Pure spinor equations to lift gauged supergravity

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A few words on supersymmetry (SUSY)

- Two species of particles: bosons and fermions
- Bosons (B) $\rightarrow$ integer spin $\rightarrow$ scalar, vector, tensor fields
  Fermions (F) $\rightarrow$ half-integer spin $\rightarrow$ spinorial fields
- SUSY exchanges bosons with fermions

$$\delta_\epsilon B \propto \bar{\epsilon} F \qquad \delta_\epsilon F \propto \epsilon \partial B$$

ϵ: SUSY parameter $\rightarrow$ spinorial object (constant in flat space)

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Closure on translations

Computing the commutator of different SUSY variations:

$$\left[\delta_{\epsilon_1}, \delta_{\epsilon_2}\right] B \propto (\bar{\epsilon}_1 \gamma^\mu \epsilon_2) \partial_\mu B$$

$\Rightarrow$ translations along $\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \equiv v^\mu$

Supersymmetric QFTs

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Two fundamental consequences

- SUSY enforces the **same** number of bosonic and fermionic d.o.f.
  $\Rightarrow$ Fields arranged in multiplets
- $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, B \propto v^{\mu}\partial_{\mu} B$
  $\rightarrow$ Translational invariance along v^{μ} is required

Localizing supersymmetry

- Till now: SUSY as a global symmetry $\rightarrow \epsilon$ is a **constant**
- Local SUSY $\Rightarrow \epsilon \rightarrow \epsilon(x)$ non-trivial function on the space-time

First consequence

$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, B \propto v^{\mu}(x)\partial_{\mu} B \Rightarrow$ Diffeomorphism invariance

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- Diffeomorphisms act on the metric $g^{\mu\nu}$
  $\rightarrow g^{\mu\nu}$ becomes a **dynamical object** $\Rightarrow$ theory of gravity

Supergravity (SUGRA): generalization of Einstein's gravity invariant under **local** SUSY transformations

An example: $4d$, $\mathcal{N} = 1$, pure SUGRA

- Gauging a global symmetry requires a gauge field
  $\rightarrow$ gravitino field ψ_μ. It is a **spinorial** gauge field (ϵ is a spinor)
- As usual we need a kinetic term for ψ_μ
  $\rightarrow$ **Rarita-Schwinger** action

$$\mathcal{L}_{\mathrm{RS}} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5 D_\rho\psi_\sigma$$

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Final Lagrangian

$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{RS}} + \mathcal{L}_{\psi^4}$$

$\mathcal{L}_{\psi^4}$ is a **quartic** interactive term for the gravitino
Necessary in order to have SUSY invariance

Generalizations

- First generalization: adding matter fields
  - chiral multiplets (ϕ^i, χ^i): ϕ^i is a complex scalars, χ^i is a Majorana fermion
  - vector multiplets (χ^M, A_μ^M): A_μ^M is a vector field, χ^M is a Majorana fermion

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$\rightarrow \phi^i$ are coordinates on a Kähler manifold

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- Second generalization: Extended SUGRA
  - Till now: only **one** SUSY parameter ϵ
    $\rightarrow$ Extended SUGRA $\Rightarrow \epsilon(x) \rightarrow \epsilon^i(x) \qquad i \leq 8$
  - Lagrangians are more and more constrained

We will focus on $4d$, $\mathcal{N} = 2$ (i.e. ϵ_1 and ϵ_2) **gauged** SUGRA

$4d$, $\mathcal{N} = 2$ gauged SUGRA

Field content

- gravity multiplet $(g^{\mu\nu}, \psi^i_\mu, A^0_\mu)$: **metric** $(g^{\mu\nu})$, 2 **gravitini** (ψ^i_μ), 1 **vector** (A^0_μ, field strength $T_{\mu\nu}$) the **graviphoton**
- n_v vector multiplets $(A^a_\mu, \lambda^{ia}, t^a)$: n_v**vector** fields (A^a_μ, field strength $G^a_{\mu\nu}$), $2n_v$ **gaugini** (λ^{ia}), n_v complex **scalars** (t^a)
- n_h hypermultiplets (k_α, q^u): $2n_h$ **hyperini** (k_α), $4n_h$ **scalars** (q^u)

