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Holography and the Quark-Gluon Plasma

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Plan

- Quark-Gluon Plasma from heavy ion collisions
- Holography
- Applications

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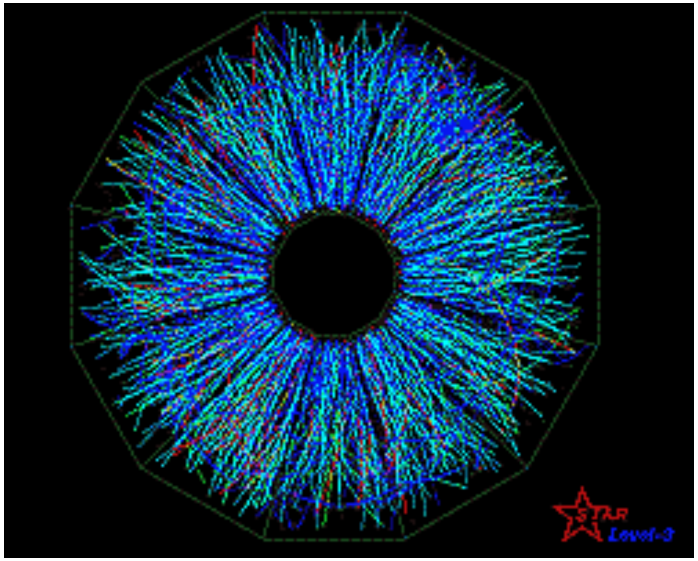
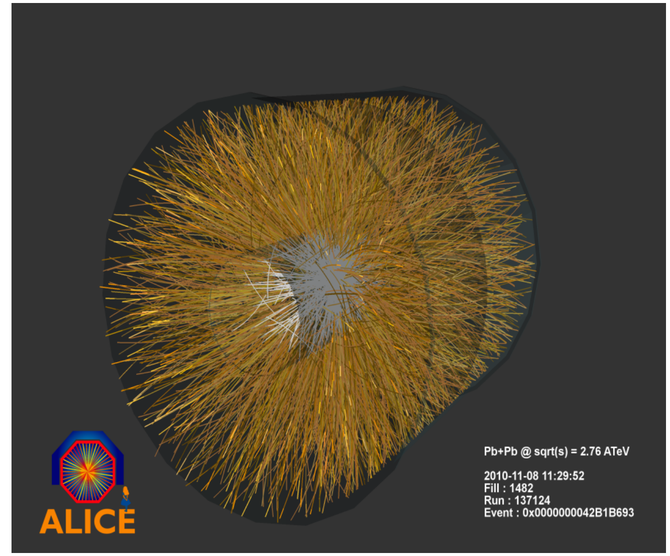
Need novel theoretical tools for real-time properties at strong coupling



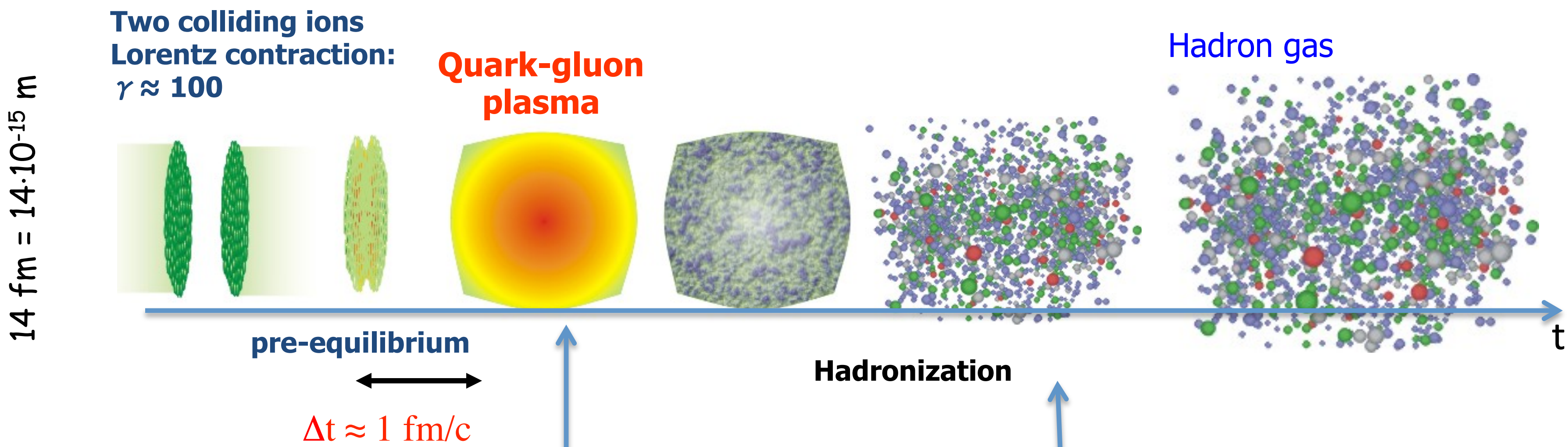


RHIC: Au+Au collisions.
200 GeV/nucleon pair.
Running since 2000.

LHC: Pb+Pb collisions.
3 TeV/nucleon pair.
Running since 2010.

High energies, heavy nuclei: Au = 197 nucleons ; Pb = 208 nucleons
Why? QCD at high energy densities=Universe a few μs old

Main picture



$\varepsilon > 5$ GeV/fm^3 $> \varepsilon_c$
$T \approx 1.5 \div 4\, T_c$
($T_c \approx 155$ MeV)
small μ_B (≈ 10 MeV)

