Dispersive treatment of the hadronic light-by-light contribution to $(g-2)_\mu$

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Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light (HLbL)
- Status of $(g-2)_\mu$
- Approaches to the calculation of HLbL

The HLbL tensor: gauge invariance and crossing symmetry

A dispersion relation for HLbL
- Master Formula
- Dispersive calculation
- Pion transition form factor
- Pion box contribution
- Pion rescattering contribution

Outlook and Conclusions

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Outlook and Conclusions

JHEP09(2014)091, arXiv:1506.01386
in collab. with M. Hoferichter, M. Procura and P. Stoffer and
PLB738(2014)6 .... +B. Kubis

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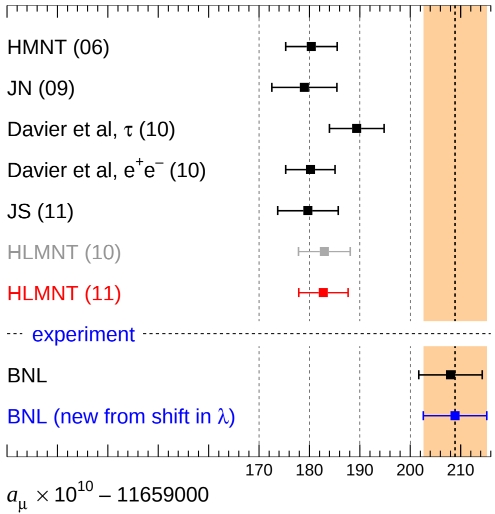
Status of $(g-2)_\mu$, experiment vs SM

Hagiwara et al. 2012

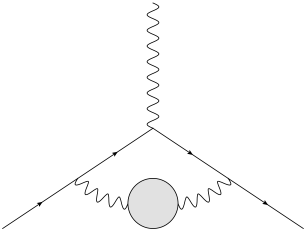


Status of $(g-2)_\mu$, experiment vs SM

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.95	0.04
electroweak, total	153.6	1.0
HVP (LO) [Hagiwara et al. 11]	6 949.	43.
HVP (NLO) [Hagiwara et al. 11]	$-98.$	1.
HLbL [Jegerlehner-Nyffeler 09]	116.	40.
HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]	12.4	0.1
HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]	3.	2.
theory	116 591 855.	59.

Hadronic light-by-light: irreducible uncertainty?

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved



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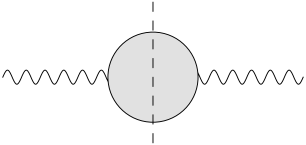
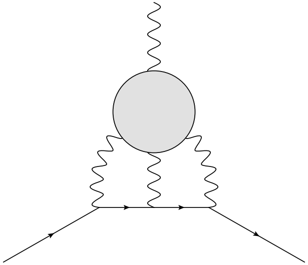


  - basic principles: unitarity and analyticity
  - direct relation to experiment: total hadronic cross section $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
  - dedicated e^+e^- program (BaBar, Belle, BESIII, CMD3, KLOE2, SND)

(but going much below 1% is hard – dealing with radiative corrections poses nontrivial problems)

Hadronic light-by-light: irreducible uncertainty?

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
- Hadronic light-by-light (HLbL) is more problematic:



  - 4-point fct. of em currents in QCD
  - "*it cannot be expressed in terms of measurable quantities*"
  - up to now, only model calculations
  - lattice QCD not yet competitive

→ T. Blum's talk

Different evaluations of HLbL

Jegerlehner Nyffeler 2009

Table 13
Summary of the most recent results for the various contributions to $a_\mu^{\text{LbL;had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	–	–	–	$2.3\pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (Ks are subdominant)
- heavier single-particle poles decreasingly important (unless one models them to resum the high-energy tail)