- **SUSY constraints**
  - t^a: coordinates on a Special Kähler manifold SK
  - q^u: coordinates on a Quaternionic manifold Q

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- **SUSY constraints**
  - t^a: coordinates on a Special Kähler manifold SK
  - q^u: coordinates on a Quaternionic manifold Q
- **Gauged SUGRA**
  SK and Q have isometries (Killing vectors k^a_Λ and k^u_Λ)
  $\rightarrow$ act **globally** on t^a and q^u $\Rightarrow$ gauge them using A^Λ_μ $(\Lambda = 0 \ldots n_v)$

$$Dt^a = dt^a + g\, A^\Lambda k^a_\Lambda \,, \qquad Dq^u = dq^u + g\, A^\Lambda k^u_\Lambda$$

Basics on string theory

- String theory is the most promising candidate to obtain unification
- Point particles (PP) are replaced by strings
  $\rightarrow$ natural cut-off scale $\Rightarrow$ quantization of gravity
- PP as normal modes of vibration
  $\rightarrow$ mass proportional to the level of the mode $\rightarrow$ **Infinite** particles

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- **Closed strings**: we have a graviton $\Rightarrow$ theory of gravity
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- To have fermions SUSY is required $\rightarrow$ Superstring theories
  $\Rightarrow$ SUGRA theories as low-energy effective-actions
- Superstring theory requires a **ten-dimensional** spacetime
  $\Rightarrow$ Ten-dimensional SUGRA

Ten dimensional Type II Supergravities

- Two different models: **Type IIA** and **Type IIB**
- 2 SUSY parameters (Majorana-Weyl): ϵ^1_+ and $\epsilon^2_\pm$ (upper index IIB)

Field content

- **Fermions**: gravitini ψ^1_{M+} and $\psi^2_{M\pm}$, dilatini λ^1_+ and $\lambda^2_\pm$
- **Bosons**: the metric g_{MN}, a 2-form B_{MN}, a scalar ϕ (called dilaton), and a collection of p-forms $C_{M_1 \ldots M_p}$, p is odd (even) for IIA (IIB)

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- **Fluxes**:
  We introduce the field-strengths for B_{MN} and $C_{M_1 \dots M_p}$

$$H = dB \qquad F_{p+1} = dC_p - H \wedge C_{p-2}$$

Hodge-duality $\rightarrow F_p = (-1)^{\lfloor \frac{p}{2} \rfloor} * F_{10-p}$

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- Type II SUGRA are not **realistic** tout-court
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Reduction recipe (KK)

Suppose a ten-dimensional manifold as

$$\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{M}_6$$

$\mathcal{M}_4$: extended usual spacetime. $\mathcal{M}_6$: compact internal space.
Ten-dimensional fields are expanded in terms of eigenmodes of an **internal** operator (e.g. the Laplacian) and **only** zero-modes are considered
$\Rightarrow$ 4d SUGRA as effective actions

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- **Problems: consistent truncations and unphysical fields**
  In general no guarantee that a solution of the reduced theory is also a solution of the original theory $\rightarrow$ consistent truncations problem
  Moreover, reducing a theory can leads to unphysical $4d$ multiplets

Consistent truncations problem: An example

- Consider two scalar fields H (heavy) and l (light)
- **Action:**

$$S = \int d^4x \, (\partial_\mu H \partial^\mu H + M^2 H^2 + \partial_\mu l \partial^\mu l + m^2 l^2 + Hl^2)$$

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The red terms are **problematic**. Set $H = 0$ in the e.o.m. $\Rightarrow l = 0$
Set $H = 0$ in the Lagrangian $\Rightarrow l$ becomes a free field
$\Rightarrow$ spurious solutions and not consistent truncation

