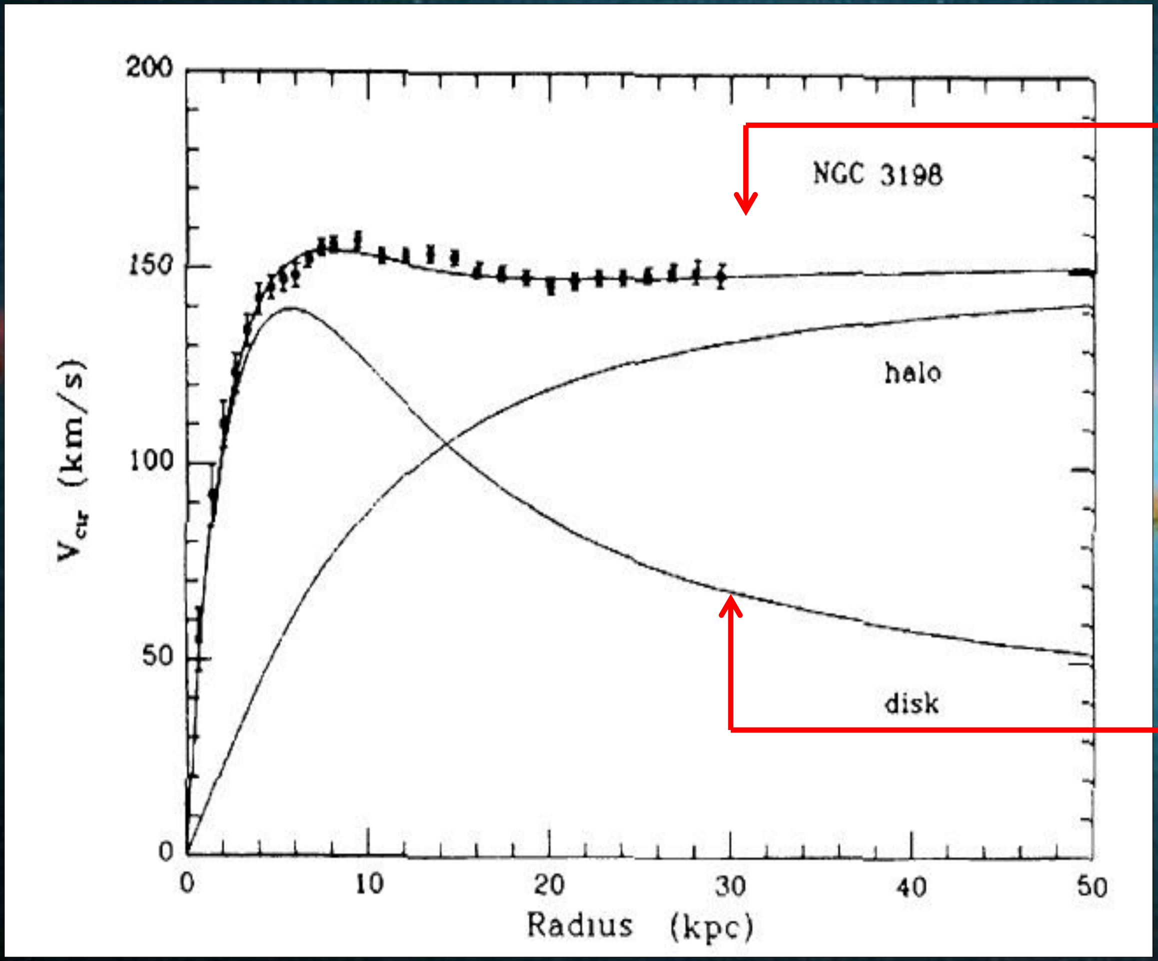
Anisotropic Systems of Fermions: Gravitational Equilibrium and Stability