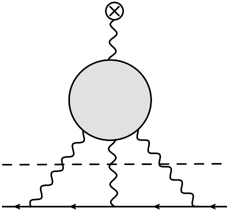
Approaches to Hadronic light-by-light

- Model calculations
  - ENJL — Bijnens, Pallante, Prades (95-96)
  - NJL and hidden gauge — Hayakawa, Kinoshita, Sanda (95-96)
  - nonlocal χQM — Dorokhov, Broniowski (08)
  - AdS/CFT — Cappiello, Cata, D'Ambrosio (10)
  - Dyson-Schwinger — Goecke, Fischer, Williams (11)
  - constituent χQM — Greynat, de Rafael (12)
  - resonances in the narrow-width limit — Pauk, Vanderhaeghen (14)
- Impact of rigorously derived constraints
  - high-energy constraints taken into account in several models above
    addressed specifically by — Knecht, Nyffeler (01)
  - high-energy constraints related to the axial anomaly — Melnikov, Vainshtein (04) and Nyffeler (09)
  - sum rules for $\gamma^*\gamma \to X$ — Pascalutsa, Pauk, Vanderhaeghen (12)
    see also: workshop MesonNet (13)
  - low-energy constraints–pion polarizabilities — Engel, Ramsey-Musolf (13)
- Lattice — Blum et al. (05,12)

Dispersive approach of Pauk-Vanderhaeghen

Pauk, Vandehaeghen, PRD 90 (2014)

- Dispersive representation of the Pauli form factor (the HLbL contribution to it)
- cut through the three photon legs $\Rightarrow$ HLbL tensor with photons on-shell $\Rightarrow$ easier to use experimental input



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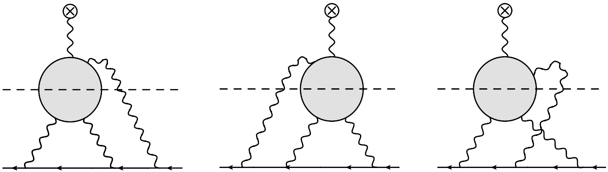

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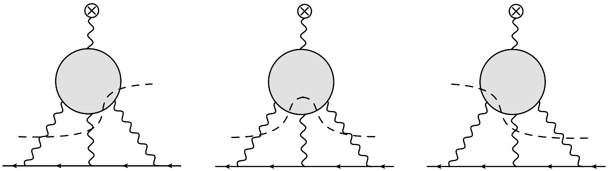

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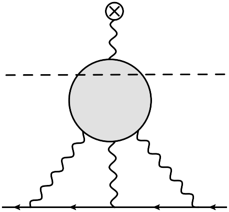

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- other cuts $\Rightarrow$ make the calculation at least as complicated as in other approaches
- for the pseudoscalar pole contribution, the dispersive calculation done this way reproduces the known result

Our approach to hadronic light-by-light

We address the calculation of the hadronic light-by-light tensor

- model independent $\Rightarrow$ rely on dispersion relations (or at least on a dispersive approach/language)
- as data-driven as possible
- takes into account high-energy constraints [OPE, perturbative QCD] (exact implementation not discussed here)

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Some notation

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\left\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \right\} | 0 \rangle$$

where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x) \gamma^\mu q_i(x)$, $i = u, d, s$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

with Mandelstam variables

$$s = (q_1 + q_2)^2 \;\; t = (q_1 + q_3)^2 \;\; u = (q_2 + q_3)^2$$

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General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \dots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \dots\}$, but in $d = 4$ only 136 are linearly independent

Eichmann *et al.* (14)

Some notation

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Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

Detour: the subprocess $\gamma^*\gamma^* \to \pi\pi$

Consider $\gamma^*(q_1,\lambda_1)\gamma^*(q_2,\lambda_2) \to \pi^a(p_1)\pi^b(p_2)$:

$$W_{ab}^{\mu\nu}(p_1,p_2,q_1) = i\int d^4x\, e^{-iq_1\cdot x}\langle\pi^a(p_1)\pi^b(p_2)|T\{j_{\rm em}^\mu(x)j_{\rm em}^\nu(0)\}|0\rangle$$