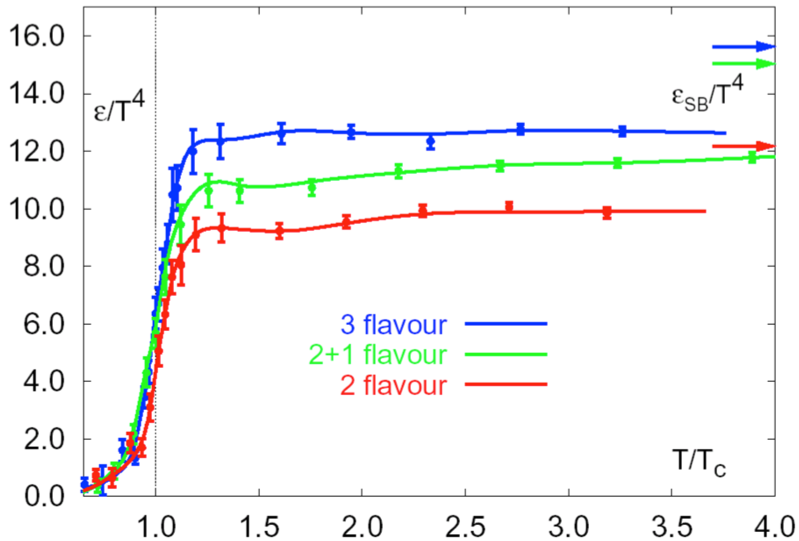
$T < T_c$

Large energy density: deconfinement

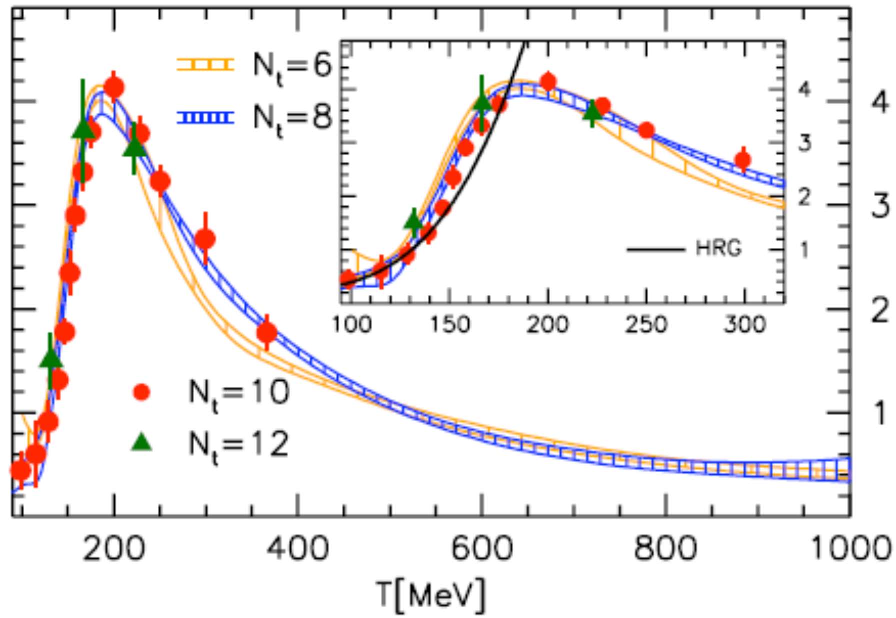
Static properties (thermodynamics)

- Lattice optimized for equilibrium, at small baryon density

Energy density ε/T^4 [F.Karsch, 2002]



Trace anomaly: (ε-3p)/T^4 [Borsanyi et al, 2010; Nf =2+1]



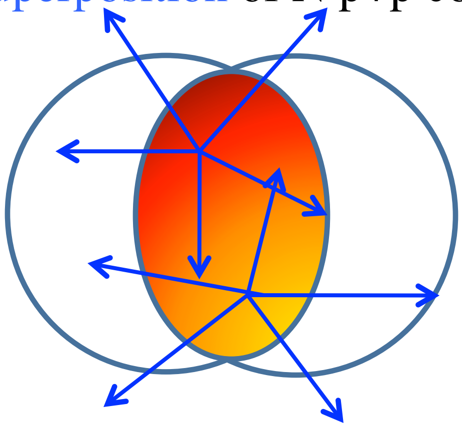
- At T<Tc: hadron gas [O(1) d.o.f.]. At T>Tc: QGP [O(Nc^2+NcNf)]
- At T >Tc , **ε/T^4 is ≈ 80%** of free quark-gluon gas
- **Quasi-conformal trend in a window of T > 2 Tc**

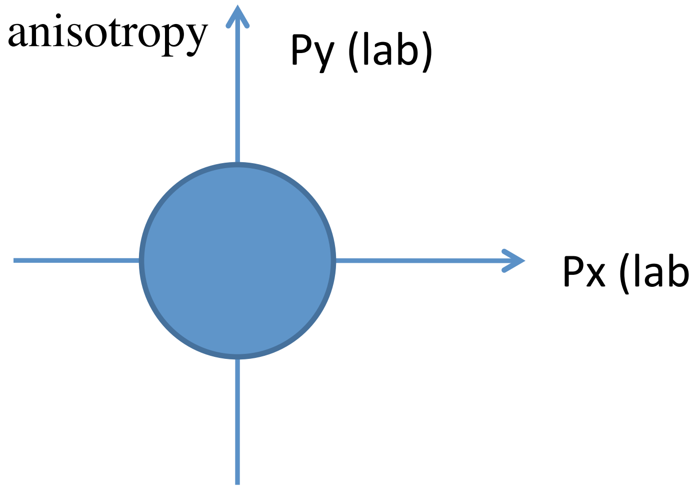
Dynamics: QGP is a strongly coupled medium

Dynamics: elliptic flow

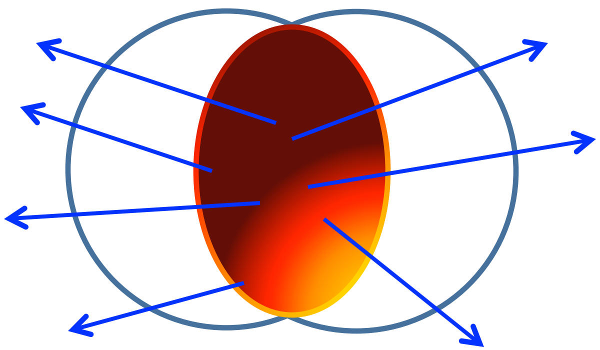
Non central ion-ion collision. Beam direction (z) orthogonal to the screen

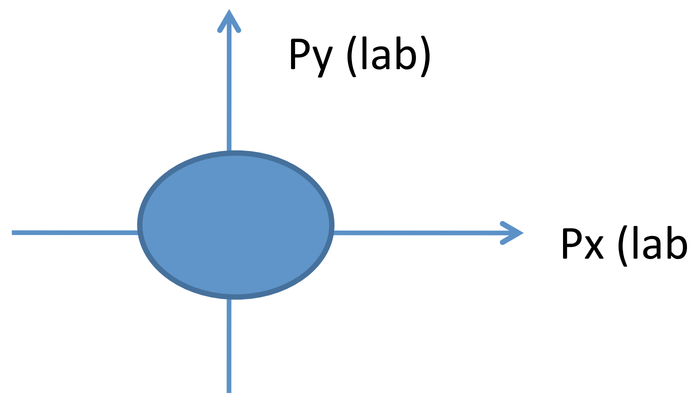
A) Superposition of N p+p collisions: $1/\sqrt{N} \rightarrow 0$ anisotropy





B) If collective mode, pressure gradients





Azimuthal anisotropy is observed: collective behavior.

Dynamics: elliptic flow

- If QGP quickly formed, elliptic flow data fit with…
- … a relativistic hydrodynamic model with…
- … a very low shear viscosity/s (s= entropy density)

$$\frac{\eta}{s} = A\frac{1}{4\pi}\left(\frac{\hbar}{K_B}\right) , \qquad A \in [1,3]$$

Cfr.: η/s (water)≈ (380) (1/4π) ; η/s (liquid He)≈ (9) (1/4π)]

- Similar values for ultracold atoms at unitarity (BEC-BCS crossover)

Recall: Hydrodynamics

- Effective theory for small frequency ($\omega << T$) , long wavelength ($|\mathbf{k}| << T$) fluctuations around local thermal equilibrium
- Expansion of $T_{\mu\nu}$ in derivatives of 4-velocity.
- Characterized by transport coefficients. First order: shear (η) and bulk (ζ) visc

- η/s: measures rate of, say, py momentum diffusion in a transverse direction x.
- I.e. how fast a stream spread out in the ocean?
- Strong (weak) coupling ➔small (large) mean free path➔η/s small (large)

- Hence: QGP is a strongly coupled liquid!