Giuseppe Alberti

Problem: Rotational Curves

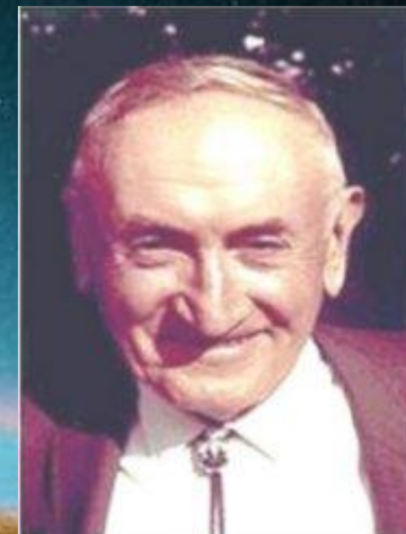


Observational Behavior

Theoretical Prediction

<u>*Rotational Curves: Possible Explanations*</u>

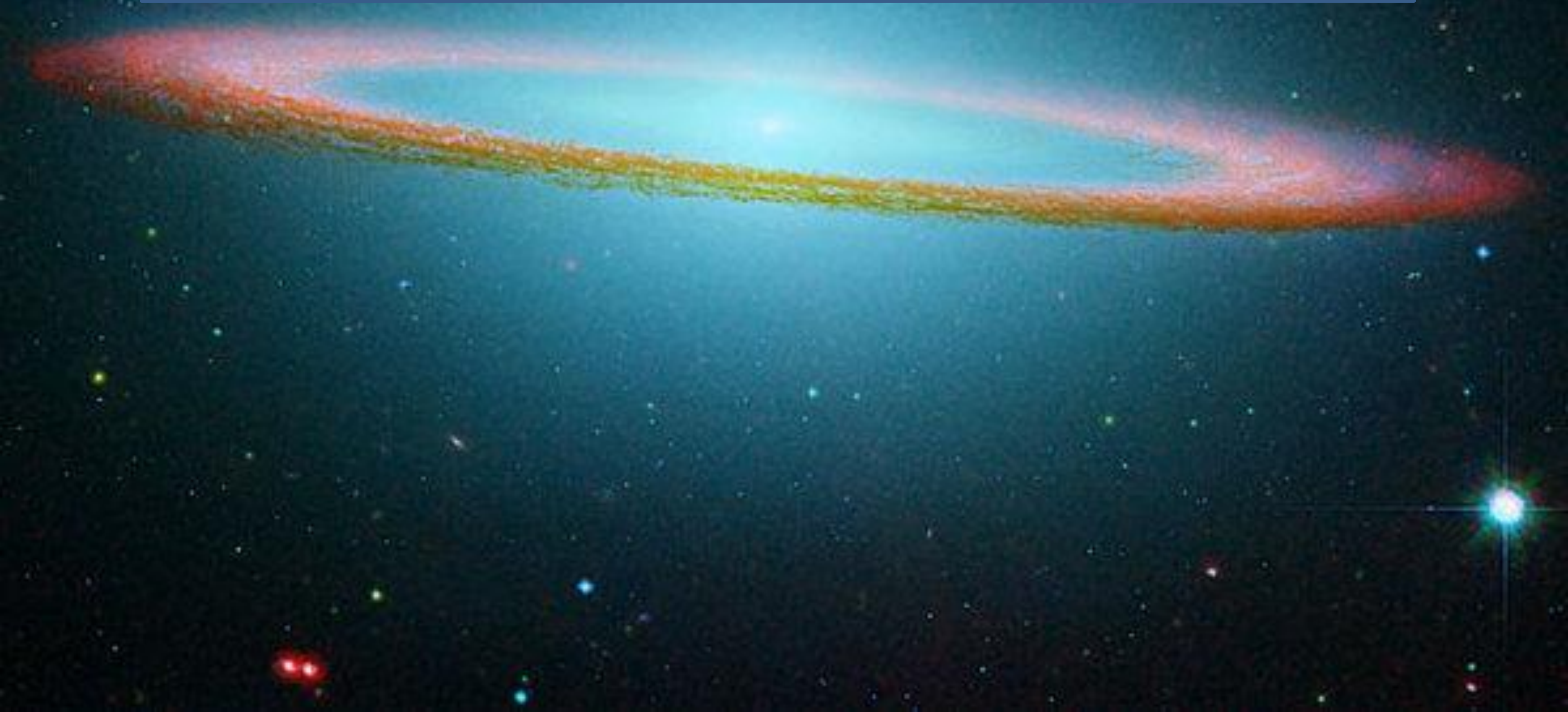
1) "Existence of non visible matter into the halos, acting only gravitationally, Dunkle Materie" (Zwicky 1933, Helvetica Physica Acta 6, 110)



2) "Modification of Newtonian law of gravity to galactic distances" (Milgrom 1983, ApJ 270, 365)

Understanding Dark Matter: Tools

Fundamental Physics: Axions, WIMPs



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N – Body Simulations

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N – Body Simulations

Fermions…?