From $4d$ to $10d$: lifting

- It is not known if $4d$ SUGRA make sense as **quantum** theories
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- **Standard approach to lift solutions**
  Performing a **consistent** truncation of Type II to $4d$ ensures that a solution of the reduced theory can be embedded in string theory.
  Performing a consistent reductions is very **difficult** in general

Out of reach in general [Kashani-Poor, Minasian, 2006]

Supersymmetric solutions and a new strategy to lift

- Solutions of equations of motion which are **invariant** under SUSY are more interesting than the others $\rightarrow$ Supersymmetric (BPS) solutions
- **Standard strategy to find them**
  We put $F = 0$ and impose $\delta F = 0$
  $\rightarrow$ Apart some subtleties these configurations solve e.o.m.

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A new approach to lifting

We rewrite the equations for ten-dimensional supersymmetry in a way formally **identical** to a corresponding system in $4d$, $\mathcal{N} = 2$ gauged supergravity. Some **additional** equations are found, they are the obstructions to lift.
This provides a way to look for lifts of BPS solutions without having to reduce or even rewrite the ten-dimensional action

Conceptual steps

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The additional equations are due to fields in $10d$ which are unphysical in $4d$, $\mathcal{N} = 2$ SUGRA. These fields belong to gravitino multiplet $\rightarrow$ well-defined only for $\mathcal{N} > 2$ SUGRA

Supersymmetry in $4d$, $\mathcal{N} = 2$ SUGRA

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k, v^x, o^x and μ are **geometrical** quantities defined by the SUSY parameters

Supersymmetry equations in $10d$

- Conditions for unbroken SUSY in $10d$ already found [Tomasiello,2011]

$$L_K g = 0\ , \qquad d\tilde{K} = \iota_K H$$

$$\boxed{d_H(e^{-\phi}\Phi) = -(\tilde{K}\wedge + \iota_K)F}$$

$$(\psi_- \otimes \overline{\epsilon_2} e_2^-\ , \pm d_H(e^{-\phi}\Phi \cdot e_2^-) + \sigma_2\Phi - 2F) = 0$$

$$(e_1^- \epsilon_1 \otimes \overline{\psi_-}\, , d_H(e^{-\phi} e_1^- \cdot \Phi) - \sigma_1\Phi - 2F) = 0 \qquad \forall \psi_- \in \Sigma_-$$

- $d_H = d + H\wedge \qquad F = \sum_k F_k \qquad \sigma_i \equiv \frac{1}{2} e^{\phi} d^\dagger (e^{-2\phi} e_i^-)$

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- Φ, K $\tilde{K}$ are geometrical quantities defined by the ϵ^i
- e_-^i are **additional** vectors $\to$ not defined by ϵ^i
- Last equations are in terms of the ten-dimensional Mukai pairing:

$$(A,B)\equiv (A\wedge\lambda(B))_{\rm top} \qquad \lambda\alpha_k \equiv (-)^{\lfloor\frac{p}{2}\rfloor}\alpha_k$$

A basis for the internal forms

- We want to **massage** SUSY conditions and make them **similar** to $4d$
  $\Rightarrow$ We need a basis for the **internal** forms

A basis: Generalized Hodge diamond (GHD)

$$
\begin{array}{ccccccc}
 & & & \phi_+ & & & \\
 & & \phi_+\gamma^{i_2} & & \gamma^{\bar{i}_1}\phi_+ & & \\
 & \phi_-\gamma^{\bar{i}_2} & & \gamma^{\bar{i}_1}\phi_+\gamma^{i_2} & & \gamma^{i_1}\bar{\phi}_- & \\
\phi_- & & \gamma^{\bar{i}_1}\phi_-\gamma^{\bar{j}_2} & & \gamma^{i_1}\bar{\phi}_-\gamma^{j_2} & & \bar{\phi}_- \\
 & \gamma^{\bar{i}_1}\phi_- & & \gamma^{i_1}\bar{\phi}_+\gamma^{\bar{j}_2} & & \bar{\phi}_-\gamma^{i_2} & \\
 & & \gamma^{i_1}\bar{\phi}_+ & & \bar{\phi}_+\gamma^{\bar{i}_2} & & \\
 & & & \bar{\phi}_+ & & &
\end{array}
$$