Recall: Hydrodynamics

- To connect with microscopic theory can use linear response

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \, \mathrm{Im}\, G^R_{xy,xy}(\omega, \mathbf{0})$$ Kubo formula

$$G^R_{xy,xy}(\omega, \mathbf{0}) = \int dt\, d\mathbf{x}\, e^{i\omega t} \theta(t) \langle [T_{xy}(t, \mathbf{x}),\, T_{xy}(0, \mathbf{0})] \rangle$$ Retarded correlator

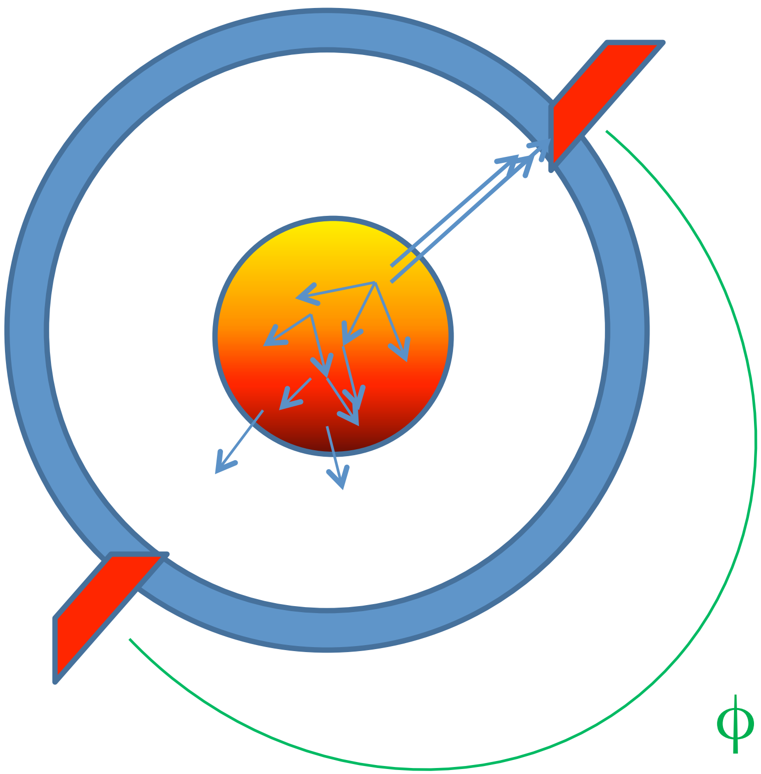
-How to compute in QCD at strong coupling? Do we get small η/s for the QGP?

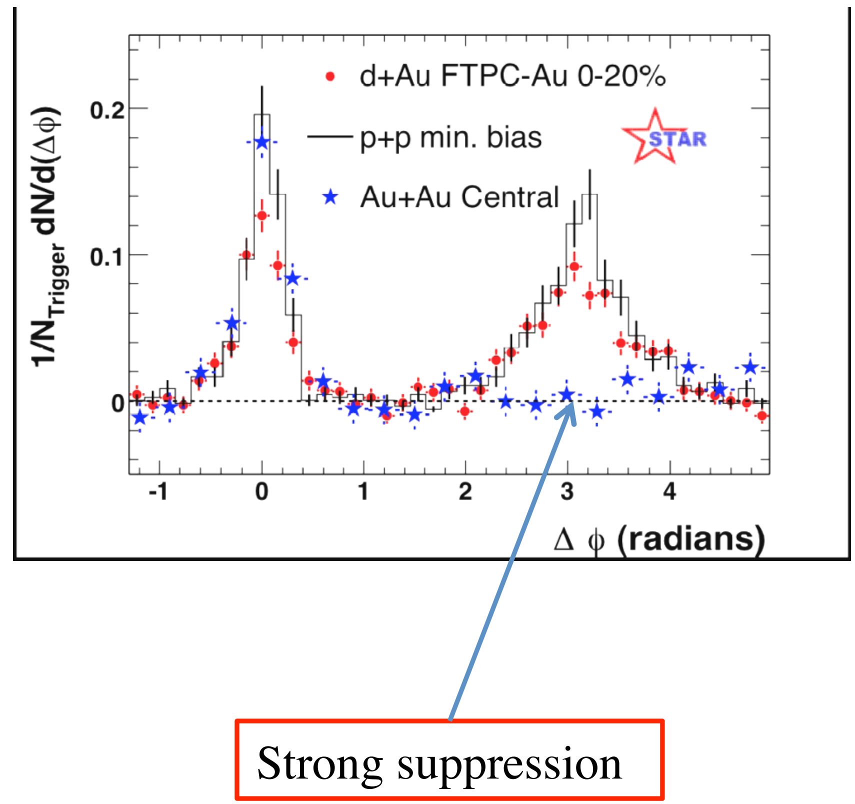
-Lattice (Euclidean) not suited.

-Extrapolation from pQCD gives wrong results by order of magnitude

Dynamics: jet quenching

Strong suppression of back-to-back jets: QGP strongly coupled, highly opaque





Dynamics: jet quenching

- Mainly due (LHC) to energy loss of partons by gluon emission
- Effect of the medium (at scales ~ T) accounted by jet quenching parameter q
- Non perturbative def. : a light-like Wilson loop in the adjoint [Wiedemann]

$$\langle W^A[C]\rangle_T \sim \exp\left[-\frac{\hat{q}}{4\sqrt{2}}L_- L^2\right]$$

- Contour C: a rectangle; L_ along lightcone; L transverse; L_>>L
- Data: $\hat{q}$ in the range $5 - 15\,\mathrm{GeV}^2/\mathrm{fm}$
- Extrapolation from pQCD: $\hat{q} < 1\mathrm{GeV}^2/\mathrm{fm}$
- Lattice not well suited: light-like WL requires Minkowski signature