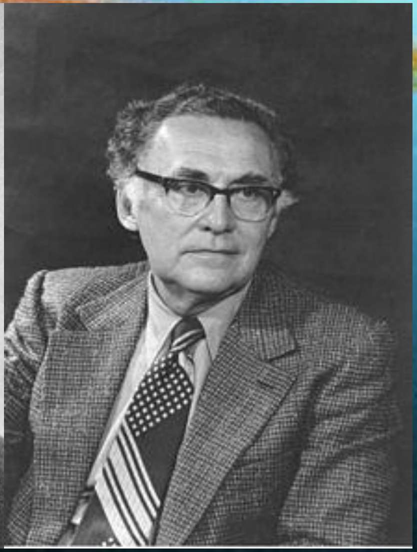
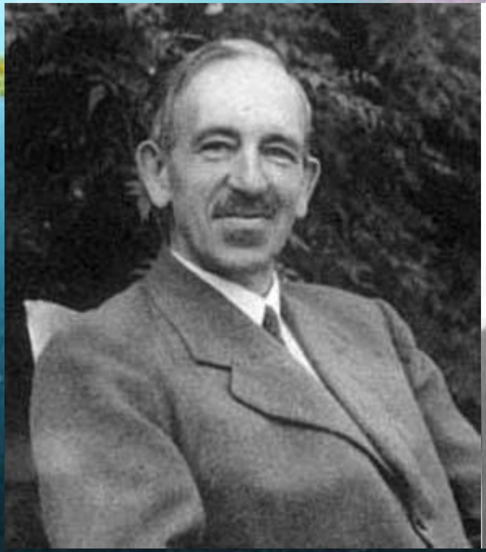
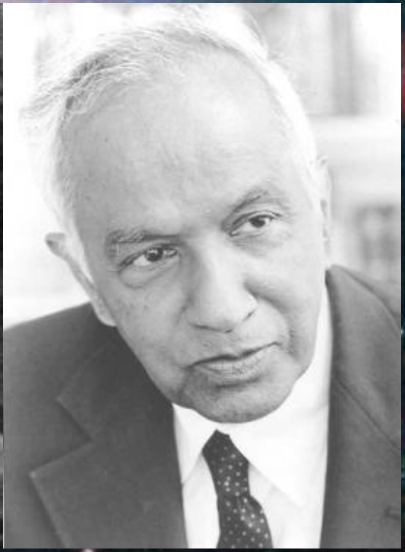
Fermions in Astrophysics

Neutron Stars

Oppenheimer & Volkoff
(1939, Phys. Rev. 55, 374)

White Dwarfs

Fowler (1926, MNRAS 87, 114)
Chandrasekhar (1931, ApJ 74, 81)

Distribution Function

For the calculations: l = 1

$$\begin{cases} f = \dfrac{g}{h^3}\left(1+\dfrac{L^2}{L_c^2}\right)^{l} \dfrac{1-e^{(E-E_c)/k_B T}}{e^{(E-\mu)/k_B T}+1} & E \le E_c \\ f = 0 & E > E_c \end{cases}$$

Anisotropy Parameter: $a = r_a / \xi$

Values Chosen:
1, 0.5, 10^{-1}, 10^{-3}, 10^{-5}

ξ has the dimension of a length and is parameterized by $\xi = (h^3/g\sigma G m^4)^{1/2}$.

Useful Variables

$$x = \frac{\varepsilon}{k_B T} = \frac{p^2}{2mk_B T}$$

Kinetic Energy normalized by $k_B T$

$$\vartheta = \frac{\mu}{k_B T}$$

Degeneracy Parameter

$$W = \frac{\varepsilon_c}{k_B T} = \frac{m\left(\Phi_R - \Phi\right)}{k_B T} = \vartheta - \vartheta_R$$

Energy Cutoff normalized by $k_B T$

$$\sigma^2 = \frac{2k_B T}{m}$$

Velocity Dispersion

$$A_k = \int_0^{\pi} \left(\sin\psi\right)^{2k+1} d\psi = 2\sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{2i+1}$$

Bronstein & Semendyaev (1985, Handbook of Mathematics)

