma electrons inside: $s_i(r) = -\rho(r)$.
For electron bunches $J_z \approx \rho \rightarrow$ they do not contribute to S.

Solution for Ψ inside the bubble:

$$\Psi(\xi, r) = \int_0^r \frac{1}{y} S_I(y) dy + \Psi_0(\xi),$$

$$S_I(r) = -\int_0^r \rho(r) r dr.$$

Forces acting on the accelerated particles depend only on one coordinate each:

$$F_r(r) = -\frac{\partial \Psi}{\partial r}, \quad F_z(\xi) = -E_z(\xi) = -\frac{\partial \Psi}{\partial \xi}.$$

Electron motion equations

In coordinates $(\xi = t - z, r)$ the Hamilton function does not explicitly depend on time, therefore its value is constant on trajectories:

$$H(P_r, P_z, r, -\xi) = \gamma - P_z - \varphi = const,$$

$\mathbf{P} = (\mathbf{p} - \mathbf{A})$ — canonical momentum, $\gamma = \sqrt{1 + \mathbf{p}^2 + \langle \mathbf{a}^2 \rangle}$ — Lorentz-factor, $\mathbf{a}$ — dimensionless vector-potential of the laser field.

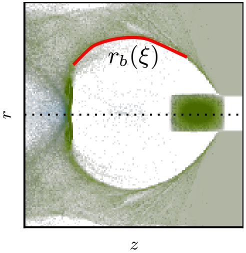
$\xi(t)$ is monotonous, thus the dependence on time can be replaced with the dependence on ξ.
Equation for a trajectory:

$$\frac{d}{d\xi}\left[(1+\Psi)\frac{dr}{d\xi}\right] = \left[\frac{1+(1+\Psi)^2}{2(1+\Psi)^2} + \frac{1}{2}\left(\frac{dr}{d\xi}\right)^2\right]\frac{\partial \Psi}{\partial r} + \frac{\partial A_z}{\partial r} + \frac{\partial A_r}{\partial \xi} - \frac{\partial}{\partial r}\frac{\langle \mathbf{a}^2 \rangle}{2(1+\Psi)}.$$

Electron sheath inner border $r_b(\xi)$ is also a trajectory!

$$A(r_b) r_b \frac{d^2 r_b}{d\xi^2} + B(r_b)\left(\frac{dr_b}{d\xi}\right)^2 + C(r_b) = \lambda(\xi),$$

$$\lambda(\xi) = -\int_0^{r_b(\xi)} J_z(\xi, r') r' dr'.$$



Approximations for the bubble shape equation

General equation:

$$A(r_b)r_b\frac{d^2r_b}{d\xi^2}+B(r_b)\left(\frac{dr_b}{d\xi}\right)^2+C(r_b)=\lambda(\xi)$$

Thin sheath approximation
The electron sheath surrounding the bubble is infinitely thin:

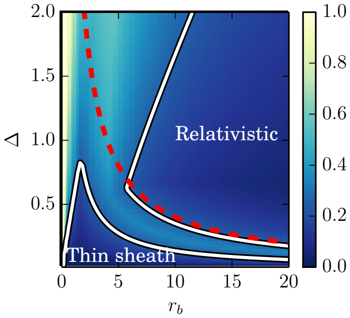
$$\Delta\to 0.$$

$$A=1-\frac{S_I}{2},\quad B=-\frac{S_I' r_b}{2},\quad C=-S_I.$$

Relativistic approximation
Electrons in the sheath are relativistic ($\Psi(\xi,r_b)\gg 1$):

$$\Delta\ll r_b,\quad \Delta\gtrsim\left|\frac{2r_b}{S_I(r_b)}\right|.$$

$$A=-\frac{S_I}{2},\quad B=-\frac{S_I' r_b}{2},\quad C=-\frac{S_I}{2}.$$

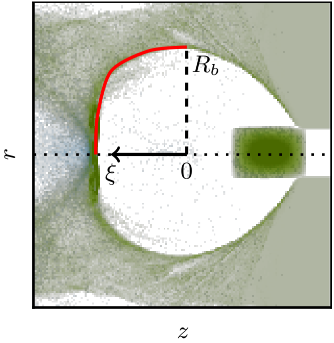


Bubble shape derivation

Equation for the shape:

$$S_I(r_b) r_b \frac{d^2 r_b}{d\xi^2} + S_I'(r_b) r_b \left(\frac{dr_b}{d\xi}\right)^2 + S_I(r_b) = -2\lambda(\xi),$$

$$S_I(r) = -\int_0^r \rho(r') r' dr', \quad \lambda(\xi) = -\int_0^{r_b} J_z(\xi, r') r' dr'.$$