Orthogonal basis: every form has vanishing six-dimensional pairing with every form except with the ones symmetric with respect to the center

Organizing fields as four-dimensional ones

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- $4d$ **scalars**

$$
\begin{aligned}
&D\bar{t}^a = -(\delta\phi_+^a, d_4\phi_+) \qquad \delta\phi_+^a = \{\gamma^{i_1}\bar{\phi}_+\gamma^{\bar{j}_2}\} \\
&D(z^\alpha + i\tilde{z}^\alpha) = (\delta\phi_-^\alpha, d_4\phi_-) \qquad \delta\phi_-^\alpha = \{\gamma^{i_1}\bar{\phi}_-\gamma^{j_2}\} \\
&D(\xi^\alpha + i\tilde{\xi}^\alpha) \equiv -\frac{1}{2}e^\phi(\delta\phi_-^\alpha, F_1) \\
&D(\xi + i\tilde{\xi}) \equiv 2e^\phi(\bar{\phi}_-, F_1) \\
&\phi, \quad da = *H_3
\end{aligned}
$$

$(\xi, \tilde{\xi}, \phi, a) \rightarrow$ **Universal hypermultiplet**
$(\xi^\alpha, \tilde{\xi}^\alpha, z^\alpha, \tilde{z}^\alpha) \rightarrow$ **Non-Universal hypermultiplets**
$\bar{t}^a \rightarrow$ complex scalars for the **vector multiplets**

Crucial observation

We are not performing any reduction. We are only rearranging fields in $4d$ language $\rightarrow$ **Consistent truncation** issues are not involved

Organizing fields as four-dimensional ones

- Using the GHD we can decompose internal fields along it
- **"Wrong" Fields:**

$$(\phi_+ \gamma^{i_2}, F_1), \qquad (\gamma^{\bar{i}_1} \phi_+, F_1), \qquad (\phi_- \gamma^{\bar{i}_2}, F_2), \qquad (\gamma^{i_1} \bar{\phi}_-, F_2)$$

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These fields are usually discarded in the reduction procedure.
$(\phi_-\gamma^{\bar{i}_2}, F_2)$ and $(\gamma^{i_1}\bar{\phi}_-, F_2)$ give rise to additional gravitino multiplets
$\Rightarrow$ **Unphysical** in $4d$
However we are not **reducing** but just **rewriting**
$\rightarrow$ We keep them

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- **Additional** equations are found $\rightarrow$ due to "Wrong" fields

SUSY in 10d and lifting conditions

- The conditions for unbroken SUSY in 10d are rewritten with the redefinitions just discussed and taking the **timelike** hypothesis
- Most of the equations take exactly the same form of the four-dimensional ones

$$
\begin{aligned}
D\mu &= S_x v_x - 2\iota_k T^+ \\
dk &= -2\mathrm{Re}(S_x \bar{o}_x + 2\bar{\mu} T^+) \\
Dv_x &= 2\epsilon_{xyz}\mathrm{Im}(\bar{S}_y o_z) \\
& i\,k \cdot Dq^v + \Omega^{xv}{}_u v_x \cdot Dq^u - g\,\bar{\mathcal{L}}^\Lambda k^v_\Lambda\,\mu = 0 \\
& 2i\bar{\mu} Dt^a - 4\iota_k G^{a+} - iW^{ax}v^x + 2W^a k = 0
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Lifting conditions without reducing any theory

Conclusions and future directions

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  - We have a new way to study the problem of lifting solutions in SUGRA
  - Equations written in terms of exterior calculus $\rightarrow$ easier to handling
  - Consistent truncations problem **overcome**