Thermodynamic Quantities

Density of State

$$n = \frac{\rho}{m} = \frac{\pi g m^3 \sigma^3}{h^3} \sum_{k=0}^{l} \binom{l}{k} \left(\frac{r}{r_a}\right)^{2k} A_k \int_0^W x^{k+\frac{1}{2}} \frac{1-e^{x-W}}{e^{x-W-\vartheta_R}+1} dx$$

Radial Pressure

$$P_{rr} = \frac{\pi g m^4 \sigma^5}{h^3} \sum_{k=0}^{l} \binom{l}{k} \left(\frac{r}{r_a}\right)^{2k} \left(A_k - A_{k+1}\right) \int_0^W x^{k+\frac{3}{2}} \frac{1-e^{x-W}}{e^{x-W-\vartheta_R}+1} dx$$

Tangential Pressure

$$P_t = \frac{\pi g m^4 \sigma^5}{2h^3} \sum_{k=0}^{l} \binom{l}{k} \left(\frac{r}{r_a}\right)^{2k} A_{k+1} \int_0^W x^{k+\frac{3}{2}} \frac{1-e^{x-W}}{e^{x-W-\vartheta_R}+1} dx$$

Equations for the Gravitational Equilibrium

$$\begin{cases} \dfrac{dP_{rr}}{dr} = -\dfrac{GM_r\rho}{r^2} - \dfrac{2}{r}(P_{rr} - P_t) \\ \dfrac{dM_r}{dr} = 4\pi r^2 \rho \end{cases}$$

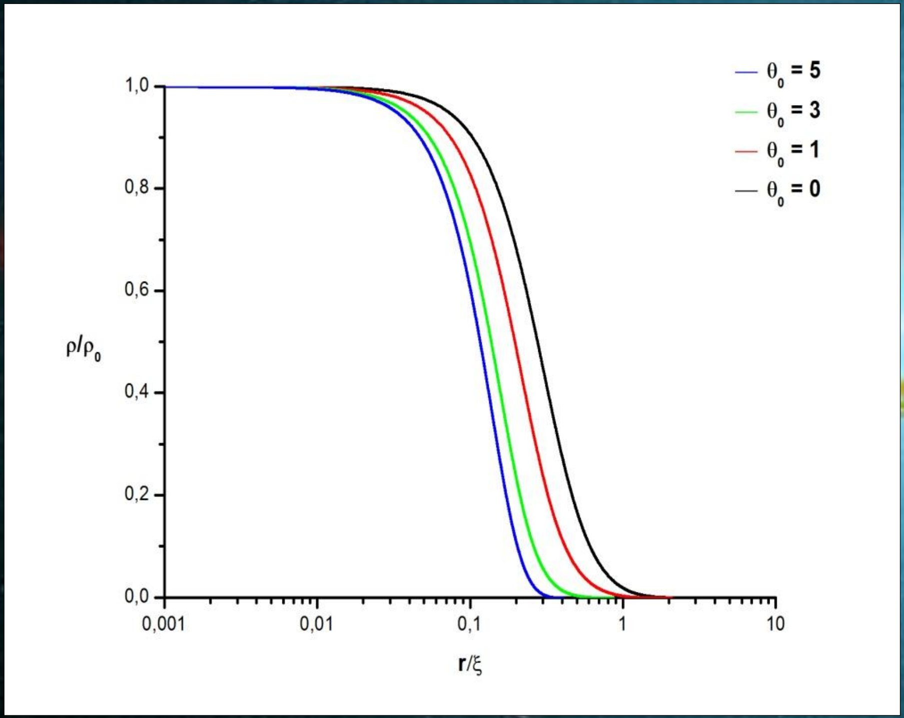
But we prefer to solve the Poisson Equation, written in terms of W instead of Φ