Longitudinal electric field inside the bubble:

$$E_z(\xi) = \frac{S_I(r_b)}{r_b} \frac{dr_b}{d\xi}.$$

Initial conditions for the equation:

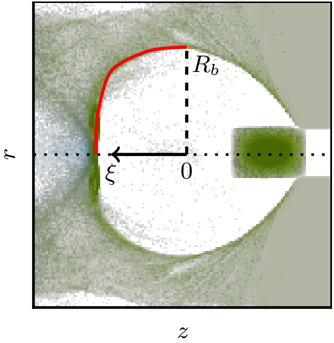
$$r_b(\xi = 0) = R_b, \quad r_b'(\xi = 0) = 0.$$

Non-loaded bubble (without accelerated bunches)

Equation for the shape:

$$S_I(r_b) r_b \frac{d^2 r_b}{d\xi^2} + S_I'(r_b) r_b \left(\frac{dr_b}{d\xi}\right)^2 + S_I(r_b) = -2\lambda(\xi),$$

$$S_I(r) = -\int_0^r \rho(r') r' dr', \quad \lambda(\xi) = -\int_0^{r_b} J_z(\xi, r') r' dr'.$$



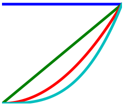
For unloaded bubble $\lambda(\xi) = 0$.
The solution is:

$$\xi(r_b) = -\int_{r_b}^{R_b} \frac{S_I(y) dy}{\sqrt{2 \int\limits_y^{R_b} \frac{S_I^2}{x} dx}}.$$

Near $\xi = 0$:

$$r_b(\xi) \approx R_b - \frac{\xi^2}{2R_b}, \quad E_z(\xi) \approx \frac{S_I(R_b)}{R_b^2} \xi \geqslant -\frac{1}{2}\xi.$$

Non-loaded bubble in plasma with a power-law density profile

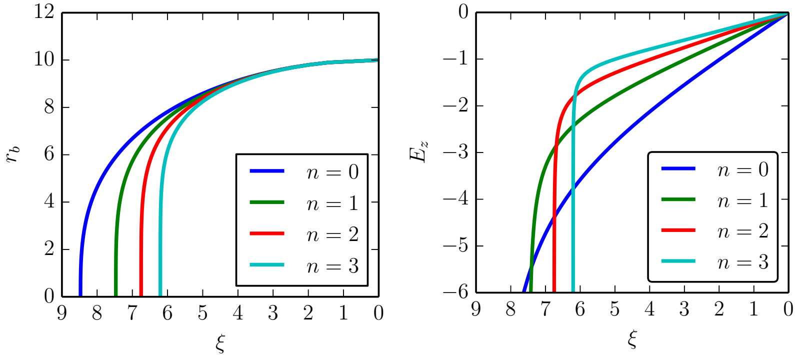


Plasma profile $\rho(r) = \left(\frac{r}{R_b}\right)^n$.

The following analytical solution can be obtained:

$$\xi(r_b) = R_b\sqrt{n+2}\frac{t^{n+3}}{n+3} \cdot {}_2F_1\left(\frac{1}{2}, \frac{n+3}{2(n+2)}; \frac{3n+7}{2(n+2)}; t^{2(n+2)}\right)\Bigg|_{\frac{r_b}{R_b}}^{1}.$$

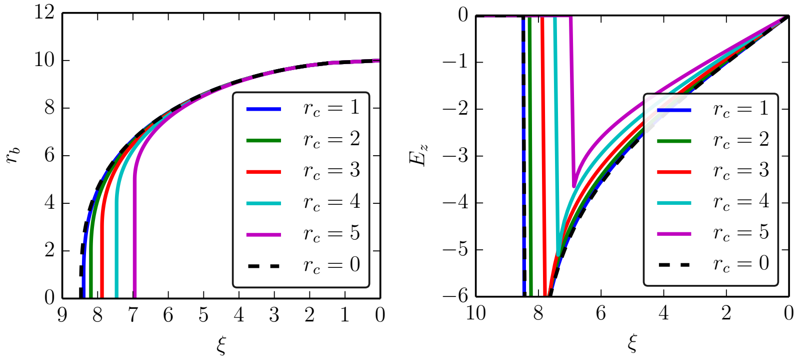
Longitudinal field near $\xi = 0$: $E_z \approx -\frac{\xi}{n+2}$.



