Emittance and phase space measurement

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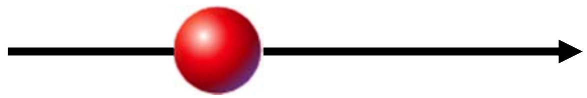
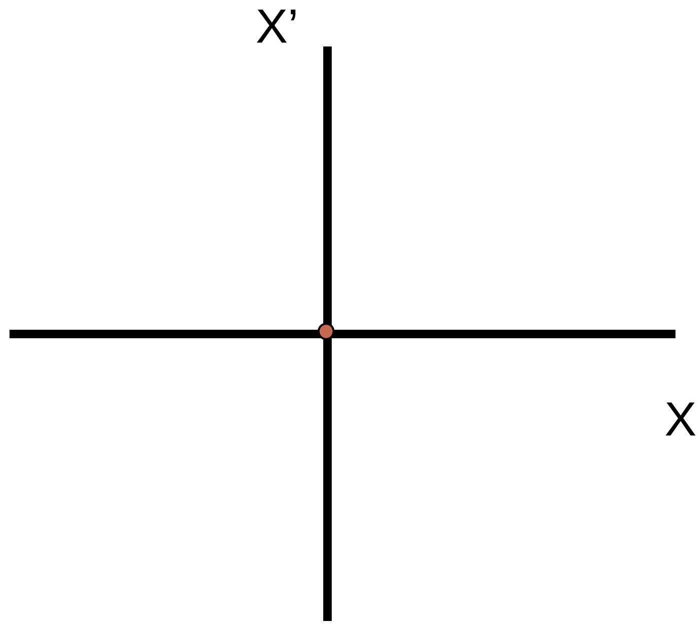
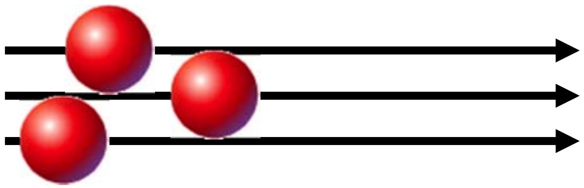
The emittance