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  - We have a new way to study the problem of lifting solutions in SUGRA
  - Equations written in terms of exterior calculus $\rightarrow$ easier to handling
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- **Further possibilities:**
  - Apply this technique to explicit solutions in $4d$, $\mathcal{N} = 2$ SUGRA $\rightarrow$ AdS/CFT applications
  - Extend this approach to other $4d$ SUGRA
  - Applications to the Attractor mechanism

From spinors to forms: bilinears method

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Bilinears method

Given a couple of spinors ϵ_1, ϵ_2 we consider the expression (in d dimensions)

$$\epsilon_1 \otimes \overline{\epsilon_2} = \sum_{k=0}^{d} \frac{1}{2^{\lfloor \frac{d}{2} \rfloor} k!} (\overline{\epsilon_2} \gamma_{M_k \ldots M_1} \epsilon_1) \gamma^{M_1 \ldots M_k}$$

This is a collection of spinor bilinears (bispinors) that can be mapped to forms via the **Clifford map**

From spinors to forms: Clifford map

- Clifford Map
  Given a bispinor $\overline{\epsilon_2}\gamma_{M_k\ldots M_1}\epsilon_1\gamma^{M_1\ldots M_k}$ we convert it to a differential k-form by simply putting

$$\overline{\epsilon_2}\gamma_{M_k\ldots M_1}\epsilon_1\gamma^{M_1\ldots M_k} \longrightarrow \alpha_k = \overline{\epsilon_2}\gamma_{M_k\ldots M_1}\epsilon_1 dx^{M_1} \wedge \cdots \wedge dx^{M_k}$$

- It can be understood as the **inverse** (and as a generalization) of the well-known slash map

$$dx^M \wedge \partial_M \longrightarrow \gamma^M \partial_M$$

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The combined use of the bilinears method and of the Clifford map allow us to convert spinorial quantity in differential forms
$\Rightarrow$ SUSY conditions mapped in equations involving **only** differential forms

Geometry defined by spinors and the timelike hypothesis

- In $4d$, $\mathcal{N} = 2$ SUGRA we have 2 SUSY parameters (Weyl) ζ_1, ζ_2
  $\rightarrow$ Majorana conjugates $\zeta^i \qquad i = 1, 2$
- ζ_i can be proportional (null case) or not (timelike case)
  Assume to be in the **timelike** case $\rightarrow$ ζ_i constitute a basis of spinors

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- **Geometrical quantities in the timelike case:**
  We consider the **bispinors**

$$\zeta_i \otimes \overline{\zeta^j} = (1 + i*) v_i{}^j = (1 + i*)(k \delta_i{}^j + v^x \, \sigma_x{}_i{}^j) \ ,$$

$$\zeta_i \otimes \overline{\zeta_j} = \mu \epsilon_{ij} (1 + i \mathrm{vol}) + o_{ij} = \mu \epsilon_{ij} (1 + i \mathrm{vol}) + o^x \epsilon_{ik} \sigma^k_{x\,j}$$

$(k,\ v^x)$ is a **quartet** of vectors, μ is a scalar, o^x is a **triplet** of 2-forms
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Consequence of the timelike hypothesis

(k, v^x) together constitute a vielbein $\rightarrow$ **Crucial** simplification

Supersymmetric configurations in $4d$

- The **fermions** are: $F = (\psi_{i\mu}, \lambda^{ia}, k_\alpha) \to$ we put $F = 0$
- Supersymmetric configurations admit a solutions to the system

$$\delta_\zeta \psi_{i\mu} = D_\mu \zeta_i + \left(T^+_{\mu\nu} \gamma^\nu \epsilon_{ij} - \frac{1}{2} \gamma_\mu S_x \sigma^x_{ij} \right) \zeta^j = 0 \ ,$$

$$\delta_\zeta \kappa_\alpha = i U_{\alpha\, iu} D q^u \zeta^i + N^i_\alpha \zeta_i = 0 \ ,$$

$$\delta_\zeta \lambda^{ia} = i D t^a \zeta^i + \left((G^{a+} + W^a) \epsilon^{ij} + \frac{i}{2} W^{a\,x} \sigma^{ij}_x \right) \zeta_j = 0$$