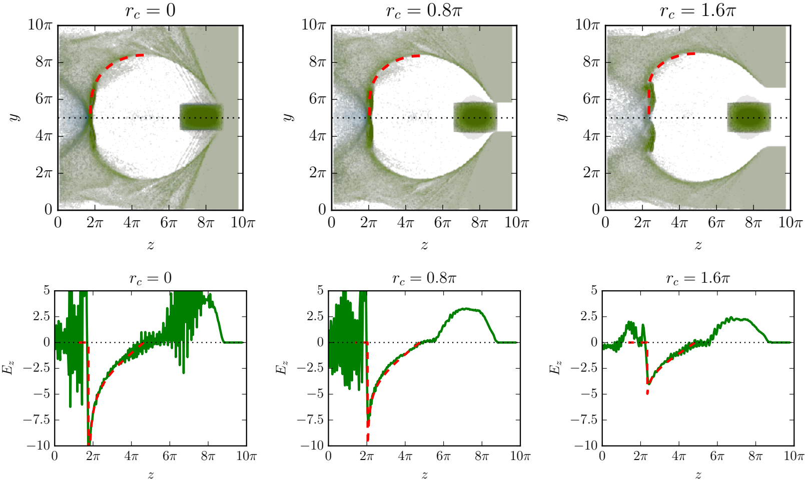
Non-loaded bubble in plasma with a vacuum channel

Plasma profile $\rho(r) = \theta(r - r_c)$, $\theta(x)$ – Heaviside step function.

Near $\xi = 0$: $E_z \approx -\frac{1}{2}\left(1 - \frac{r_c^2}{R_b^2}\right)\xi$.



Non-loaded bubble in plasma with a vacuum channel: 3D PIC simulations



Bunch profile for homogeneous accelerating field

Tzoufras et al., *Beam loading by electrons in nonlinear plasma wakes*, Phys. Plasmas **16**, 2009

ξ_t — electron bunch injection coordinate.
Condition for the bunch profile $\lambda(\xi)$:

$$E_z(\xi > \xi_t) = -E_t = E_z(\xi_t).$$

Parametric solution ($\lambda(\xi) = \Lambda(r_b(\xi))$):

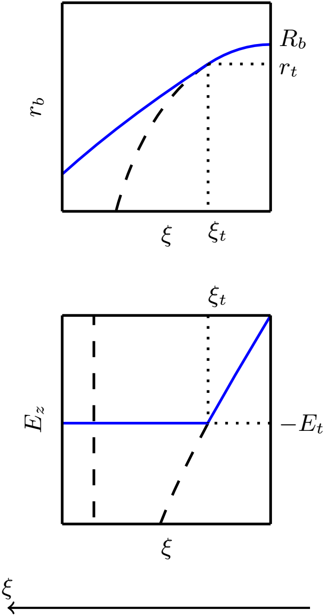
$$E_t \cdot (\xi - \xi_t) = \int_{r_t}^{r_b} \frac{S_I(x)}{x} dx,$$

$$\Lambda(r_b) = -\frac{S_I(r_b)}{2} - \frac{E_t^2 r_b^2}{2S_I(r_b)}.$$

Total charge in the bunch:

$$Q_{max} E_t = 2\pi \int_0^{R_b} \frac{S_I^2}{x} dx.$$

Maximum efficiency is 100%.



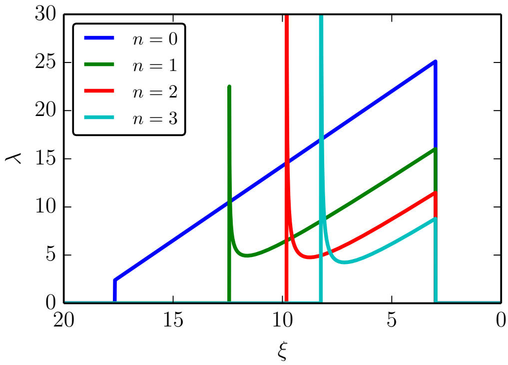
Homogeneous accelerating field in plasma with a power-law profile

Plasma profile $\rho(r) = \left(\frac{r}{R_b}\right)^n$.