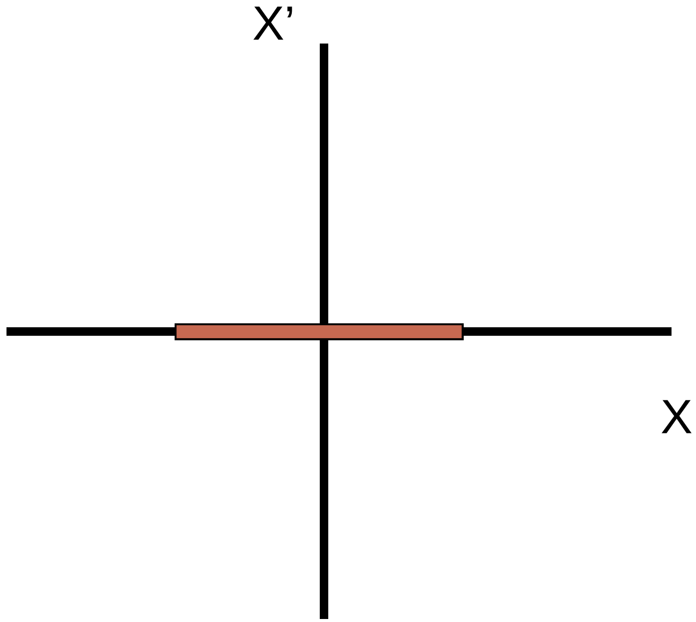
Single Particle Trace Space



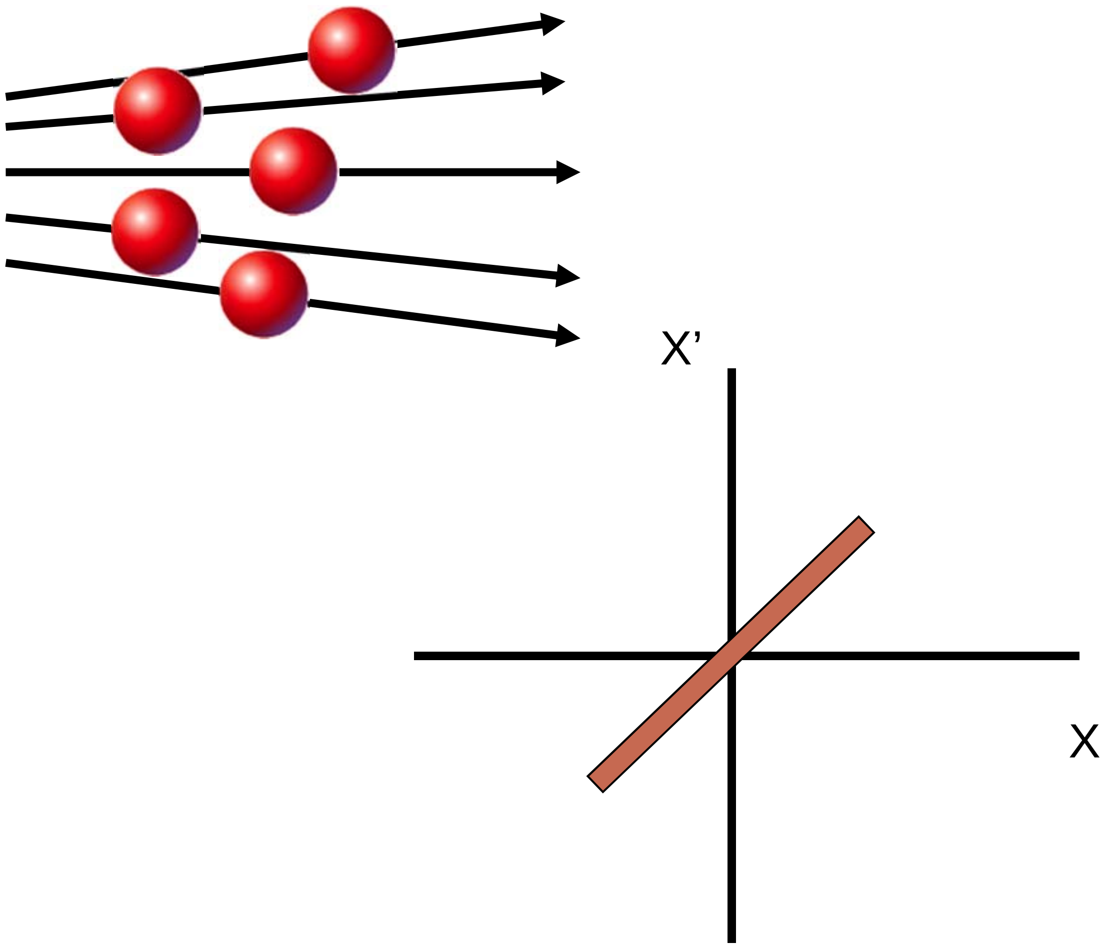


Trace space of a parallel beam

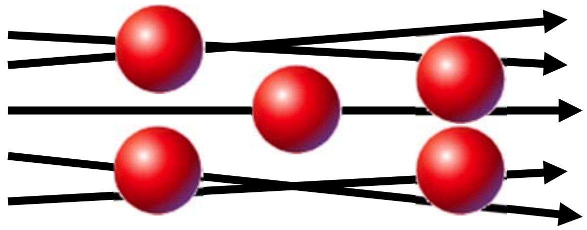


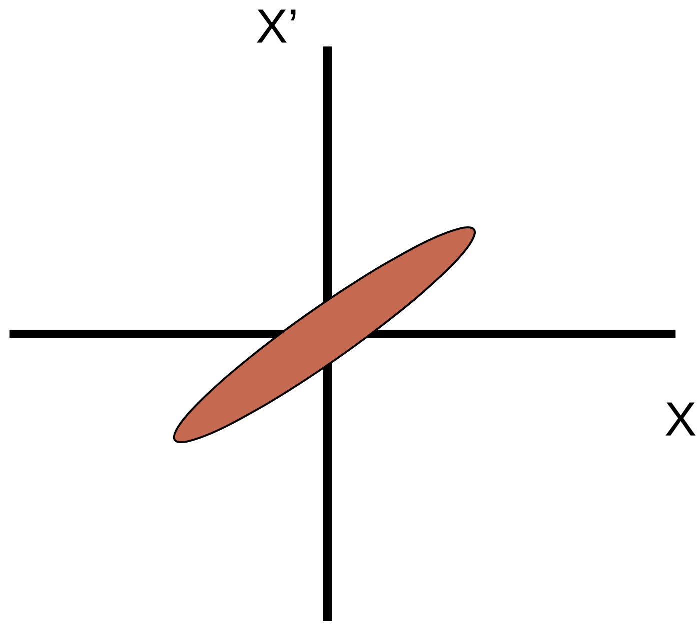


Trace space laminar beam

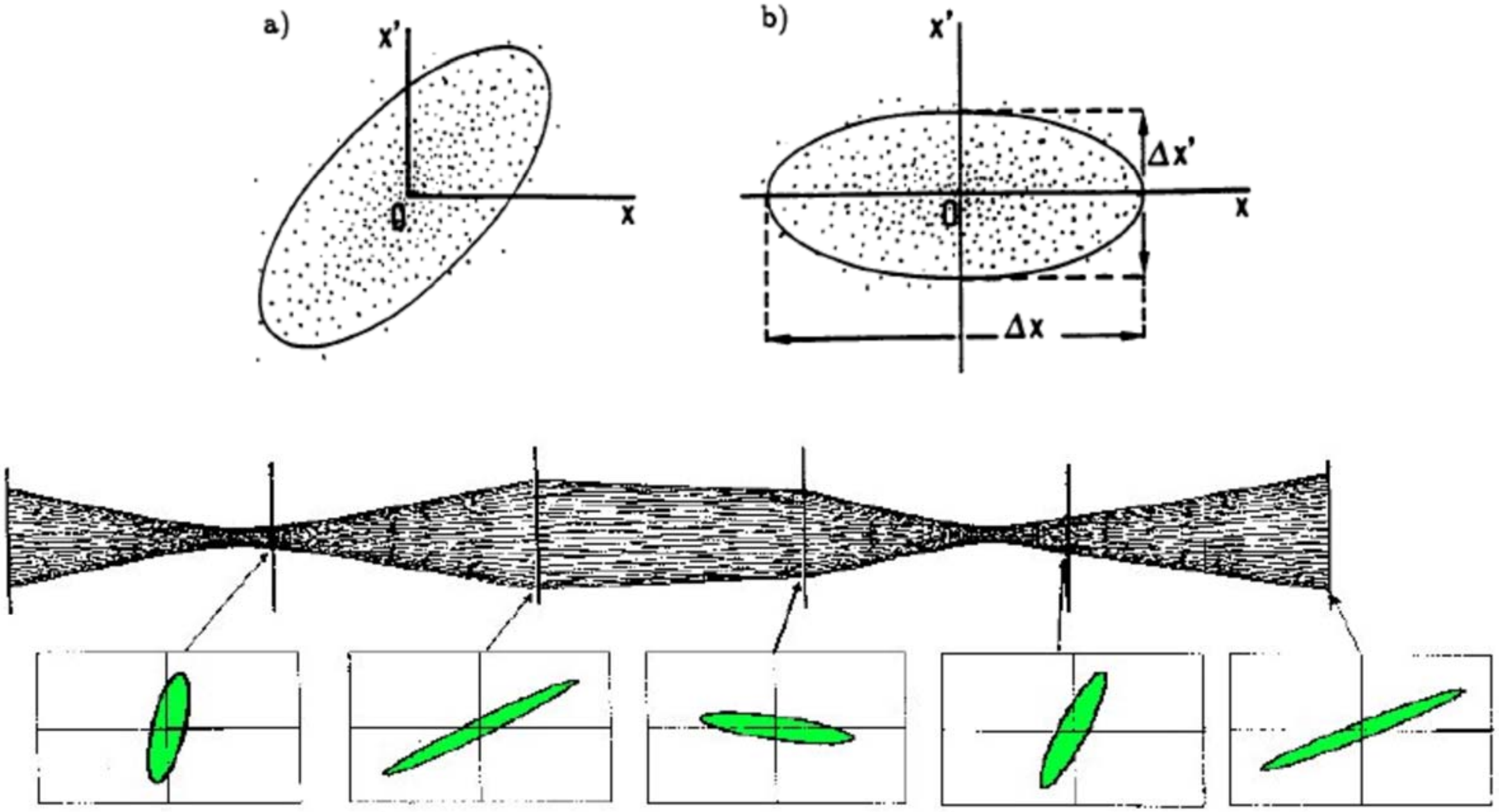


Trace space of non laminar beam





Trace space in a transport line



Ellipse equation

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\left(\Psi(s)+\phi\right)$$

$$\Psi(s) = \int \frac{1}{\beta(s)} ds$$

$$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\left(\Psi(s)+\phi\right) + \sin\left(\Psi(s)+\phi\right)\right]$$

$$\cos\left(\Psi(s)+\phi\right) = \frac{x}{\sqrt{\varepsilon}\sqrt{\beta(s)}}.$$

$$\sin\left(\Psi(s)+\phi\right) = \frac{\sqrt{\beta(s)}x'}{\sqrt{\varepsilon}} + \frac{\alpha(s)x}{\sqrt{\varepsilon}\sqrt{\beta(s)}}$$

$$\frac{x^2}{\beta(s)} + \left(\frac{\alpha(s)}{\sqrt{\beta(s)}}x + \sqrt{\beta(s)}x'\right)^2 = \varepsilon$$

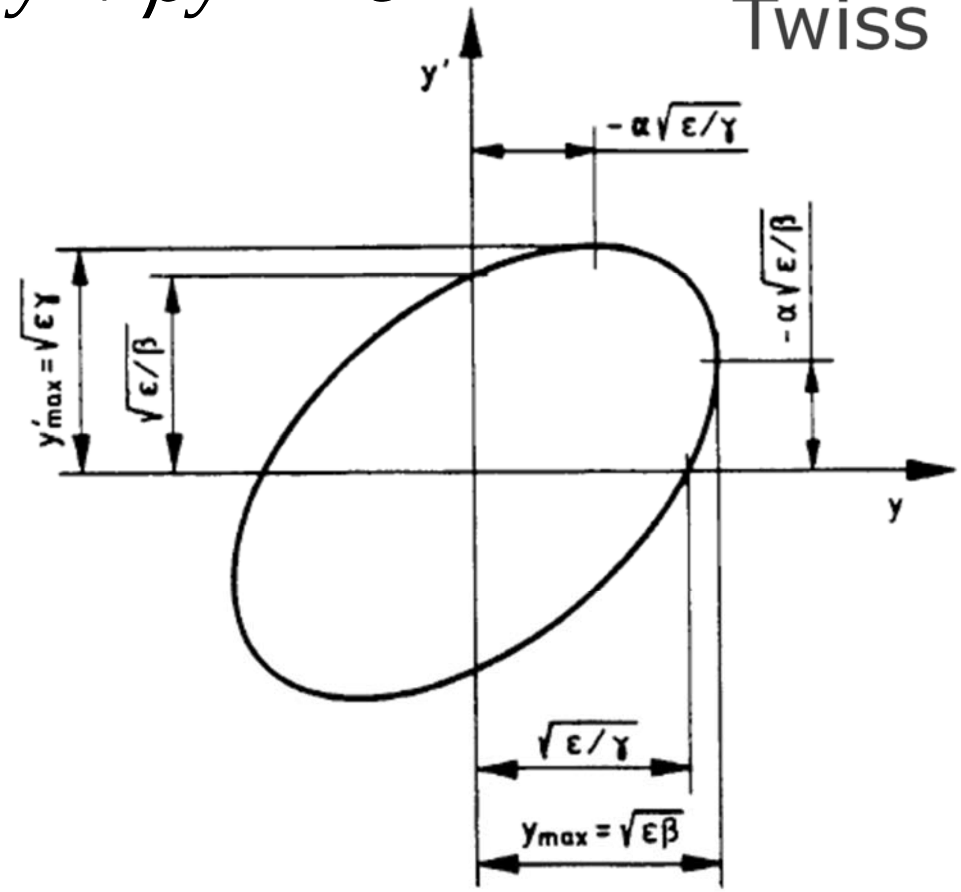
$$\gamma(s) \equiv \frac{1+\alpha^2(s)}{\beta(s)}$$

Ellipse equation

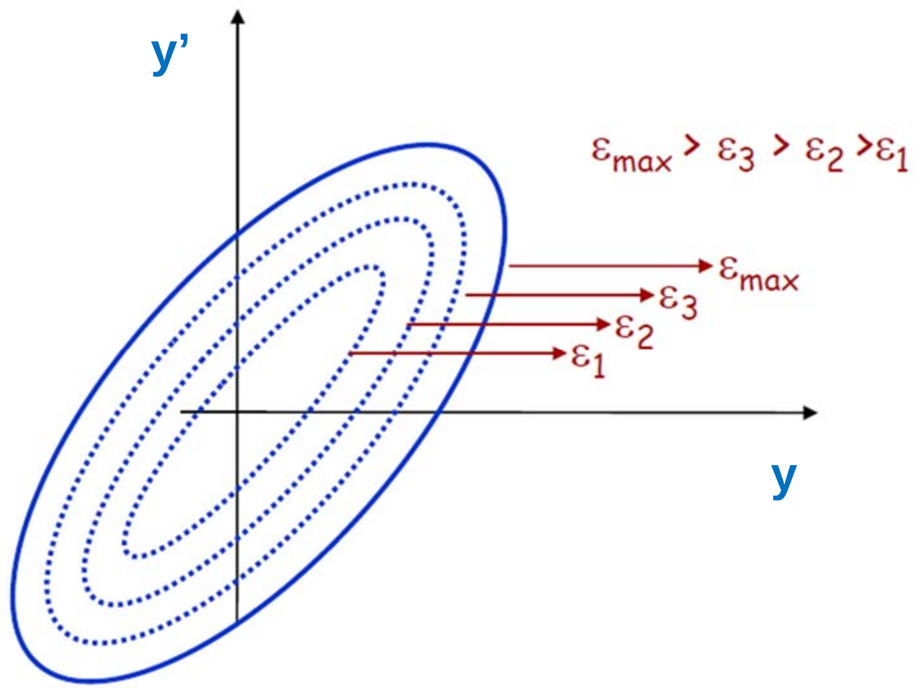
$$\gamma y^2 + 2\alpha y y' + \beta y'^2 = \varepsilon$$

Twiss parameters

$$\beta\gamma - \alpha^2 = 1$$



Several particles



- The outermost ellipse can be used to represent the whole beam

Emittance of a beam

- The phase space surface occupied by a beam is a convenient figure of merit to designate the quality of a beam. This quantity is the emittance ε_x and is represented by an ellipse that contains the whole particle distribution in the phase space (x,x'), such that $\boldsymbol{A = \pi\varepsilon_x}$.
- An analogous definition holds for the (y,y') and (z,z') planes. The original choice of an elliptical shape comes from the fact that when linear focusing forces are applied to a beam, the trajectory of each particle in phase space lies on an ellipse, which may be called the trajectory ellipse.