General tensor decomposition (q_i, $i=1,\ldots,3$, $q_3 = p_2 - p_1$):

$$W^{\mu\nu} = g^{\mu\nu}W_1 + \sum_{i,j} q_i^\mu q_j^\nu W_2^{ij}$$

gives ten independent scalar functions.
Gauge invariance requires:

$$q_1^\mu W_{\mu\nu} = q_2^\nu W_{\mu\nu} = 0$$

Gauge invariance: Bardeen-Tung-Tarrach approach

Consider the projector

Bardeen, Tung (68)

$$I^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}$$

which satisfies

$$I_\mu{}^\lambda W_{\lambda\nu} = W_{\mu\lambda} I^\lambda{}_\nu = W_{\mu\nu}, \qquad q_1^\mu I_{\mu\nu} = q_2^\nu I_{\mu\nu} = 0$$

and contract it twice with $W_{\mu\nu}$, leaving it invariant:

$$W_{\mu\nu} = I_{\mu\mu'} I_{\nu'\nu} W^{\mu'\nu'} = \sum_{i=1}^{5} \bar{T}^i_{\mu\nu} \bar{A}_i = \sum_{i=1}^{5} T^i_{\mu\nu} A_i$$

The $\bar{A}_i$ are free of kinematic singularities, but have zeros. To remove the zeros from the $\bar{A}_i \Rightarrow$ remove the poles from the $\bar{T}_i^{\mu\nu}$

Gauge invariance: Bardeen-Tung-Tarrach approach

$$T_1^{\mu\nu} = q_1 \cdot q_2 g^{\mu\nu} - q_2^\mu q_1^\nu,$$
$$T_2^{\mu\nu} = q_1^2 q_2^2 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_2^\nu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu,$$
$$T_3^{\mu\nu} = q_1^2 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_3^\nu - q_1^2 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_1^\mu q_1^\nu,$$
$$T_4^{\mu\nu} = q_2^2 q_1 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_2^\nu - q_2^2 q_3^\mu q_1^\nu - q_1 \cdot q_3 q_2^\mu q_2^\nu,$$
$$T_5^{\mu\nu} = q_1 \cdot q_3 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_3^\nu - q_1 \cdot q_3 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_3^\mu q_1^\nu,$$

This is a basis of gauge-invariant tensors, but for $q_1 \cdot q_2 = 0$ it becomes degenerate: need one more structure: Tarrach (75)

$$T_6^{\mu\nu} = \left(q_1^2 q_3^\mu - q_1 \cdot q_3 q_1^\mu\right)\left(q_2^2 q_3^\nu - q_2 \cdot q_3 q_2^\nu\right)$$

Back to hadronic light-by-light

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with: GC, Hoferichter, Procura, Stoffer (2015) $\to$ talk by P. Stoffer