Plan

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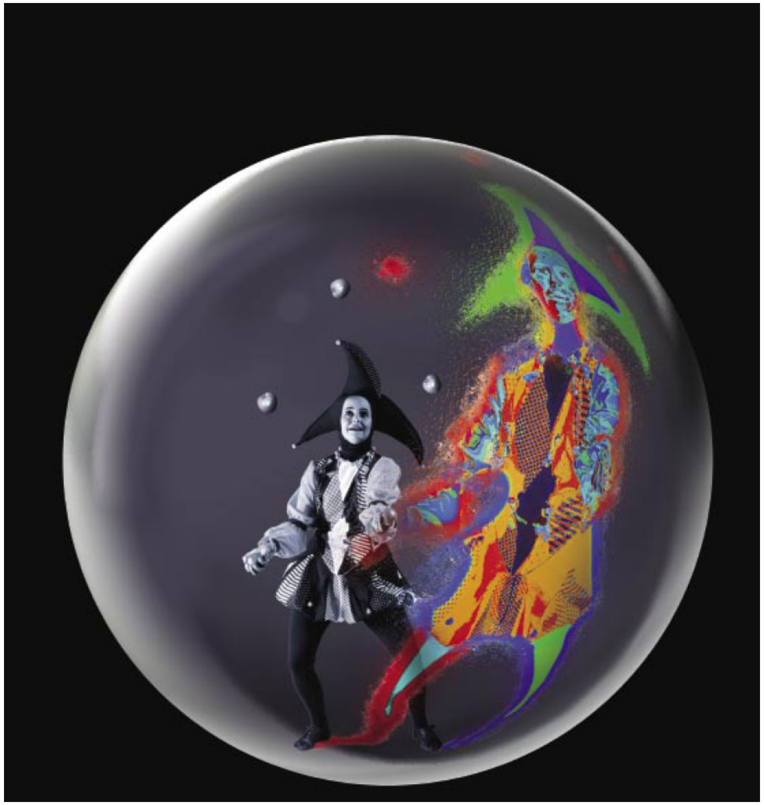
Provides novel tools for strongly coupled QFT, both in and out equilibrium

" Ordinary quantum field theories are secretly quantum theories of gravity in at least one higher dimension"



Heuristic Hint 1

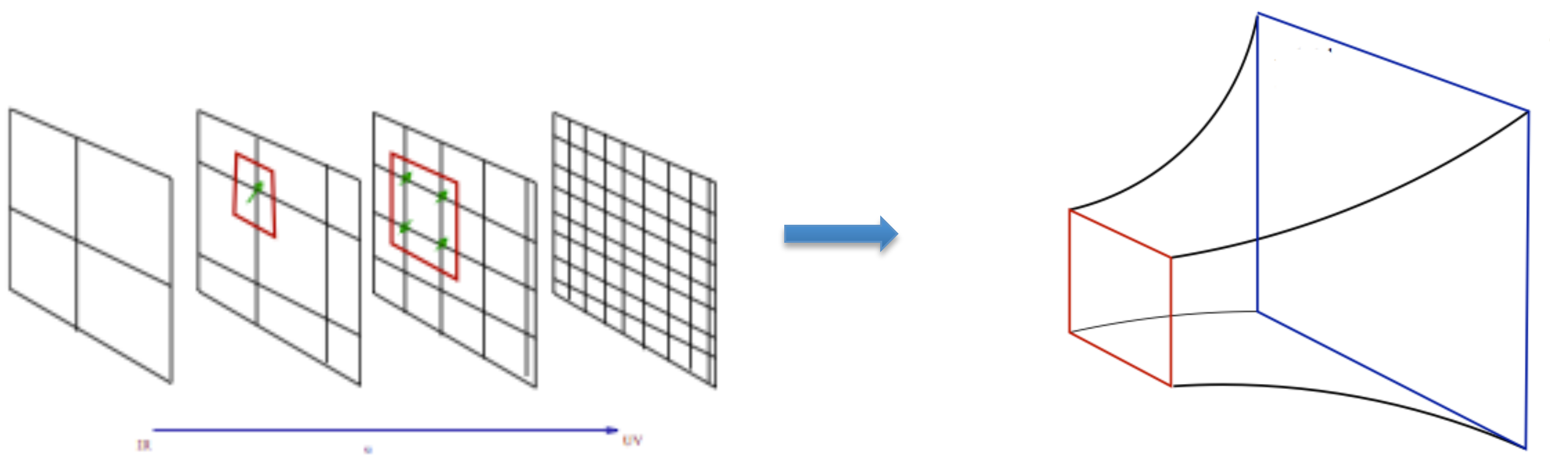
- Renormalization Group equations

$$u\frac{dg}{du} = \beta(g)$$

local in the scale u.

- Idea: RG flow of a D-dim QFT as "foliation" in D+1 dims .

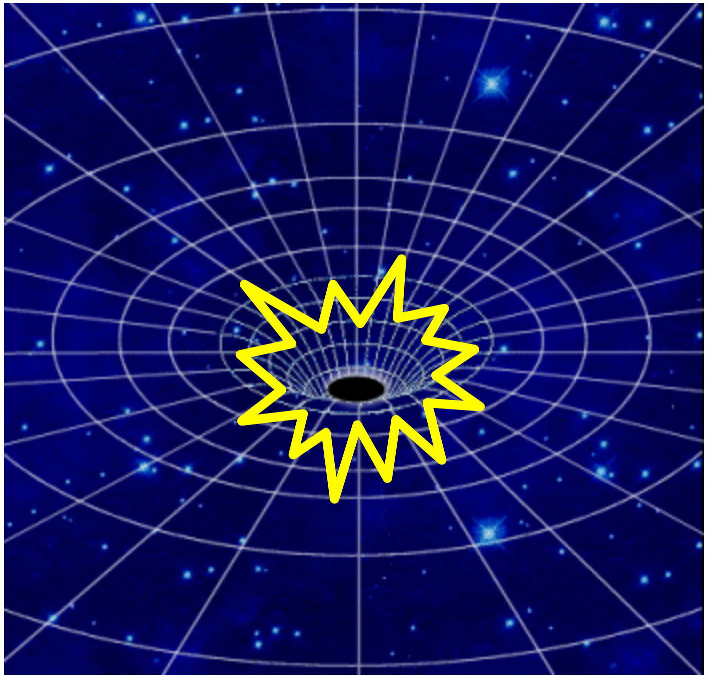
RG scale u = Extra dimension



Heuristic Hint 2

- Effective description in D+1 must have same number of d.o.f. as the QFT in D-dims
- Gravity is a good candidate: it is "holographic"
- See black hole physics

Black holes… are not so black



Quantum effects: emit thermal radiation.

Obey laws of thermodynamics

Entropy scales like the area of the event horizon and not as the enclosed volume!
[Bekenstein, Hawking 1974]

Quantum gravity, whatever it is, is holographic.
Degrees of freedom in a d+1 dimensional spacetime volume encoded by some theory on the d-dimensional boundary.
[' t Hooft, Susskind, 1994]

- But still… (quantum) gravity in D+1 so different from a QFT in D!
- Any possible connection should work in a very subtle way
- In fact…

Certain regimes where the QFT is strongly interacting, mapped into classical (i.e. weakly interacting) gravity!