Analytical solutions for the bubble shape and the electron bunch profile:

$$r_b(\xi) = \left(r_t^{n+2} - R_b^n E_t(\xi - \xi_t)(n+2)^2\right)^{1/(n+2)},$$

$$\lambda(\xi) = \frac{r_t^{n+2}}{2(n+2)R_b^n} - \frac{(n+2)E_t}{2}(\xi - \xi_t) + \frac{R_b^n E_t^2 (n+2)}{2}\frac{1}{r_b^n(\xi)}.$$



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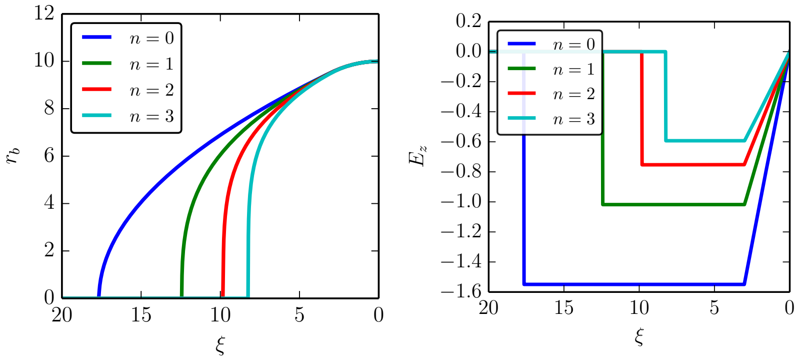
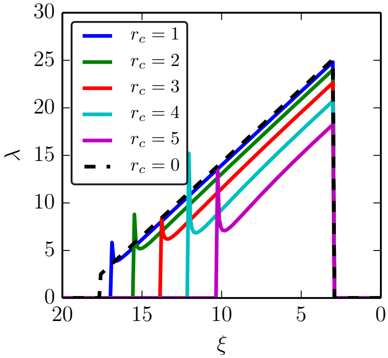
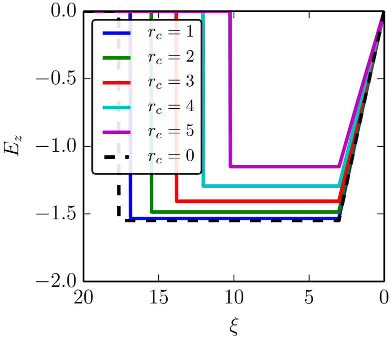


Homogeneous accelerating field in plasma with a vacuum channel

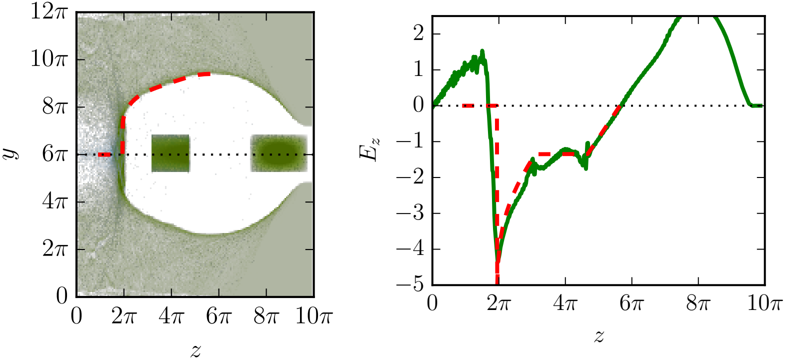
Plasma profile $\rho(r) = \theta(r - r_c)$.





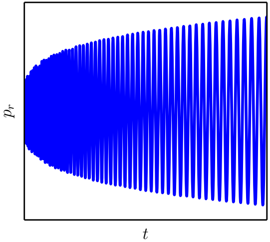
Parameters: $\xi_t = 3$, $R_b = 10$.

Homogeneous accelerating field in plasma with a vacuum channel: 3D PIC simulations



Influence of the vacuum channel on betatron oscillations and betatron radiation

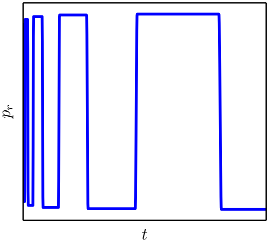
Homogeneous plasma



$p_\perp \propto \gamma^{1/4} \gg 1$ – the radiation is in synchrotronous regime.

Kostyukov et al., *X-ray generation in an ion channel*, Phys. Plasmas **10**, 2003

Plasma with a channel



$p_\perp \approx const$ and $p_\perp < 1$ is possible. The electron radiates only when it reaches the walls of the channel.

Conclusions

- A theory of a bubble in radially-inhomogeneous plasma is developed.
- It is shown that the bubble shape equation allows two approximations making its consideration by analytical methods possible. The areas of applicability of these approximations are studied.
- The shape of a non-loaded bubble is described.
- The electron bunch profile providing homogeneous accelerating field in a loaded bubble is found for an arbitrary plasma profile.
- General results are applied to special cases of plasma with a power-law profile $\rho(r) = (r/R_b)^n$ and plasma with a vacuum channel $\rho(r) = \theta(r - r_c)$.
- The results for plasma with a vacuum channel are verified by 3D PIC simulations.