Liouville

- The utility of a such a description derives from the discovery by Liouville that the density in phase space of a system of non-interacting particles subject to a Hamiltonian (such as that of an electromagnetic field) is constant in time. Accordingly, the extent of the beam in phase space, termed its *emittance*, is also constant in time, at least under ideal conditions.

Liouville Theorem

- Using x,y,z and p_x, p_y, p_z
- Defining the 6D phase space (x, p_x, y, p_y, z, p_z)
- **Liouville Theorem**: under the influence of Hamiltonian forces the particle density in phase space ρ *(x,* p_x*, y,* p_y*, z,* p_z*)* of a system of identical particles stays constant.

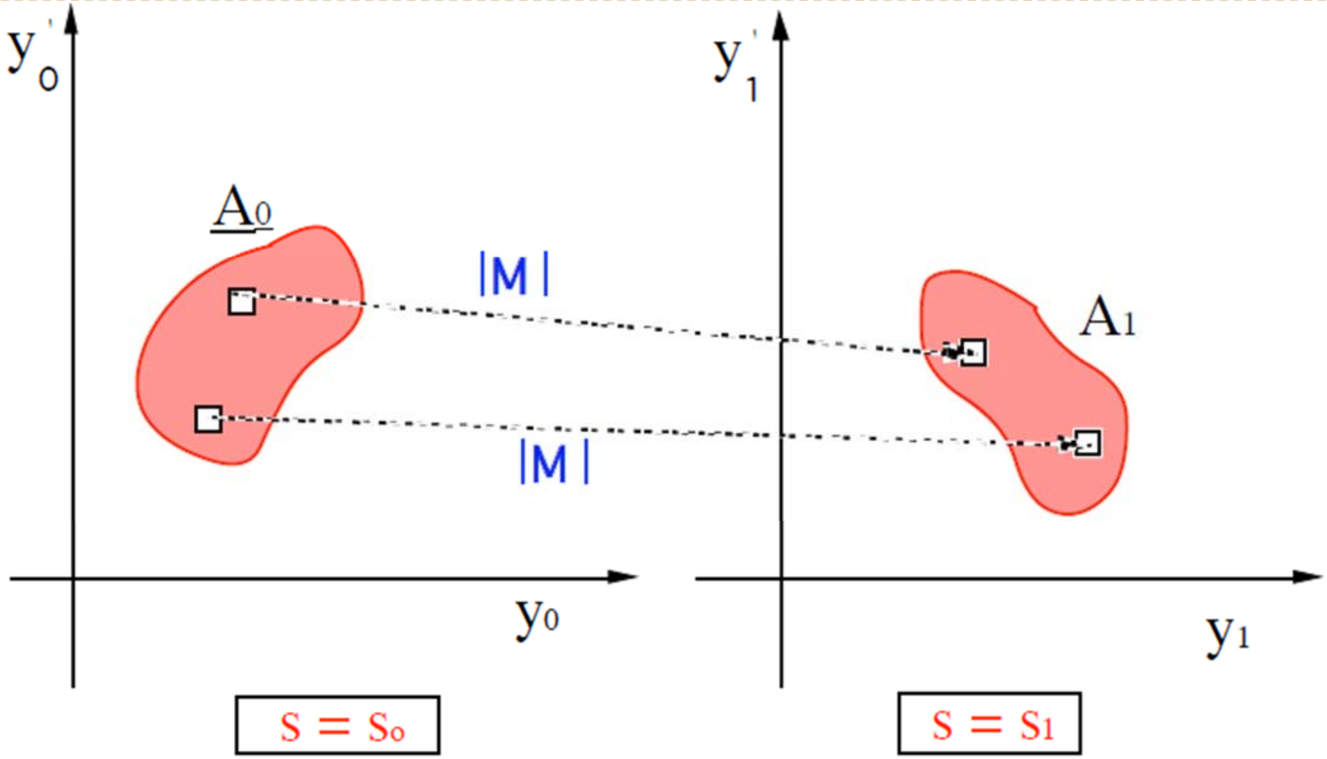
$$\frac{d}{dt}(\rho dV) = \frac{d\rho}{dt}\delta V + \rho\frac{d\delta V}{dt} = 0$$

$$\frac{d\delta V}{dt} = \frac{d}{dt}\int d^3 q_i d^3 p_i = 0$$

Decoupling

- Usually, the longitudinal motion along the beam axis is decoupled from the motion in the plane transverse to the beam axis
- In that case, the 6D phase space can split in a longitudinal phase space (2D) and a transverse phase space (4D)
- But, if the transverse motion can be decomposed into two independent motion along two orthogonal directions, the transverse phase space can also be split into two 2D phase spaces.

Only the 6D volume is invariant



- The hypersurface limiting a finite volume is not invariant in general. Its geometrical form is modified, but in a way such that the enclosed volume is conserved.

Trace space vs phase space

- The phase space is (x, p_x, y, p_y, z, p_z)
- The trace space is (x, x', y, y', z, z')
- The phase space is constant, the trace space not.
- But…

$$\int p_x(x)\, dx = cost_x \;;\; \int p_y(y)\, dy = cost_y \;;\; \int p_z(z)\, dz = cost_z.$$

$$p_x = m_o\, c\,\gamma_{rel}\, \beta_x$$

$$x' = \frac{dx}{ds} = \frac{dx}{dt} \cdot \frac{dt}{ds} = \frac{\beta_x}{\beta_s}$$

Normalized emittance

$$\int p_x \, dx = m_o c \int \gamma_{rel} \beta_x \, dx = m_o c \int \gamma_{rel} \beta_s \, x' dx =$$

$$m_o c \gamma_{rel} \beta_s \int x' dx = m_o c \gamma_{rel} \beta_s \varepsilon_x = \text{constant}$$

$$\varepsilon_{x,N} = \gamma_{rel} \beta \varepsilon_x$$

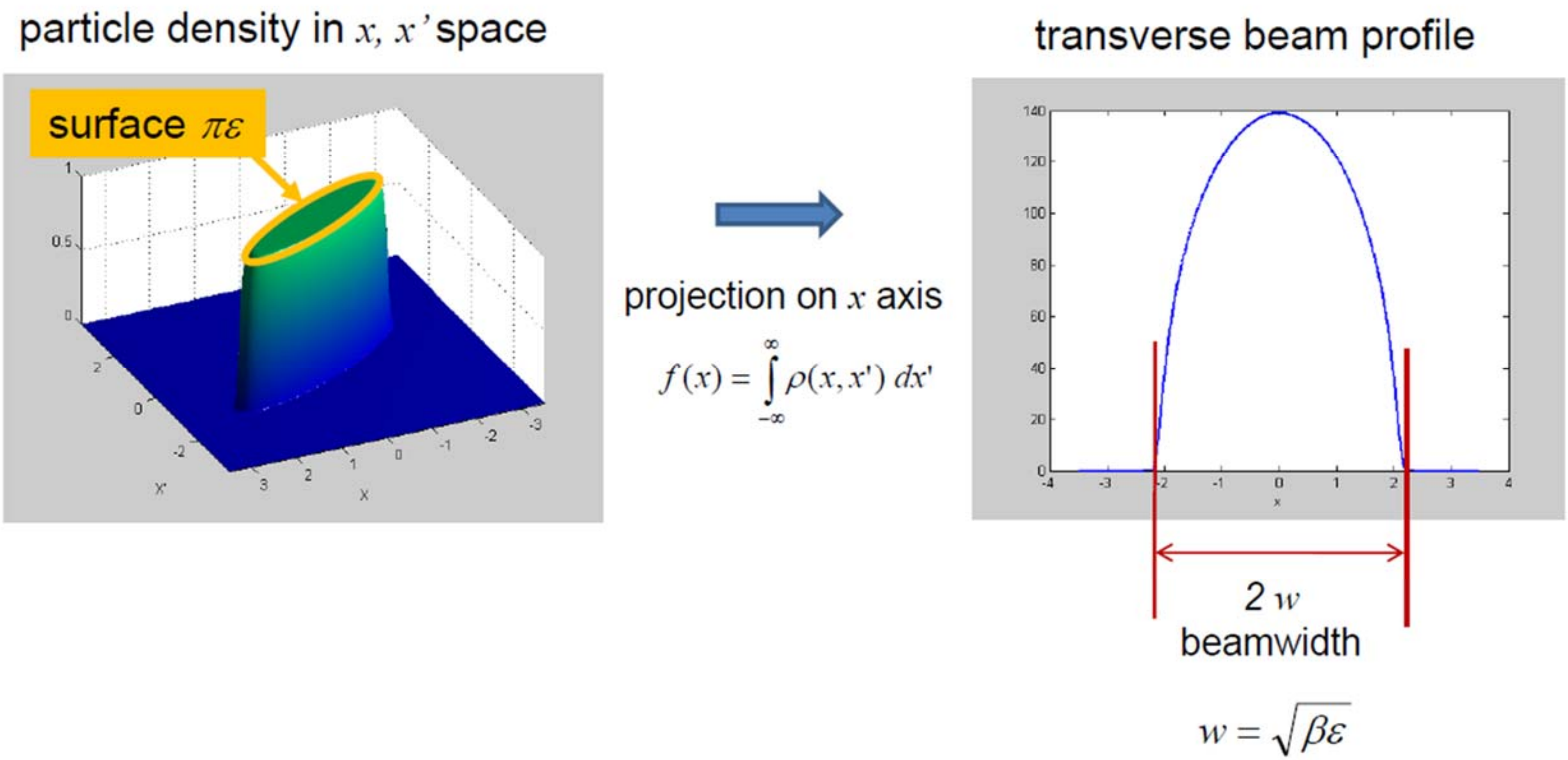
- This quantity is called normalized emittance
- If the energy is constant also ε_x is conserved. ε_x is called geometrical emittance. So we can use the trace space instead of the phase space
- Important: when we consider a beam of particles they must have the same energy in order to preserve the normalized emittance