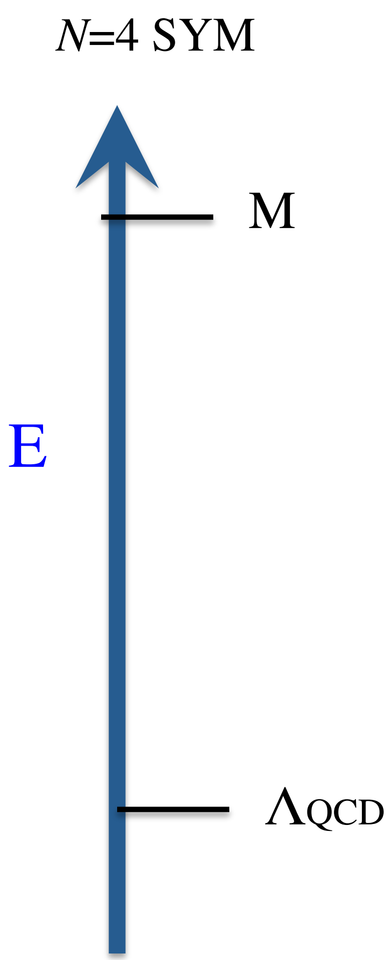
Certain regimes where the QFT is strongly interacting, mapped into classical (i.e. weakly interacting) gravity!

- First explicit example from string theory [Maldacena 1997]:

$\mathcal{N} = 4$ $SU(N_c)$ SYM in $D = 4$ dual to gravity on $AdS_5 \times S^5$.
Classical gravity regime: $N_c \gg 1, \quad \lambda = g_{YM}^2 N_c \gg 1$.

- An enormous amount of validity checks has been provided
- Extended to many other QFTs, including confining ones

Unfortunately holographic QCD is an hard task!



$$\Lambda_{QCD} \sim M \exp\left(-\frac{a}{g_{YM}^2(M)N_c}\right)$$

Decoupling: $g_{YM}^2(M)N_c \ll 1$

Classical gravity: $g_{YM}^2(M)N_c \gg 1$

Moreover: high spin resonances not accounted by just gravity.

Need a complete stringy description. Technically hard.

Therefore

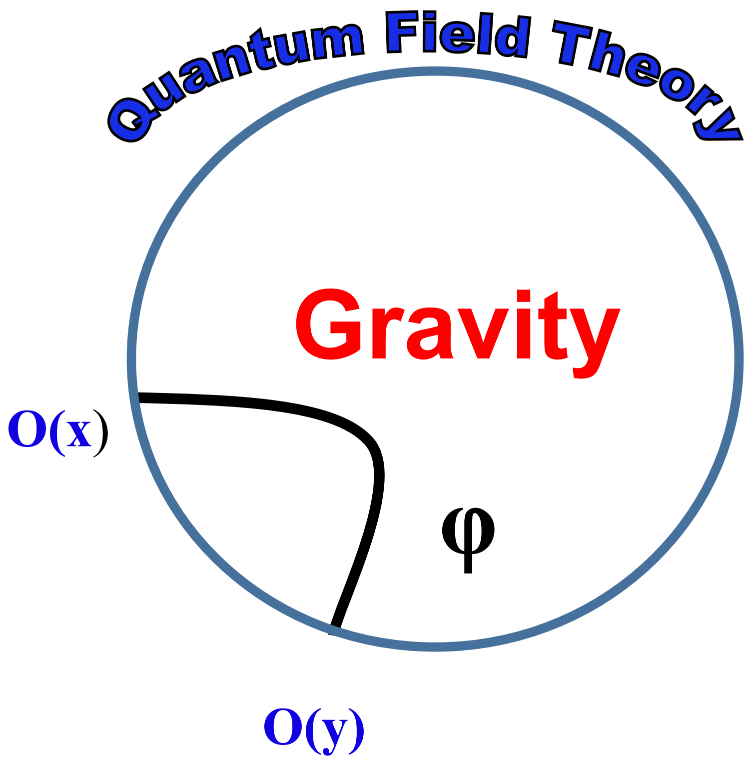
- Price: classical gravity allows us to holographically describe dual QFTs which are not QCD. Toy models.

- However: there are phases of QCD (e.g. at T>Tc) for which holographic models provide good benchmarks

- Key: universality. Some dynamical properties not so tied to microscopic details

- Gain: calculability. Can explore regimes (e.g. finite baryon density, real-time issues) at strong coupling, otherwise hard to access with standard theoretical tools.

How to compute?



RG scale E ➔ radial extra dim. r

QFT vacuum ➔ Gravity background

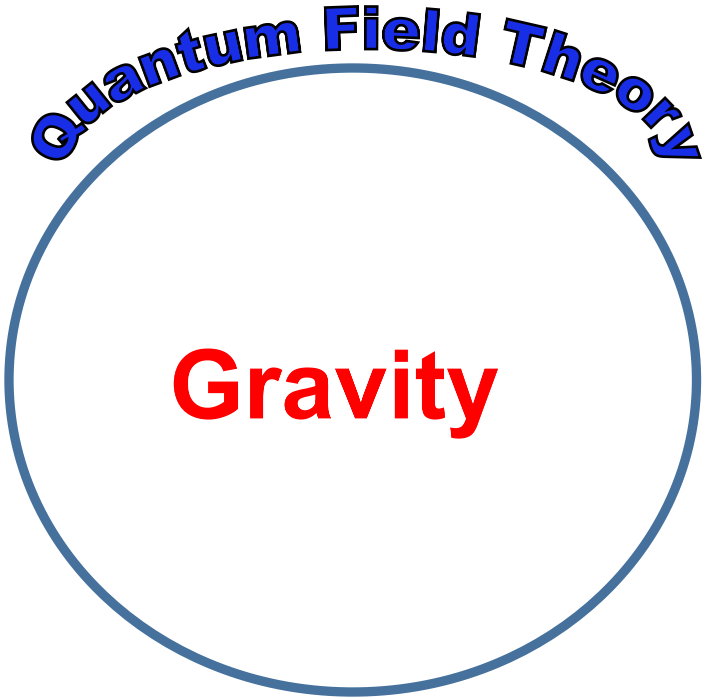
Operator O(x) ➔ Gravity field φ(x,r)

<O(x) O(y) > ➔ On-shell action for φ

Can compute these at strong coupling just from classical gravity equations of motion!

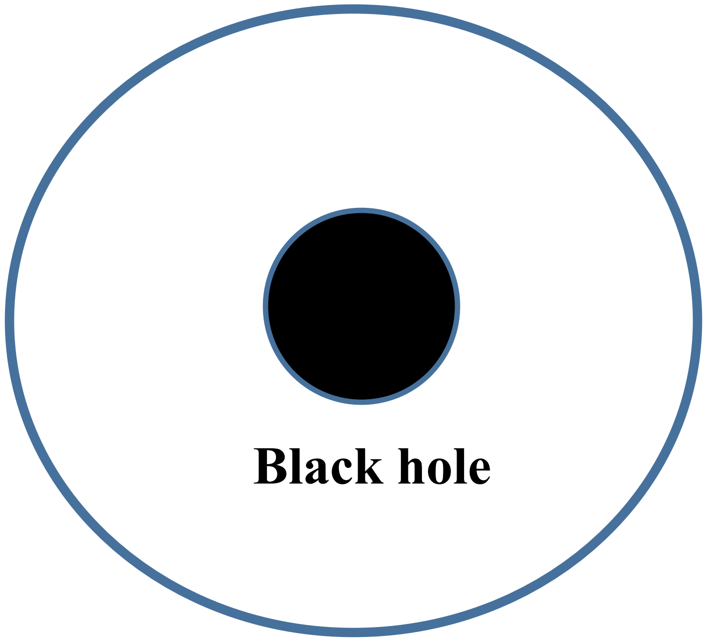
[Witten; Gubser, Klebanov, Polyakov, 1998]