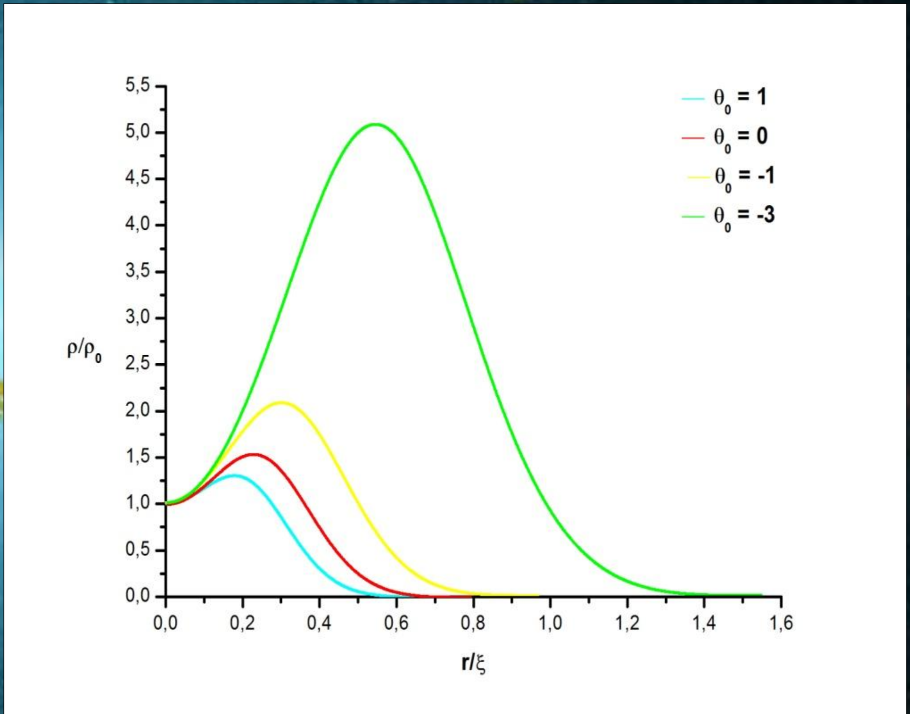
$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G\rho \quad \Rightarrow \quad \frac{d^2W}{dr^2} + \frac{2}{r}\frac{dW}{dr} = -\frac{8\pi G}{\sigma^2}\rho$$