- 43 basis tensors (BT)
- 11 additional ones (T)
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$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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$$
\begin{aligned}
T_1^{\mu\nu\lambda\sigma} &= \epsilon^{\mu\nu\alpha\beta}\epsilon^{\lambda\sigma\gamma\delta} q_{1\alpha} q_{2\beta} q_{3\gamma} q_{4\delta},\\
T_4^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_4^\lambda q_3^\sigma - q_3\cdot q_4 g^{\lambda\sigma}\right),\\
T_7^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_1\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_1^\sigma q_1\cdot q_3 - q_1^\lambda q_1^\sigma q_3\cdot q_4\right),\\
T_{19}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_2^\sigma q_1\cdot q_3 - q_1^\lambda q_2^\sigma q_3\cdot q_4\right),\\
T_{31}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2^\lambda q_1\cdot q_3 - q_1^\lambda q_2\cdot q_3\right)\left(q_2^\sigma q_1\cdot q_4 - q_1^\sigma q_2\cdot q_4\right),\\
T_{37}^{\mu\nu\lambda\sigma} &= \left(q_3^\mu q_1\cdot q_4 - q_4^\mu q_1\cdot q_3\right)\left(q_3^\nu q_4^\lambda q_2^\sigma - q_4^\nu q_2^\lambda q_3^\sigma + g^{\lambda\sigma}\left(q_4^\nu q_2\cdot q_3 - q_3^\nu q_2\cdot q_4\right)\right.\\
&\quad \left. + g^{\nu\sigma}\left(q_2^\lambda q_3\cdot q_4 - q_4^\lambda q_2\cdot q_3\right) + g^{\lambda\nu}\left(q_3^\sigma q_2\cdot q_4 - q_2^\sigma q_3\cdot q_4\right)\right),\\
T_{49}^{\mu\nu\lambda\sigma} &= q_3^\sigma\left(q_1\cdot q_3 q_2\cdot q_4 q_4^\mu g^{\lambda\nu} - q_2\cdot q_3 q_1\cdot q_4 q_4^\nu g^{\lambda\mu} + q_4^\mu q_4^\nu\left(q_1^\lambda q_2\cdot q_3 - q_2^\lambda q_1\cdot q_3\right)\right.\\
&\quad \left. + q_1\cdot q_4 q_3^\mu q_4^\nu q_2^\lambda - q_2\cdot q_4 q_4^\mu q_3^\nu q_1^\lambda + q_1\cdot q_4 q_2\cdot q_4\left(q_3^\nu g^{\lambda\mu} - q_3^\mu g^{\lambda\nu}\right)\right)\\
&\quad - q_4^\lambda\left(q_1\cdot q_4 q_2\cdot q_3 q_3^\mu g^{\nu\sigma} - q_2\cdot q_4 q_1\cdot q_3 q_3^\nu g^{\mu\sigma} + q_3^\mu q_3^\nu\left(q_1^\sigma q_2\cdot q_4 - q_2^\sigma q_1\cdot q_4\right)\right.\\
&\quad \left. + q_1\cdot q_3 q_4^\mu q_3^\nu q_2^\sigma - q_2\cdot q_3 q_3^\mu q_4^\nu q_1^\sigma + q_1\cdot q_3 q_2\cdot q_3\left(q_4^\nu g^{\mu\sigma} - q_4^\mu g^{\nu\sigma}\right)\right)\\
&\quad + q_3\cdot q_4\left(\left(q_1^\lambda q_4^\mu - q_1\cdot q_4 g^{\lambda\mu}\right)\left(q_3^\nu q_2^\sigma - q_2\cdot q_3 g^{\nu\sigma}\right) - \left(q_2^\lambda q_4^\nu - q_2\cdot q_4 g^{\lambda\nu}\right)\left(q_3^\mu q_1^\sigma - q_1\cdot q_3 g^{\mu\sigma}\right)\right).
\end{aligned}
$$

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- 43 basis tensors (BT)
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- all remaining 47 can be obtained by crossing transformations of these 7

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

The 54 scalar functions Π_i are free of kinematic singularities and zeros and as such are amenable to a dispersive treatment

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HLbL contribution to a_μ

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2) = -k^\rho \frac{\partial}{\partial k^\sigma} \Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2).$$

Contribution to a_μ: $m := m_\mu$

$$a_\mu = \frac{-1}{48m} \mathrm{Tr}\left\{ (\not{p}+m)[\gamma^\rho,\gamma^\sigma](\not{p}+m)\Gamma^{\mathrm{HLbL}}_{\rho\sigma}(p) \right\}$$

$$\Gamma_{\rho\sigma} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2} \frac{\gamma^\mu (\not{p}+\not{q}_1+m)\gamma^\lambda(\not{p}-\not{q}_2+m)\gamma^\nu}{((p+q_1)^2-m^2)((p-q_2)^2-m^2)} \times \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2)\Big|_{k=0}$$

The BTT method allows us to take the limit $k_\mu \to 0$ explicitly at this point (no kinematic singularities!)

Master Formula

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1,q_2;p)\hat{\Pi}_i(q_1,q_2,-q_1-q_2)}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the Π_i
- 5 integrals can be performed with Gegenbauer polynomial techniques

GC, Hoferichter, Procura, Stoffer (2015)

Master Formula

After performing the 5 integrations:

→ talk by P. Stoffer

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \times \\ \times \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where Q_i^μ are the Wick-rotated[a] four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

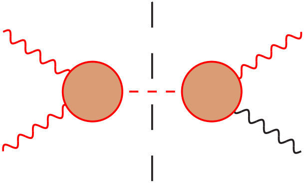
GC, Hoferichter, Procura, Stoffer (2015)

[a] Wick rotation can be performed safely here, even in the presence of anomalous cuts.