$U_{\alpha iu}$ is a **vielbein** on Q; S_x, N_{α^i}, W^a and W^{ax} related to the **gauging**

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These are **spinorial differential equations** $\to$ very difficult to handling

Rewriting SUSY conditions

- Using the geometrical quantities defined by the ζ_i in the timelike case we can recast SUSY conditions in the appealing form

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Geometrical quantities in $10d$

- Type II SUGRA: $(\epsilon_1, \epsilon_2) \to$ **Geometrical quantities** like in $4d$
- **Vectors:**

$$(K_i)^M = \frac{1}{32}\bar{\epsilon}_i\gamma^M\epsilon_i \qquad K \equiv \frac{1}{2}(K_1 + K_2) \qquad \tilde{K} \equiv \frac{1}{2}(K_1 - K_2)$$

- **Main geometrical object:**

$$\Phi \equiv \epsilon_1\bar{\epsilon}_2$$

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- **A source of difficulties:**
  In general ϵ_i **don't** define a complete **vielbein**$\to$ Φ is not sufficient
  $\Rightarrow$ we introduce two **additional** vectors e_i^- satisfying

$$(e_i^-)^2 = 0 \ , \qquad e_i^- \cdot K_i = 1$$

Factorization $10 \to 4 + 6$

- Suppose a metric like

$$ds^2_{10}(x, y) = ds^2_4(x) + ds^2_6(x, y)$$

$\Rightarrow$ **gamma matrices**: $\Gamma^{(10)}_\mu = \gamma^{(4)}_\mu \otimes 1^{(6)} \;, \quad \Gamma^{(10)}_m = \gamma^{(4)}_5 \otimes \gamma^{(6)}_m$

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- Spinorial Ansatz:
  we decompose ten-dimensional spinors as

$$\begin{aligned}\epsilon_1 &= \zeta_1(x)\,\eta^1_+(x, y) + \zeta^1(x)\,\eta^1_-(x, y)\\ \epsilon_2 &= \zeta_2(x)\,\eta^2_\mp(x, y) + \zeta^2(x)\,\eta^2_\pm(x, y)\end{aligned}$$

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This is not the most general Ansatz. One could use more general Ansatz
η^i imply that the internal manifold has $SU(3)\times SU(3)$-structure
$\to$ **usual** in reductions to $N=2$ SUGRA

Geometry in $10 \to 4 + 6$

- **vectors:**

$$K_i \to \frac{1}{2} k_i \Rightarrow K \to \frac{1}{2} k,\ \tilde{K} \to \frac{1}{2} v^3$$

- **Bispinor:**

$$\begin{aligned}\Phi &= 2\mathrm{Re}[\mp\zeta_1\overline{\zeta^2} \wedge \phi_\mp + (\zeta_1\overline{\zeta_2}) \wedge \phi_\pm] \\ &= 2\mathrm{Re}\left[\mp(v + i * v) \wedge \phi_\mp + \Big(\mu\,(1 + i\mathrm{vol}_4) + \omega\Big) \wedge \phi_\pm\right]\end{aligned}$$

$v = v^1 + i\, v^2$ analogue to the $4d$ vectors, $\phi_\pm \equiv \eta^1_+ \eta^{2\dagger}_\pm$

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- **Internal pure spinors:**
  $\phi_\pm$: **polyforms** of even (odd) degree $\to$ six-dimensional pure spinors
  They are central for vacuum solutions
  $\Rightarrow$ $10d$ system as a **generalization** of the vacua system

Fluxes in $10 \to 4+6$ and SUSY conditions

- F decomposes according to the number of **external** indices

$$F = F_0 + F_1 + F_2 + F_3 + F_4$$

- The same decomposition can be applied to H

$$H = H_0 + H_1 + H_2 + H_3$$

- **Assumption**
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SUSY conditions can be rewritten. The resulting system is not as pleasant as in $4d$. But in the **timelike** case the situation is completely different