Adiabatic damping

$$\varepsilon_{x,N}=\gamma_{rel}\beta\varepsilon_x$$

- As $\beta\gamma$ increases proportionally to the particle momentum p, the emittance ε decrease as 1/p
- It is called adiabatic damping
- Another point of view:
  - p_s increases, while p_x not. So the slope $x'=p_x/p_s$ is decreased

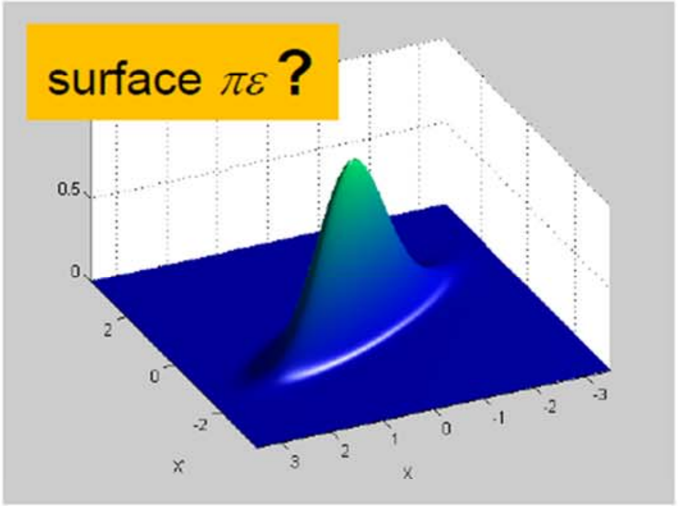
Sharp boundary



- So far we implicitly assumed that beam occupies *x, x′* space area with a sharp boundary

Real beam

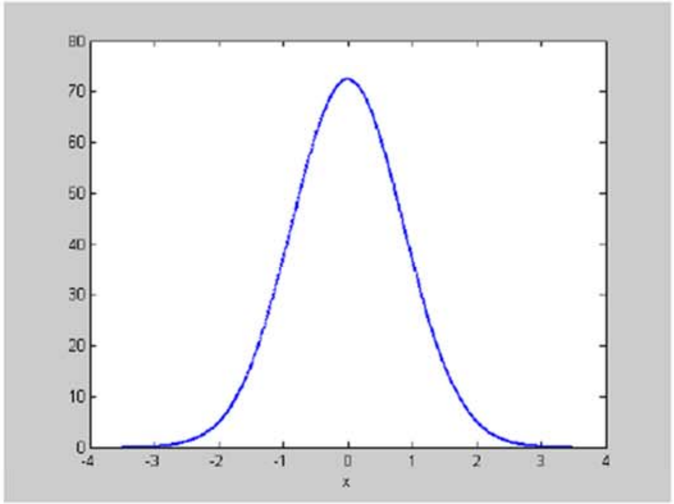
particle density in x, x' space



projection on x axis

$$f(x) = \int_{-\infty}^{\infty} \rho(x, x')\, dx'$$

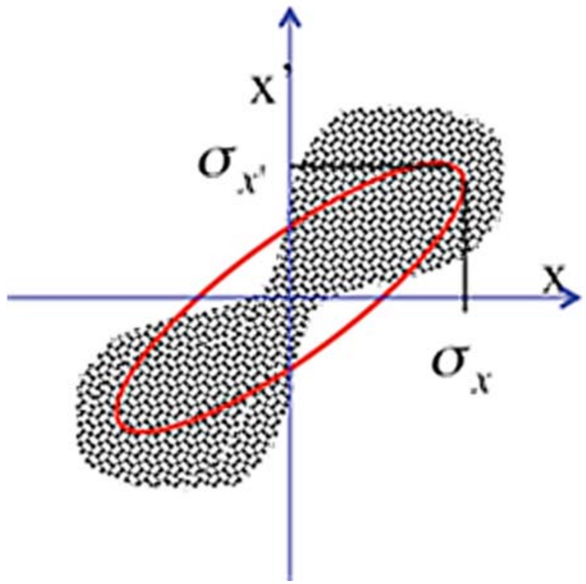
transverse beam profile



beamwidth ?

A “Real” beam

- Nonlinear field components can stretch and distort the particle distribution in the phase space and the beam loses its laminar behavior. A realistic phase space distribution is often well different by a regular ellipse



RMS emittance

- We introduce, therefore, a definition of emittance that measures the beam quality rather than the phase space area.
- *RMS (root mean square) emittance*

$$\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 = \varepsilon_{x,rms}$$

Definition of second momentum

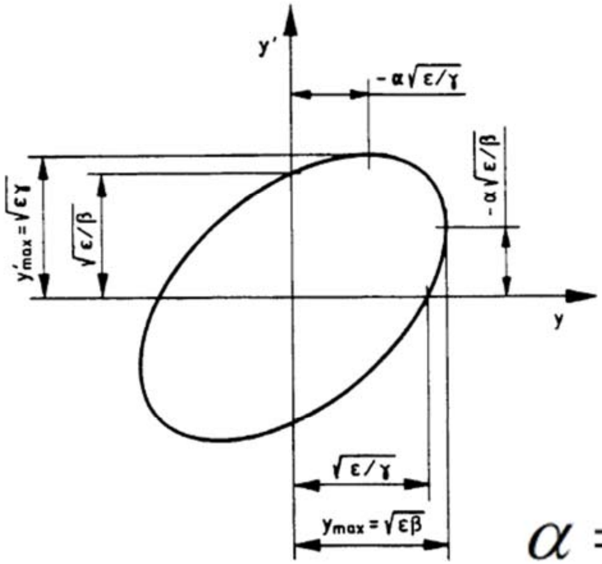
$$\sigma_x = \sqrt{\beta_x \varepsilon_{x,rms}}$$

$$\sigma_{x'} = \sqrt{\gamma_x \varepsilon_{x,rms}}$$

$$\sigma_x^2(z) = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x, x', z)\, dx dx'$$

$$\sigma_{x'}^2(z) = \langle x'^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x'^2 f(x, x', z)\, dx dx'$$

RMS emittance



$$\langle x^2 \rangle = \beta\varepsilon$$

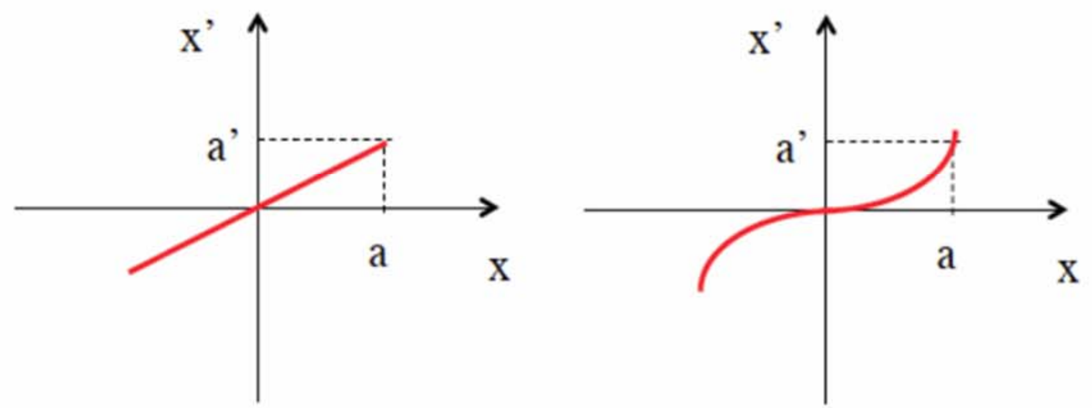
$$\langle x'^2 \rangle = \gamma\varepsilon$$

$$\alpha = -\frac{1}{2}\beta' = -\frac{1}{2}\frac{d}{dz}\frac{\langle x^2 \rangle}{\varepsilon} = -\frac{1}{\varepsilon}\langle xx' \rangle$$

$$\beta\gamma - \alpha^2 = 1$$

$$\varepsilon_{rms} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle\langle x'^2 \rangle - \langle xx' \rangle^2\right)}$$