QFT vacuum ➔ Gravity background



QFT vacuum ➔ Gravity background

QFT at finite temperature $Z = Tr e^{-\frac{H}{T}}$

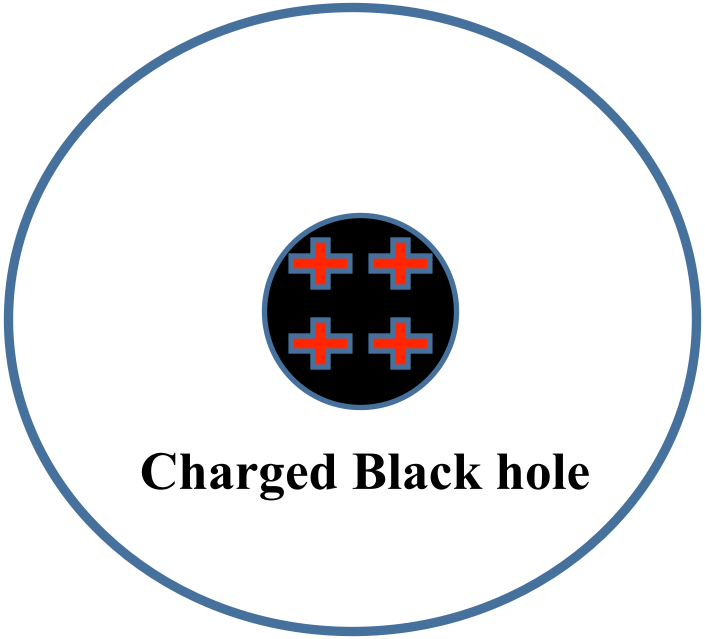


[Witten 98]

Log Z ≈ - S[gravity on shell]

QFT vacuum ➔ Gravity background

QFT at finite temperature and density $Z = Tr e^{-\frac{H-\mu N}{T}}$



QFT density = electric flux on the boundary

Log Z ≈ - S[gravity on shell]

CFT d

$$x_\mu \to \lambda x_\mu$$
$$E \to \lambda^{-1} E$$

⟺

AdS d+1

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{R^2}{r^2} dr^2$$

E ≈ r (AdS radius)

CFT d

$$x_\mu \to \lambda x_\mu$$
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⟺

AdS d+1

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{R^2}{r^2} dr^2$$

E ≈ r (AdS radius)

CFT at finite T

$$Z = Tr e^{-\frac{H}{T}}$$

⟺

AdS black hole

$$ds^2 = \frac{r^2}{R^2}\left[-b[r]dt^2 + dx_i dx_i\right] + \frac{R^2}{r^2}\frac{dr^2}{b[r]}$$

$$b[r] = 1 - \frac{r_h^d}{r^d}$$

$$T_{CFT} = T_{BH} = \frac{r_h\, d}{4\pi R^2};\qquad S_{CFT} = S_{BH} = \frac{A_h}{4G_N} \sim V_{d-1} T^{d-1}$$

CFT at finite T and μ

$$Z = Tr e^{-\frac{H - \mu N}{T}}$$

⟺

Charged RN-AdS black hole

$$A_t \sim \mu - \frac{\rho}{r^{d-2}}$$

Operator O(x) ➔ Gravity field ϕ(x,r)

- Example 1(stress tensor): $T^{\mu\nu}(x) \to g_{\mu\nu}(x,r)$
- Example 2 (conserved current): $J^{\mu}(x) \to A_{\mu}(x,r)$

$$\langle e^{-\int d^d x \phi_0(x)\mathcal{O}(x)} \rangle_{QFT} \approx e^{-S_{gravity}[\phi_0(x)]}$$

$$\lim_{r\to\infty}\phi(x,r) = \phi_0(x)$$ (schematically)

- Solve gravity e.o.m. with that boundary condition
- Plug into the gravity action : $S_{gravity}[\phi_0]$ on-shell.

Boundary value of field = source for corresponding operator

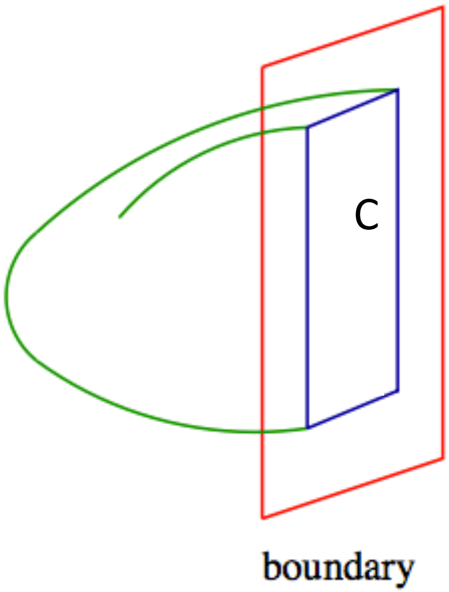
$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \frac{\delta^2 S_{grav}[\phi_0]}{\delta\phi_0(x)\delta\phi_0(y)}|_{\phi_0=0}$$

- Can compute (Euclidean and real-time) correlators at strong coupling!

Wilson loops

$$W_R[C] = Tr_R P e^{i\int_C A_\mu dx^\mu}$$

- R= representation (fundamental, adjoint…);
- C= contour (e.g. a rectangle T, L in Euclidean): W=exp[-T E[L]]
- Holographically:



$$W_F[C] \sim e^{-[\text{Minimal Area Surface with boundary C}]}$$

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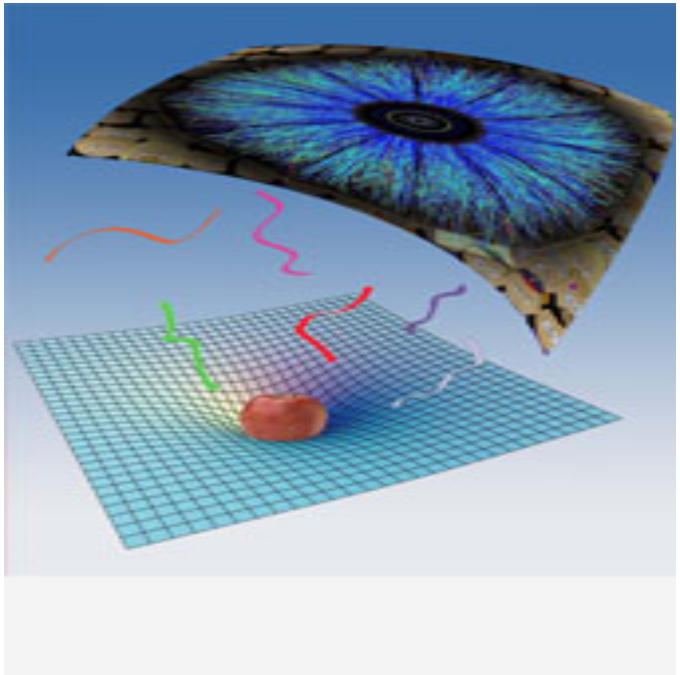