Results of Numerical Integration

a = 0.5, W_0 = 7

a = 0.1, W_0 = 3



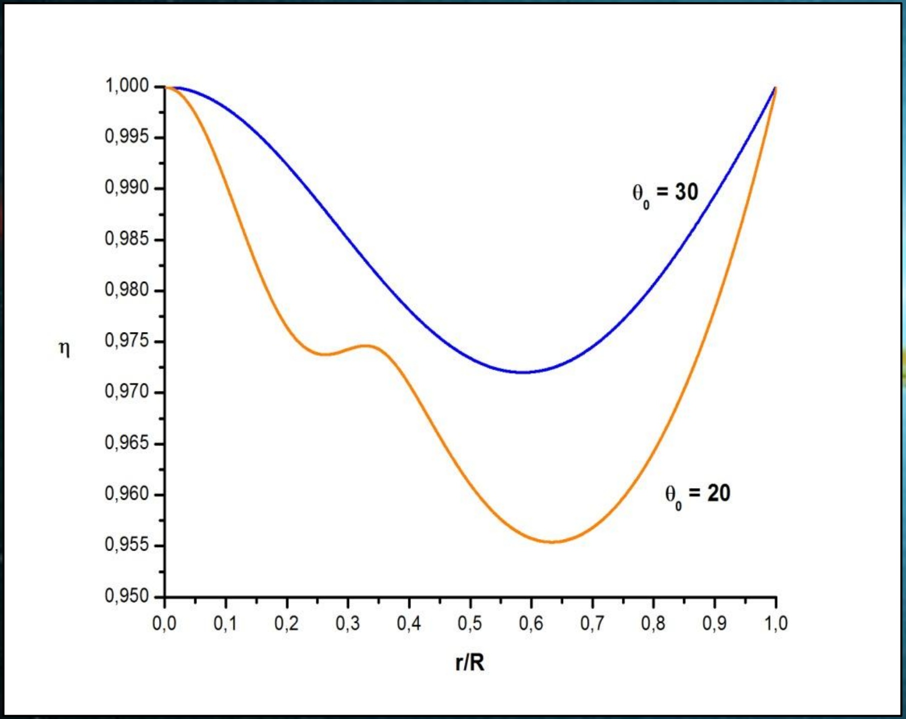


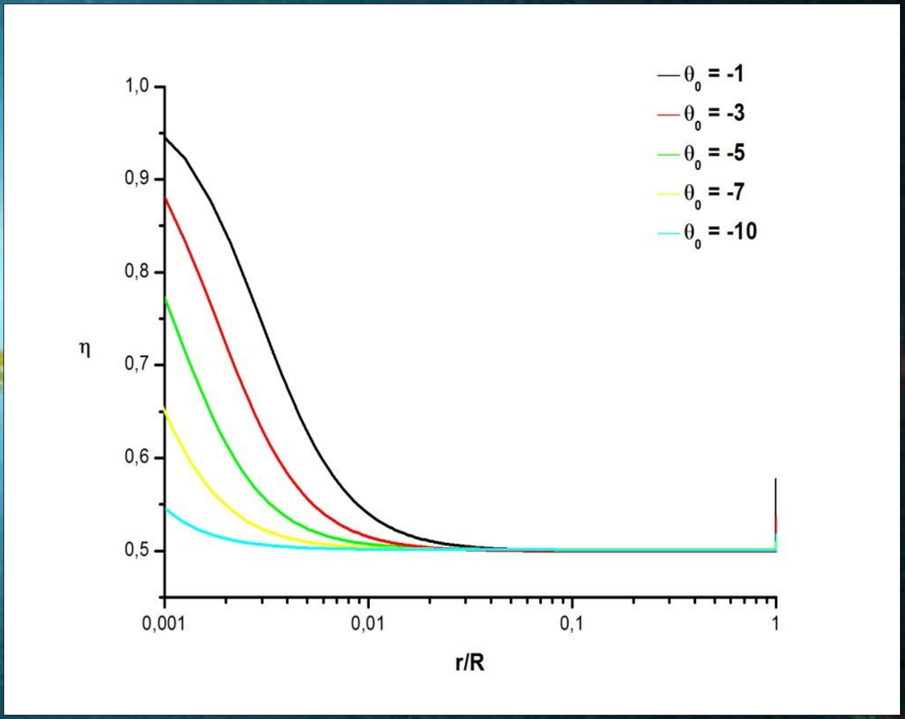
The density profile, in the isotropic limit (left panel), recovers the behavior found by Ruffini & Stella (1983, A & A 119, 35). When the level of anisotropy increases (right panel) we get the behavior of "hollow systems" (Ralston & Smith 1991, ApJ 367, 54; Nguyen & Pedraza 2013, Phys. Rev. D 88, 064020).

Results of Numerical Integration

$a = 1, W_0 = 30$

$a = 10^{-3}, W_0 = 10$





$\eta = P_{rr} / P_t$ (anisotropy level) is a gauge of the kind of orbits. Values of η close to 1 indicate the prevalence of a radial motion, values close to 0.5 a tangential one. See also Bisnovatyi – Kogan et al. (2009, ApJ 703, 628).

Limits on Particles Mass

From the distribution function, in the limit of full degeneracy ($\theta \to W$, $W \to \infty$), it is possible to derive an expression in which the mass is explicitly related to the anisotropy radius r_a .

$$m^4 \geq \frac{3\rho_c h^3}{4\pi g \sigma^3 W_c^{3/2}} \frac{1}{1 + \frac{2}{5} W_c \frac{r_c^2}{r_a^2}}$$

W_c and ρ_c are the values of W and ρ at the core radius r_c .

Limits on Particles Mass (in eV)

σ = 100 km/s
$\rho_c = 10^{-29}$ g/cm^3

W_0	$a = 10^{-5}$	$a = 10^{-3}$	$a = 10^{-1}$	$a = 0.5$	$a = 1$
10	9.58 x 10^{-2}	2.98 x 10^{-1}	9.31 x 10^{-1}	1.12 x 10^{0}	1.13 x 10^{0}
15	7.99 x 10^{-2}	2.53 x 10^{-1}	7.95 x 10^{-1}	9.66 x 10^{-1}	9.78 x 10^{-1}
20	7.21 x 10^{-2}	2.25 x 10^{-1}	7.09 x 10^{-1}	8.68 x 10^{-1}	8.80 x 10^{-1}
30	6.03 x 10^{-2}	1.91 x 10^{-1}	6.01 x 10^{-1}	7.45 x 10^{-1}	7.57 x 10^{-1}
50	4.86 x 10^{-2}	1.54 x 10^{-1}	4.87 x 10^{-1}	6.13 x 10^{-1}	6.26 x 10^{-1}