Importance of RMS emittance



$$x' = Cx^n$$

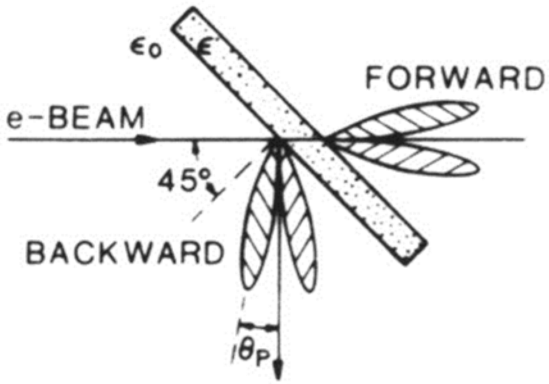
$$\varepsilon_{rms}^2 = C\sqrt{\langle x^2\rangle\langle x^{2n}\rangle - \langle x^{n+1}\rangle^2} \qquad \begin{cases} n = 1 & \Rightarrow \quad \varepsilon_{rms} = 0 \\ n > 1 & \Rightarrow \quad \varepsilon_{rms} \neq 0 \end{cases}$$

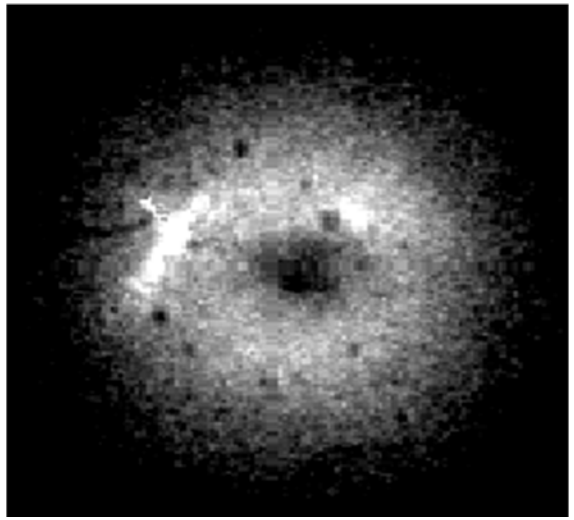
Even when the phase-space area is zero, if the distribution lies on a curved line its rms emittance is not zero. The rms emittance depends not only on the area occupied by the beam in phase space but also on distortions produced by non-linear forces

Measuring beam size

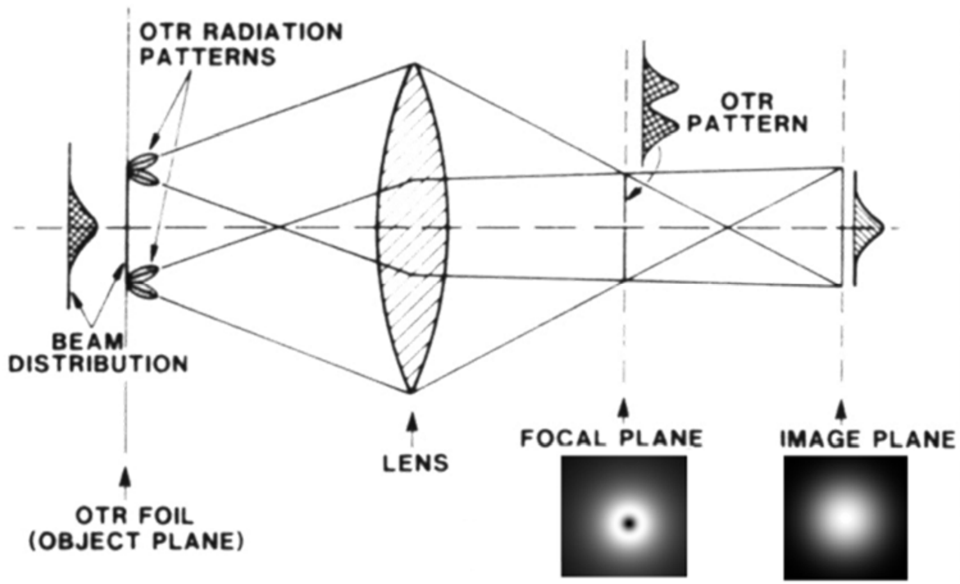
OTR (Optical Transition Radiation)

- It is emitted when a particle cross the boundary between two medium with different index of refraction
- The radiation is emitted in a narrow cone with aperture $2/\gamma$
- It is weak (~ 1 photon per 100 electrons in the optical wavelenght)
- It is linear in the number of particles
- It is a surface effect and so it is "prompt" if compared with the time structure of the beam



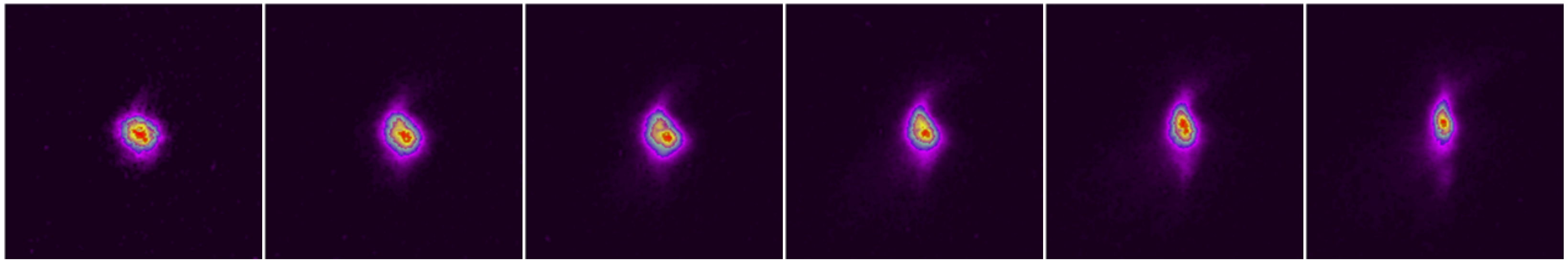


OTR for beam diagnostic



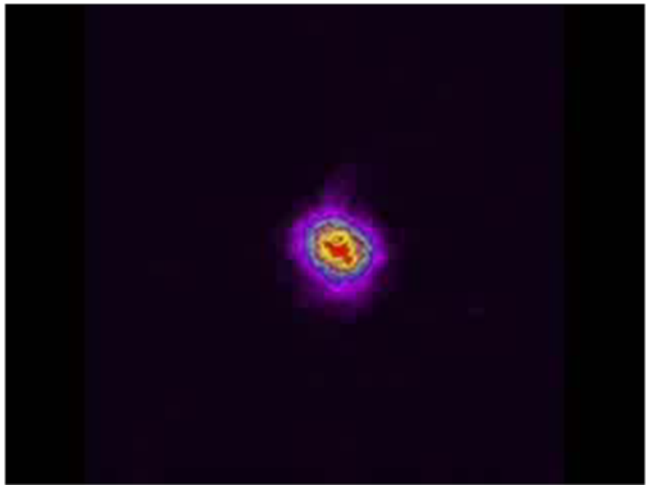
- Both the angular distribution (giving information on the energy and the angular divergence of the beam) and the beam image (giving information on the beam size) can be measured (optical resolution, diffraction limited λ/θ)

Time resolved measurements



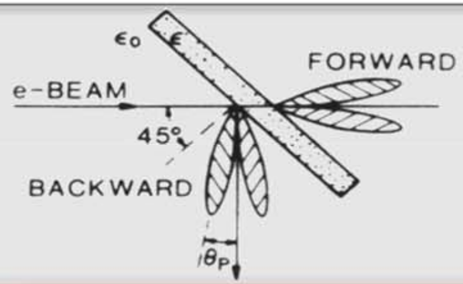
Beam evolution along the macrobunch, step 1 μs

The movie

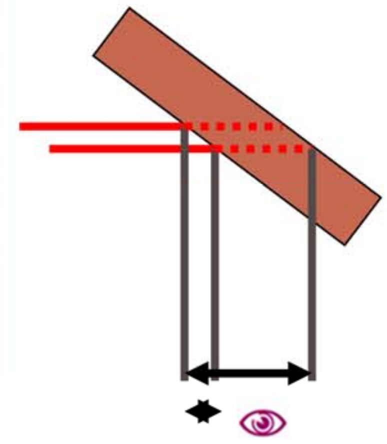


Intercepting devices

- OTR monitors
  - High energy (>tens of MeV), high charge (>hundreds of pC)
  - No saturation
  - Resolution limit closed to optical diffraction limit
  - Surface effect

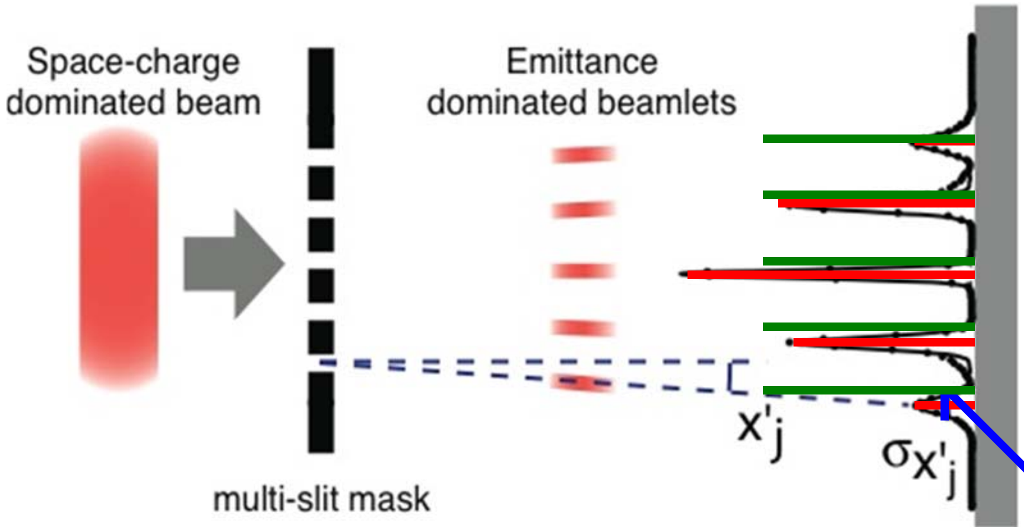


- Scintillator (like YAG:CE)
  - Large number of photons
  - Resolution limited to grain dimension (down to few microns)
  - Saturation depending of the doping level
  - Bulk effect
  - Thin crystal to prevent blurring effect
- Wire scanner
  - Multiple scattering reduced
  - Higher beam power
  - Multishot measurement
  - 1 D
  - Complex hardware installation