- Strongly coupled thermal QFT ➔ Black Hole in higher dim.
- QFT Thermodynamics ➔Black Hole thermodynamics.
- Hydrodynamics ➔ Fluctuations around black hole background

1. A toy model for a quark+gluon plasma

[F.B. , Cotrone, Mas, Paredes, Ramallo, Tarrio 09 ; F.B. , Cotrone, Mas, Mayerson, Tarrio 10]

- SU(Nc) Yang Mills coupled with massless fields:

- 6 real scalars in the adjoint
- 4 Weyl fermions in the adjoint
- Nf fermions in the (anti)fundamental (quarks)
- Nf scalars in the (anti) fundamental (squarks)

} *N*=4 SYM + Nf hypers

- At finite T and finite quark chemical potential μ
- Parameters:

$$\lambda_h = g^2_{YM}(T) N_c \gg 1, \quad N_c \gg 1 \qquad \epsilon_h = \frac{\lambda_h}{8\pi^2}\frac{N_f}{N_c} \ll 1$$

$$\delta = \frac{4}{\sqrt{\lambda_h}}\frac{\mu}{T}\left(1 - \frac{5}{24}\epsilon_h\right) \ll 1$$

Just to present simpler expression.
Results extended to any δ.
[F.B., Cotrone, Tarrio, to appear]

Consistent Thermodynamics

$$s = \frac{1}{2}\pi^2 N_c^2 T^3 \left[1 + \frac{\epsilon_h}{2}(1+\delta^2) + \frac{7\epsilon_h^2}{24}(1+\delta^2)\right]$$

$$\varepsilon = \frac{3}{8}\pi^2 N_c^2 T^4 \left[1 + \frac{\epsilon_h}{2}(1+2\delta^2) + \frac{\epsilon_h^2}{3}(1+\frac{7}{4}\delta^2)\right]$$

$$p = \frac{1}{8}\pi^2 N_c^2 T^4 \left[1 + \frac{\epsilon_h}{2}(1+2\delta^2) + \frac{\epsilon_h^2}{6}(1+\frac{7}{2}\delta^2)\right]$$

- $\varepsilon(N_f = 0) = 3p(N_f = 0)$ consistently with CFT
- $\varepsilon \approx 0.75\,\varepsilon(\lambda = 0)$ similarly to QGP at $T \in [1.5, 3]T_c$
- Trace anomaly of order εh^2. Massless dynamical flavors break conformality.

Jet quenching parameter

Evaluating light-like Wilson loop holographically (minimal area)
[for N=4 SYM, see: Liu, Rajagopal, Wiedemann 06]

$$\hat{q} = \frac{\pi^{3/2}\sqrt{\lambda_h}\,\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} T^3 \left[1 + \frac{1}{8}(2+\pi)\epsilon_h + 0.56\epsilon_h^2\right]$$

Comparing with Nf=0 theory at fixed T and fixed energy density, get that quarks enhance jet quenching: they have larger cross section than gluons

[F.B. Cotrone, Mas, Paredes, Ramallo, Tarrio 09; Magana, Mas. Mazzanti, Tarrio 12]

Extrapolating to QGP: Nc=Nf=3, λ=6π, T=300 MeV, get

q≈ 4÷5 GeV^2/fm (right in the ballpark of data)

2. Non-conformal plasmas: a simple bottom-up approach

- (Toy) Model QCD in the Quark-Gluon-Plasma phase as ….
- … a strongly coupled large N QFT at finite temperature…
- … with conformality slightly broken by….
- … a marginally relevant operator (as TrF^2)….
- … dual to a scalar field in 5d

Accounted by a simple effective dual gravity model in 5d ($\gamma \ll 1$)

$$S = \frac{1}{16\pi G_N}\int d^5x\sqrt{-\det g}\left[R[g] - \frac{1}{2}(\partial\phi)^2 + \frac{12}{L^2}e^{\gamma\phi}\right]$$

- If $\gamma = 0$ it has an AdS5 vacuum (L=AdS radius) with ϕ=const
- If $\gamma \ll 1$, $\phi \approx -3\gamma \log(r)$ i.e. logarithmic running with RG scale
- Conformality broken. E.g. speed of sound: $c_s^2 = \frac{1}{3} - \frac{\gamma^2}{2}$

Hydrodynamics: the shear viscosity

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt\, d\vec{x}\, e^{i\omega t} \langle [T_{xy}(t,\vec{x}), T_{xy}(0,\vec{0})] \rangle$$

- Compute correlator following basic holographic formula
- Source term for Txy: external metric gxy.

- Essentially compute graviton absorption cross section from the black hole

$$\frac{\eta}{s} = \frac{1}{4\pi} \frac{\hbar}{K_B}$$

[Policastro, Son, Starinets, 2001; Kovtun, Son, Starinets 2004]:

- Universal: for any isotropic QFT plasma with 2der. gravity dual
- Right in the ballpark of estimated QGP value
- (Not a bound. 1/N and 1/λ corrections can lower it.)

Second order hydrodynamics

[Baier,Romatschke,Son,Starinets,Stephanov 2008; Romatschke 2009; Bhattacharyya, Hubeny, Minwalla, Rangamani 2008]

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu} + \pi^{\mu\nu} + \Delta^{\mu\nu}\Pi$$

Expansion in gradients of u^μ and $\varepsilon \longrightarrow s$ (via thermodynamics):

$$\begin{aligned}\pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} + \eta\tau_\pi\left[\langle D\sigma^{\mu\nu}\rangle + \frac{\nabla\cdot u}{3}\sigma^{\mu\nu}\right] + \kappa\left[R^{<\mu\nu>} - 2u_\alpha u_\beta R^{\alpha<\mu\nu>\beta}\right]\\ &+\lambda_1\sigma^{<\mu}_\lambda\sigma^{\nu>\lambda} + \lambda_2\sigma^{<\mu}_\lambda\Omega^{\nu>\lambda} + \lambda_3\Omega^{<\mu}_\lambda\Omega^{\nu>\lambda} + \kappa^* 2u_\alpha u_\beta R^{\alpha<\mu\nu>\beta}\\ &+\eta\tau^*_\pi\frac{\nabla\cdot u}{3}\sigma^{\mu\nu} + \lambda_4\nabla^{<\mu}\log s\nabla^{\nu>}\log s\end{aligned}$$

For conformal fluids: $\left(\tau^*_\pi, \kappa^*, \lambda_4\right) = 0$

$$\begin{aligned}\Pi &= -\zeta(\nabla\cdot u) + \zeta\tau_\Pi D(\nabla\cdot u) + \xi_1\sigma^{\mu\nu}\sigma_{\mu\nu} + \xi_2(\nabla\cdot u)^2 + \xi_3\Omega^{\mu\nu}\Omega_{\mu\nu}\\ &+\xi_4\nabla^\perp_\mu\log s\nabla^\mu_\perp\log s + \xi_5 R + \xi_6 u^\alpha u^\beta R_{\alpha\beta}\end{aligned}$$

For conformal fluids: $\zeta = 0$

η = shear viscosity, ζ = bulk viscosity, τ_π, τ_Π = relaxation times.