σ = 100 km/s
$\rho_c = 10^{-26}$ g/cm^3

W_0	$a = 10^{-5}$	$a = 10^{-3}$	$a = 10^{-1}$	$a = 0.5$	$a = 1$
10	5.38 x 10^{-1}	1.68 x 10^{0}	5.23 x 10^{0}	6.30 x 10^{0}	6.36 x 10^{0}
15	4.50 x 10^{-1}	1.42 x 10^{0}	4.47 x 10^{0}	5.41 x 10^{0}	5.51 x 10^{0}
20	4.06 x 10^{-1}	1.26 x 10^{0}	4.00 x 10^{0}	4.88 x 10^{0}	4.94 x 10^{0}
30	3.40 x 10^{-1}	1.07 x 10^{0}	3.37 x 10^{0}	4.19 x 10^{0}	4.25 x 10^{0}
50	2.73 x 10^{-1}	8.67 x 10^{-1}	2.74 x 10^{0}	3.46 x 10^{0}	3.53 x 10^{0}

σ = 100 km/s
$\rho_c = 10^{-23}$ g/cm^3

W_0	$a = 10^{-5}$	$a = 10^{-3}$	$a = 10^{-1}$	$a = 0.5$	$a = 1$
10	3.03 x 10^{0}	9.44 x 10^{0}	2.94 x 10^{1}	3.54 x 10^{1}	3.58 x 10^{1}
15	2.53 x 10^{0}	7.99 x 10^{0}	2.51 x 10^{1}	3.05 x 10^{1}	3.10 x 10^{1}
20	2.28 x 10^{0}	7.10 x 10^{0}	2.25 x 10^{1}	2.75 x 10^{1}	2.78 x 10^{1}
30	1.91 x 10^{0}	6.04 x 10^{0}	1.89 x 10^{1}	2.36 x 10^{1}	2.39 x 10^{1}
50	1.54 x 10^{0}	4.87 x 10^{0}	1.54 x 10^{1}	1.95 x 10^{1}	1.98 x 10^{1}

Limits on Particles Mass (in eV)

σ = 100 km/s
$\rho_c = 10^{-20}$ g/cm^3

W_0	$a = 10^{-5}$	$a = 10^{-3}$	$a = 10^{-1}$	$a = 0.5$	$a = 1$
10	1.70×10^1	5.31×10^1	1.65×10^2	1.99×10^2	2.01×10^2
15	1.42×10^1	4.49×10^1	1.41×10^2	1.71×10^2	1.74×10^2
20	1.28×10^2	3.99×10^1	1.26×10^2	1.54×10^2	1.56×10^2
30	1.08×10^1	3.40×10^1	1.07×10^2	1.32×10^2	1.34×10^2
50	8.63×10^0	2.74×10^1	8.65×10^1	1.09×10^2	1.11×10^2

σ = 100 km/s
$\rho_c = 10^{-17}$ g/cm^3

W_0	$a = 10^{-5}$	$a = 10^{-3}$	$a = 10^{-1}$	$a = 0.5$	$a = 1$
10	9.58×10^1	2.99×10^2	9.30×10^2	1.12×10^3	1.13×10^3
15	8.01×10^1	2.53×10^2	7.95×10^2	9.63×10^2	9.80×10^2
20	7.22×10^1	2.25×10^2	7.11×10^2	8.68×10^2	8.79×10^2
30	6.05×10^1	1.91×10^2	5.99×10^2	7.45×10^2	7.56×10^2
50	4.86×10^1	1.54×10^2	4.87×10^2	6.16×10^2	6.27×10^2

Conclusions and Future Perspectives

We can divide the configurations in three zones:

1. ***An internal region in which the motion is isotropic,***
2. ***An intermediate region in which anisotropy prevails,***
3. ***A region of frontier where the motion is isotropic.***

The behavior of density profiles show the existence of hollow systems by indicating that the presence of the anisotropy is the "source" of this kind of configurations.
Possible application of the model to dSph Galaxies.

Conclusions and Future Perspectives

1. ***Extension of the model to General Relativity;***
   (ALMOST READY)

2. ***Study of the dynamical stability in terms of the adiabatic exponent ($\langle\gamma\rangle \geq \gamma_{cr}$);***

3. ***Study of the thermodynamical stability.***

For more details, see arXiv: 1402.0756