Transverse emittance measurement

Emittance measurement with space charge



To measure the emittance for a space charge dominated beam the used technique is know 1-D pepper-pot

The emittance can be reconstructed from the second momentum of the distribution

$$\varepsilon = \sqrt{< x^2 >< x'^2 > - < xx' >^2}$$

C. Lejeune and J. Aubert, Adv. Electron. Electron Phys. Suppl. A **13**, 159 (1980)

Design issues

- The beamlets must be emittance dominated

$$\sigma_x'' = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3} + \frac{I}{\gamma^3 I_0 (\sigma_x + \sigma_y)}$$

Martin Reiser, Theory and Design of Charged Particle Beams (Wiley, New York, 1994)

- Assuming a round beam

$$R_0 = \frac{I \sigma_0^2}{2\gamma I_0 \varepsilon_n^2} \qquad \sigma_x = \frac{d}{\sqrt{12}}$$

- *d* must be chosen to obtain $R_0 << 1$, in order to have a beam emittance dominated

Design issues (2)

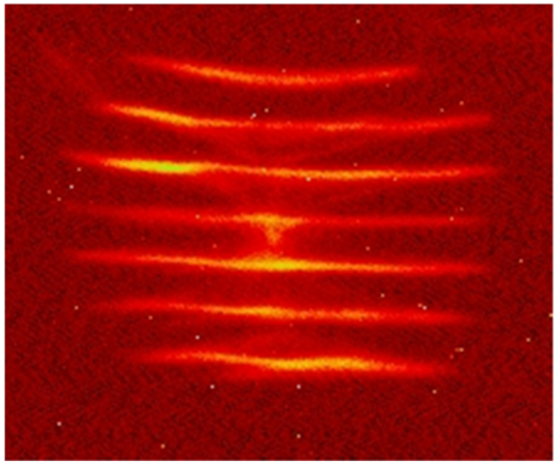
- The contribution of the slit width to the size of the beamlet profile should be negligible
- The material thickness (usually tungsten) must be long enough to stop or heavily scatter beam at large angle
- But the angular acceptance of the slit cannot be smaller of the expected angular divergence of the beam

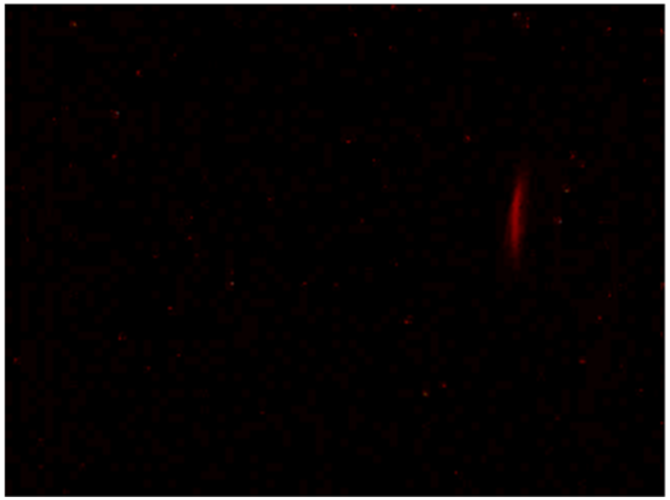
$$\sigma = \sqrt{(L \cdot \sigma')^2 + \left(\frac{d^2}{12}\right)}$$