Get all the transport coefficients!

[Romatschke 2009; F.B., Cotrone, Tarrio; F.B. , Cotrone, 2010]

$$\delta_{cb} = (1 - 3c_s^2) \ll 1$$

Expected. Universal. [Kovtun, Son, Starinets, 04]

Saturates Buchel's bound [Buchel, 07]

$\frac{\eta}{s}$	$\frac{1}{4\pi}$	$T\tau_\pi$	$\frac{2-\log 2}{2\pi} + \frac{3(16-\pi^2)}{64\pi}\delta_{cb}$	$\frac{T\kappa}{s}$	$\frac{1}{4\pi^2}\left(1 - \frac{3}{4}\delta_{cb}\right)$
$\frac{T\lambda_1}{s}$	$\frac{1}{8\pi^2}\left(1 + \frac{3}{4}\delta_{cb}\right)$	$\frac{T\lambda_2}{s}$	$-\frac{1}{4\pi^2}\left(\log 2 + \frac{3\pi^2}{32}\delta_{cb}\right)$	$\frac{T\lambda_3}{s}$	0
$\frac{T\kappa^*}{s}$	$-\frac{3}{8\pi^2}\delta_{cb}$	$T\tau_\pi^*$	$-\frac{2-\log 2}{2\pi}\delta_{cb}$	$\frac{T\lambda_4}{s}$	0
$\frac{\zeta}{\eta}$	$\frac{2}{3}\delta_{cb}$	$T\tau_\Pi$	$\frac{2-\log 2}{2\pi}$	$\frac{T\xi_1}{s}$	$\frac{1}{24\pi^2}\delta_{cb}$
$\frac{T\xi_2}{s}$	$\frac{2-\log 2}{36\pi^2}\delta_{cb}$	$\frac{T\xi_3}{s}$	0	$\frac{T\xi_4}{s}$	0
$\frac{T\xi_5}{s}$	$\frac{1}{12\pi^2}\delta_{cb}$	$\frac{T\xi_6}{s}$	$\frac{1}{4\pi^2}\delta_{cb}$		

Get all the transport coefficients!

[Romatschke 2009; F.B., Cotrone, Tarrio; F.B. , Cotrone, 2010]

$$\delta_{cb} = (1 - 3c_s^2) \ll 1$$

Shear and bulk relaxation times differ. Difference increases as conformality breaking effects get stronger. Should be taken into account in QCD hydro codes. Difference enhanced near Tc.

$\frac{\eta}{s}$	$\frac{1}{4\pi}$	$T\tau_\pi$	$\frac{2-\log 2}{2\pi} + \frac{3(16-\pi^2)}{64\pi}\delta_{cb}$	$\frac{T\kappa}{s}$	$\frac{1}{4\pi^2}\left(1 - \frac{3}{4}\delta_{cb}\right)$
$\frac{T\lambda_1}{s}$	$\frac{1}{8\pi^2}\left(1 + \frac{3}{4}\delta_{cb}\right)$	$\frac{T\lambda_2}{s}$	$-\frac{1}{4\pi^2}\left(\log 2 + \frac{3\pi^2}{32}\delta_{cb}\right)$	$\frac{T\lambda_3}{s}$	0
$\frac{T\kappa^*}{s}$	$-\frac{3}{8\pi^2}\delta_{cb}$	$T\tau_\pi^*$	$-\frac{2-\log 2}{2\pi}\delta_{cb}$	$\frac{T\lambda_4}{s}$	0
$\frac{\zeta}{\eta}$	$\frac{2}{3}\delta_{cb}$	$T\tau_\Pi$	$\frac{2-\log 2}{2\pi}$	$\frac{T\xi_1}{s}$	$\frac{1}{24\pi^2}\delta_{cb}$
$\frac{T\xi_2}{s}$	$\frac{2-\log 2}{36\pi^2}\delta_{cb}$	$\frac{T\xi_3}{s}$	0	$\frac{T\xi_4}{s}$	0
$\frac{T\xi_5}{s}$	$\frac{1}{12\pi^2}\delta_{cb}$	$\frac{T\xi_6}{s}$	$\frac{1}{4\pi^2}\delta_{cb}$		

Possible benchmarks for initial conditions (large T) on hydro coefficients of the QCD quark-gluon plasma?

- From lattice [Borsany et al 2010] : $c_s^2(T \sim 1.5 T_c) \sim 0.26$ (RHIC)

$\frac{\eta}{s}$	$\frac{1}{4\pi}$	$T\tau_\pi$	0.228	$\frac{T\kappa}{s}$	0.021
$\frac{T\lambda_1}{s}$	0.015	$\frac{T\lambda_2}{s}$	-0.023	$\frac{T\lambda_3}{s}$	0
$\frac{T\kappa^*}{s}$	-0.008	$T\tau_\pi^*$	-0.046	$\frac{T\lambda_4}{s}$	0
$\frac{\zeta}{\eta}$	0.147	$T\tau_\Pi$	0.208	$\frac{T\xi_1}{s}$	0.001
$\frac{T\xi_2}{s}$	0.001	$\frac{T\xi_3}{s}$	0	$\frac{T\xi_4}{s}$	0
$\frac{T\xi_5}{s}$	0.002	$\frac{T\xi_6}{s}$	0.006		

Many other directions

- Meson melting, phase transitions [Mateos, Myers, Thomson et al 07]
- Drag force on heavy quarks [Herzog, Karch, Kovtun, Kozcaz, Yaffe; Gubser 06]
- Photon spectrum (from retarded correlator of Jμ) [Yaffe et al 06]
- Thermalization: BH formation from colliding shock waves in AdS [Gubser et al 08; Chesler, Yaffe 11] or time-dependend backgrounds [Kraps et al 10]
- More realistic models: Sakai-Sugimoto; improved holographic bottom-up approaches [Kiritsis et al 06-12]
- Novel hydro transport coefficients driven by anomalies [Kharzeev, Son, 11]

Summary

- Holography: a novel theoretical framework for strongly coupled QFTs
- A novel set of tools which could complement well established ones especially for problems concerning e.g.

  - Real-time issues (transport properties)
  - Out of equilibrium physics
  - Finite density (cfr. sign problem in lattice)

- Often analytic control on the models. Novel intuitions.
- Still limited to effective toy models.
- Sometimes useful benchmarks on universal behaviors

Thank you