An example of reduction: Calabi-Yau

- Well-known reductions are when $\mathcal{M}_6$ is a Calabi-Yau three-fold

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- Well-known reductions are when $\mathcal{M}_6$ is a Calabi-Yau three-fold
- **Definition**
  A Calabi-Yau three-fold has 3 features:
  - It is complex: $y^1 \dots y^6$ arranged in $(z^i, z^{\bar{\imath}})$, $i, \bar{\imath} = 1 \dots 3$
  - It is Kähler: non-vanishing components of the metric are $g_{i\bar{\imath}} = \partial_i \partial_{\bar{\imath}} K$
  - It is Ricci-flat $\Rightarrow$ Exists a metric g s.t. $R_{i\bar{\imath}} = 0$

Theorem

We have $J \equiv \frac{1}{2} J_{i\bar{\imath}} dz^i \wedge dz^{\bar{\imath}}$ and $\Omega \equiv \frac{1}{6} \Omega_{i_1 i_2 i_3} dz^{i_1} \wedge dz^{i_2} \wedge dz^{i_3}$ s.t.:

$$dJ = 0 \qquad d\Omega = 0 \qquad J \wedge \Omega = 0 \qquad -\frac{1}{6} J^2 = -\frac{i}{8} \Omega \wedge \bar{\Omega} \neq 0$$

An example of reduction: Calabi-Yau

- To perform the reduction we need an **internal** differential operator
- We take the Laplacian $\nabla^2 \rightarrow$ Harmonic forms

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Harmonic p-forms are in correspondence with the **de Rham cohomology** $\Rightarrow$ p-forms which are d-closed but not d-exact $\to H^p(\mathcal{M}_6) = H^{q,p-q}(\mathcal{M}_6)$

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- **Graphical instrument: Hodge diamond**
  It collects the spaces $H^{p,q}$ in a graphical manner

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
\longrightarrow
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

An example of reduction: Calabi-Yau

- We now develop the **light** spectrum using the example of Type IIA
- Introduce
  ω^i, $i = 1 \ldots h^{1,1}$ **basis** of forms in $H^{1,1}$
  α_A, β^A, $A = 0 \ldots h^{2,1}$ **basis** in $H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$

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- **Splitting of forms:**

$$
\begin{aligned}
B &\equiv B_{0,2} + B_{2,0} \longrightarrow B_{0,2} = b_i \omega^i, \quad d_4 B_{2,0} = *da \\
C_1 &\equiv C_{1,0} = A_{0\mu} dx^\mu \\
C_3 &\equiv C_{0,3} + C_{1,2} = \zeta^A \alpha_A + \tilde{\zeta}_A \beta^A + A_{i\mu} dx^\mu \wedge \omega^i
\end{aligned}
$$

  first (second) index denotes the number of external (internal) indices
- **Deformations of J and Ω:**

$$
\begin{aligned}
&\delta\Omega \in H^{2,1} \to \delta\Omega = v^a \alpha_a + \tilde{v}_a \beta^a \qquad a = 1 \dots h^{2,1} \\
&\delta J \in H^{1,1} \to \delta J = z_i \omega^i
\end{aligned}
$$

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$$v^a, \quad \tilde{v}^a, \quad b_i, \quad z_i, \quad \phi, \quad a, \quad \zeta^A, \quad \tilde{\zeta}_A$$

$\Rightarrow (2h^{1,1}) + 4(h^{2,1} + 1)$ real scalars

- **Vectors:**

$$A_{\mu 0}, \qquad A_{\mu i}$$

$\Rightarrow (h^{1,1} + 1)$ vectors

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We conclude that the $4d$ spectrum reproduce a $4d$, $\mathcal{N} = 2$ SUGRA with $h^{1,1}$ **vector multiplets** and $h^{2,1} + 1$ **hypermultiplets**.
One can show that the reduced theory is an $\mathcal{N} = 2$ ungauged SUGRA.
However, for general manifold $\mathcal{M}_6$, the situation is not clear.