$$L >> \frac{d}{\sigma' \cdot \sqrt{12}}$$

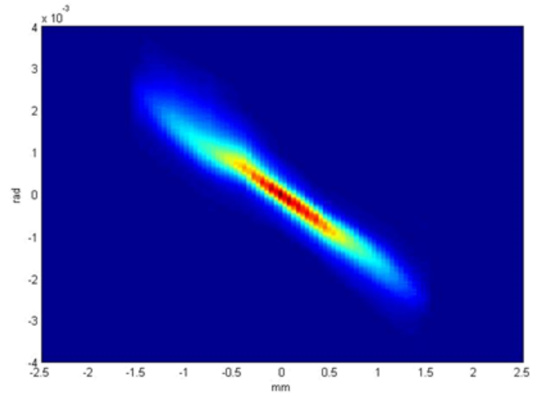
$$l < \frac{d}{2\sigma'}$$

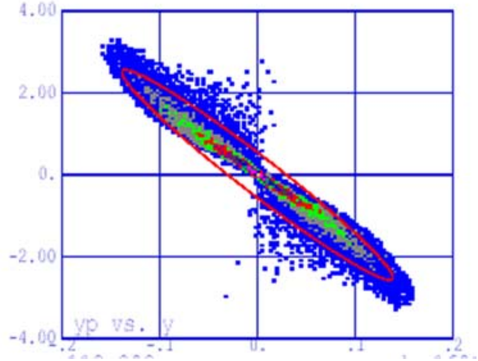
Examples

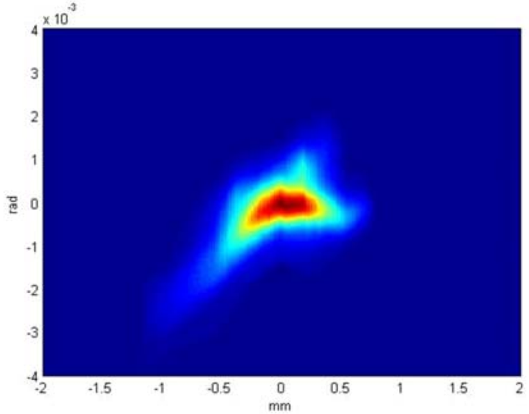


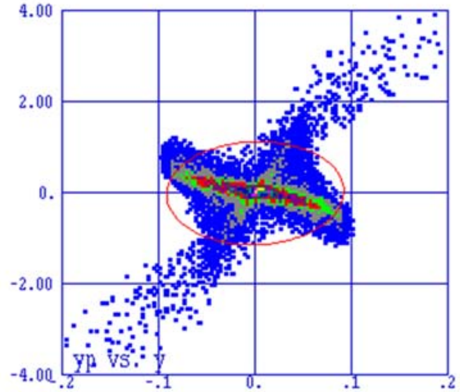


Phase space mapping

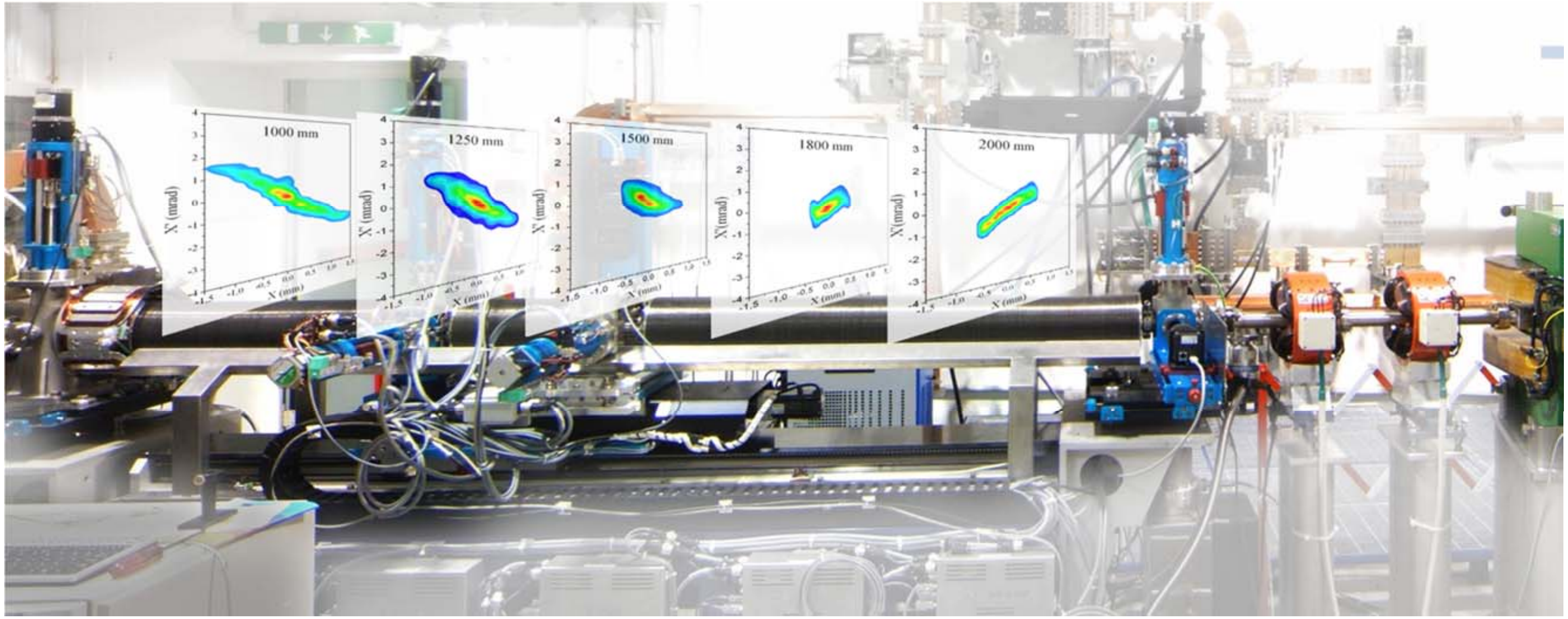




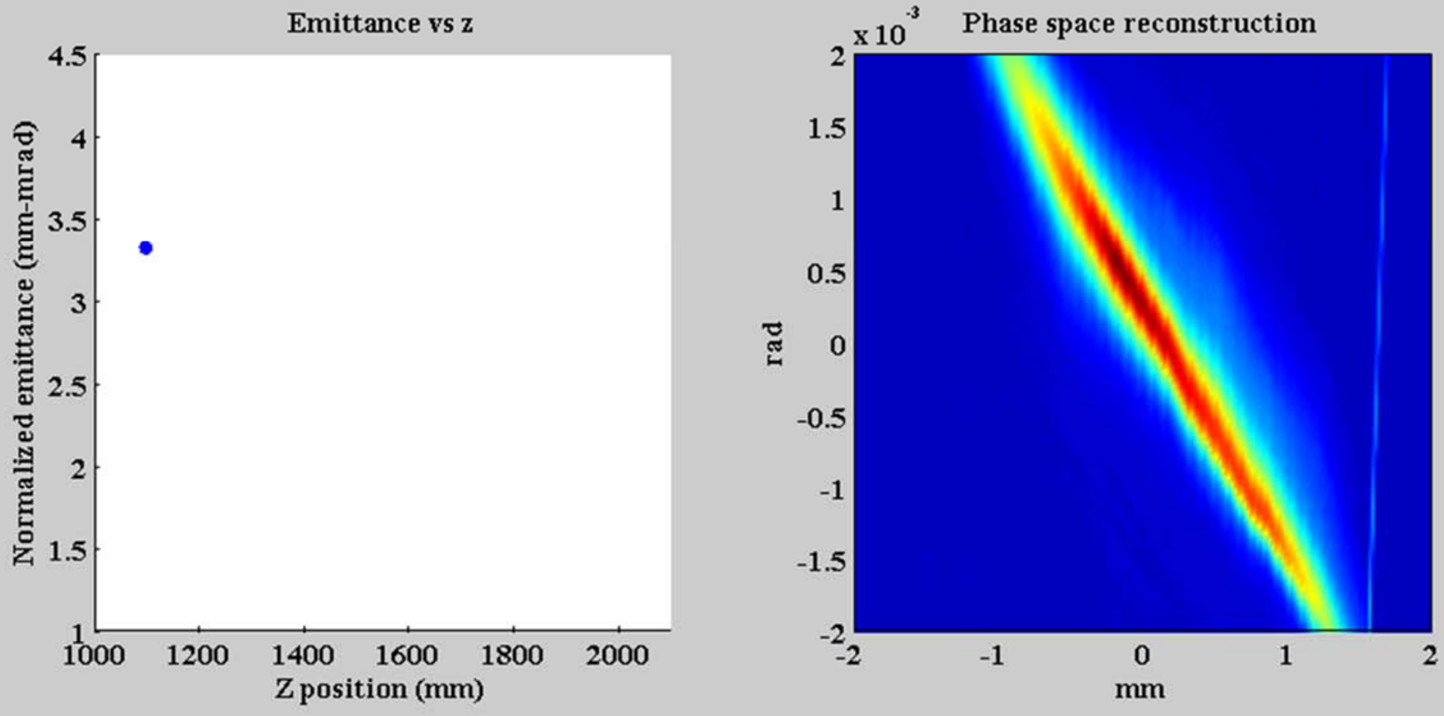




Phase space evolution

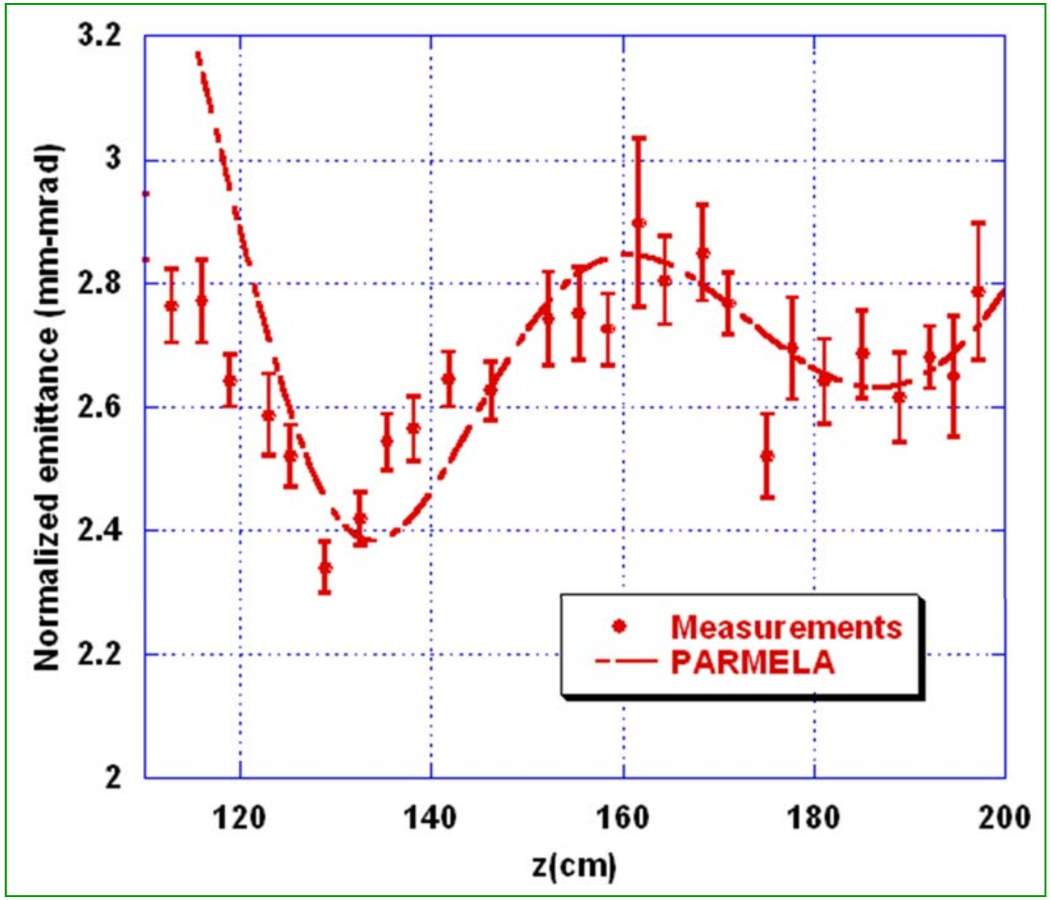


A. Cianchi et al., "High brightness electron beam emittance evolution measurements in an rf photoinjector", Physical Review Special Topics Accelerator and Beams 11, 032801,2008



Emittance vs z
Normalized emittance (mm-mrad)
4.5
4
3.5
3
2.5
2
1.5
1
1000
1200
1400
1600
1800
2000
Z position (mm)
Phase space reconstruction
x 10^-3
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
rad
-2
-1
0
1
2
mm

Double Minimum Signature



M. Ferrario et al.,
"Direct measurement of double emittance minimum in the SPARC high brightness photoinjector",
PRL **99,** 234801 (2007)

Emittance measurement without space charge

- The most used techniques for emittance measurements are quadrupole scan and multiple monitors

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon = \gamma_0 x_0{}^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2$$

$$M(s_1 s_2) = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & S'C + SC' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

Beam Matrix

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

$$\sigma_{11} x^2 + 2\sigma_{12} x x' + \sigma_{22} x'^2 = 1$$

$$\sigma_1 = M \sigma_0 M^T$$

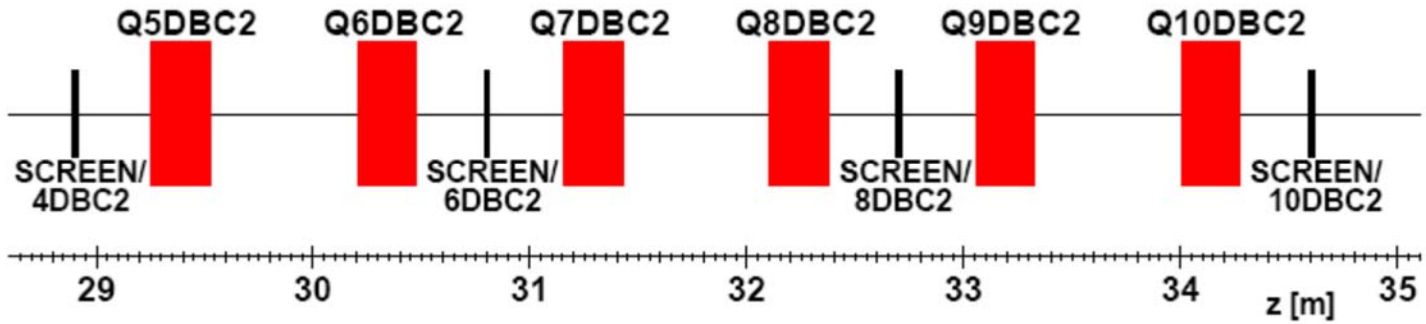
Multiple screens

$$\sigma_{i,11} = C_i^2 \sigma_{11} + 2 S_i C_i \sigma_{12} + S_i^2 \sigma_{22}$$

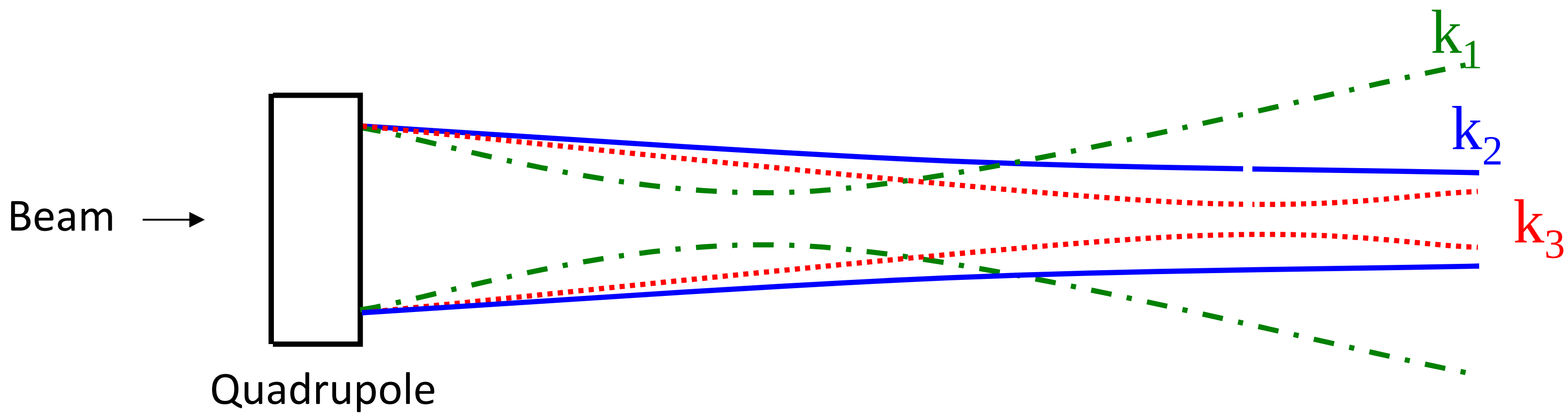
- There are 3 unknown quantities
- $\sigma_{i,11}$ is the rms beam size
- C_i and S_i are the element of the transport matrix
- We need 3 measurements in 3 different positions to evaluate the emittance

Example: FLASH @ DESY

- M. Minty, F. Zimmermann, "Measurement and control of charged particle beams", Springer (2003)
- DESY-Technical Note 03-03 , 2003 (21 pages) Monte Carlo simulation of emittance measurements at TTF2 P. Castro



Quadrupole scan

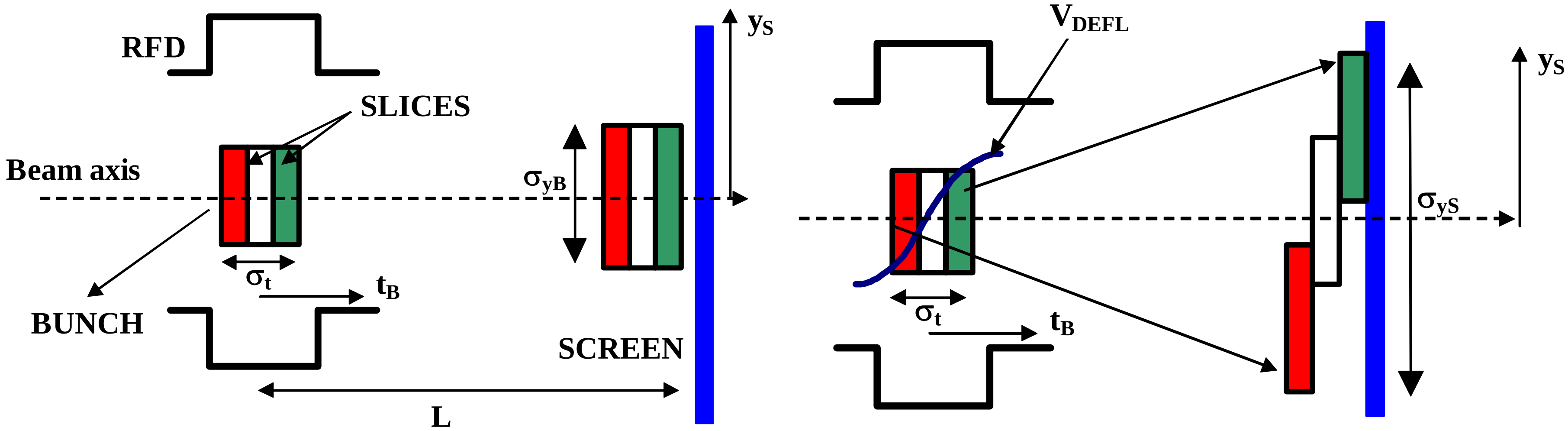


$$\sigma_{11} = C^2(k)\sigma_{11} + 2C(k)S(k)\sigma_{12} + S^2(k)\sigma_{22}$$

- **It is possible to measure in the same position changing the optical functions**
- **The main difference with respect to the multi screen measurements is in the beam trajectory control and in the number of measurements**

Bunch length and longitudinal phase space

RF deflector



- The transverse voltage introduces a linear correlation between the longitudinal and the transverse coordinates of the bunch

RFD

$$\Delta x'(z) = \frac{eV_0}{pc}\sin(kz+\varphi) \approx \frac{eV_0}{p_z c}\left[\frac{2\pi}{\lambda} z\cos\varphi + \sin\varphi\right] \qquad |z| << \lambda/2\pi$$

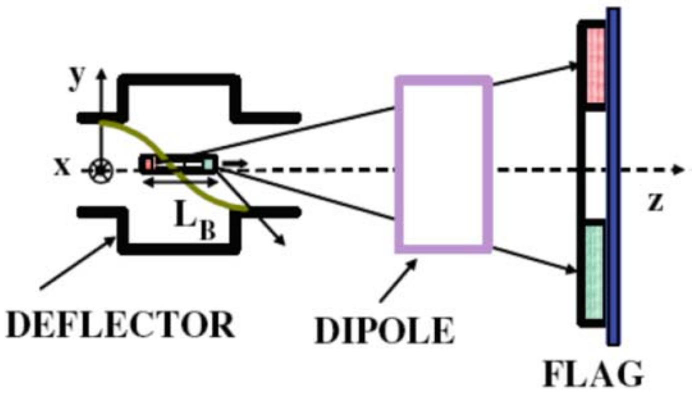
$$\Delta x(z) = \frac{eV_0}{pc}\sqrt{\beta_d \beta_s}\,\sin\Delta\Psi\left[\frac{2\pi}{\lambda} z\cos\varphi + \sin\varphi\right]$$

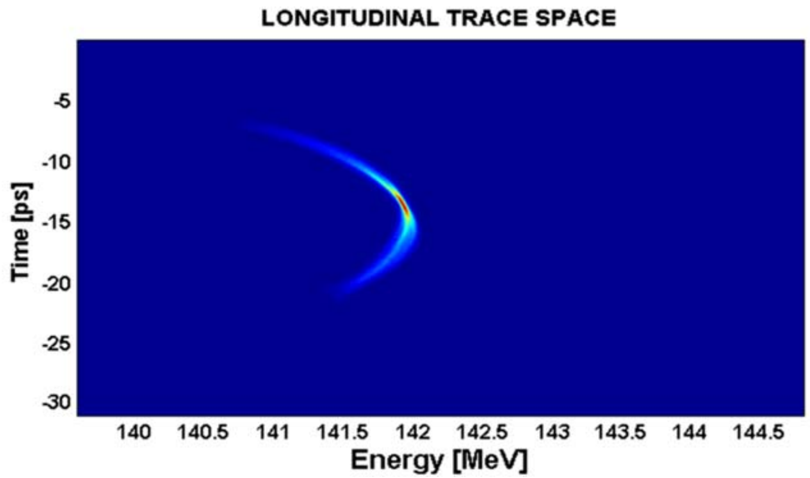
$$\sigma_x = \sqrt{\sigma^2{}_{x0} + \sigma_z^2 \beta_d \beta_s \left(\frac{2\pi e V_0}{\lambda pc}\sin\Delta\Psi\cos\varphi\right)^2} \qquad \sigma^2{}_{x0} = \sqrt{\frac{\beta_s \varepsilon_N}{\gamma}}$$

$$eV_0 >> \frac{\lambda}{2\pi\sigma_z}\frac{1}{|\sin\Delta\Psi\cos\varphi|}\sqrt{\frac{\varepsilon_N\, pcmc^2}{\beta_d}}$$

- P. Emma, J. Frisch, P. Krejcik ," *A Transverse RF Deflecting Structure for Bunch Length and Phase Space Diagnostics* " ,LCLS-TN-00-12, 2000
- D. Alesini, "*RF deflector based sub-ps beam diagnostics: application to FEL and Advanced accelerators*", International Journal of Modern Physics A, 22, 3693 (2007)

Longitudinal phase space





- Using together a RFD with a dispersive element such as a dipole
- Fast single shot measurement