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: known

Projection on the BTT basis: done

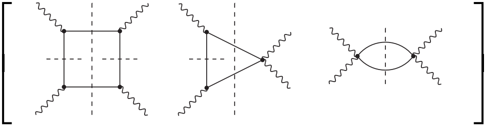
Our master formula=explicit expressions in the literature

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

In JHEP '14:

$$F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2) \times$$ 

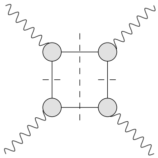
Contribution with two simultaneous cuts
– analytic properties like the box diagram in sQED
– triangle and bulb diagram required by gauge invariance
– multiplication with F_π^V gives the correct q^2 dependence
Claim: FsQED is not an approximation!

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Now, with BTT:



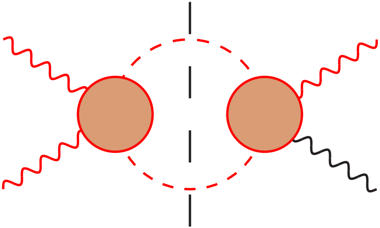
- we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole
- we have explicitly checked that this is identical to FsQED

Proven: FsQED is not an approximation!

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



The "rest" with 2π intermediate states has cuts only in one channel and will be calculated dispersively after partial-wave expansion

Setting up the dispersive calculation

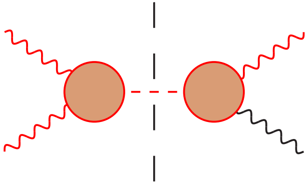
We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Contributions of cuts with anything else other than one and two pions in intermediate states will be neglected for the time being

Dispersive analysis of the pion transition form factor

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



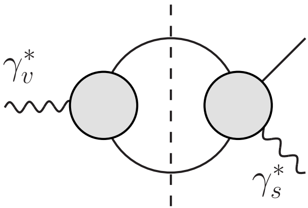
Pion pole: $\Pi_i^{\pi^0\text{-pole}}(s,t,u) = \frac{\rho_{i;s}}{s-M_\pi^2} + \frac{\rho_{i;t}}{t-M_\pi^2} + \frac{\rho_{i;u}}{u-M_\pi^2}$

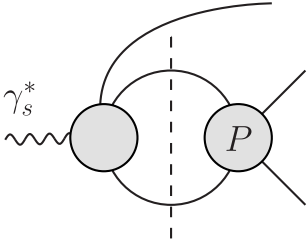
$$\rho_{i,s} = \delta_{i1}\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2),$$
$$\rho_{i,t} = \delta_{i2}\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_3^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,q_4^2),$$
$$\rho_{i,u} = \delta_{i3}\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_4^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,q_3^2),$$

Dispersive analysis of the pion transition form factor

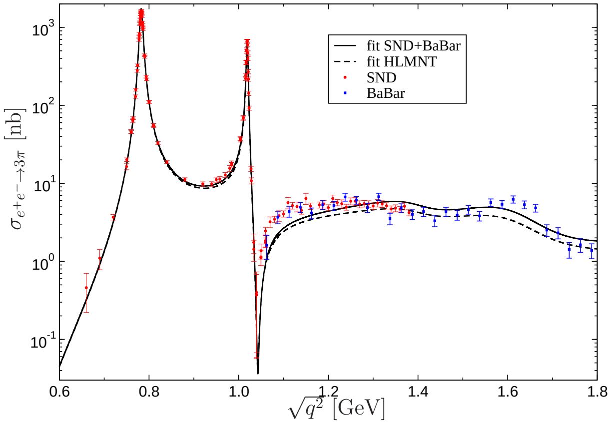
Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

- To calculate the pion-pole contribution the crucial ingredient is the pion transition form factor
- a dispersive representation thereof requires as input:
  - the pion vector form factor [dispersive repr. well known]
  - the $\gamma^* \to 3\pi$ amplitude [analyzed dispersively in this work]
  - the $\pi\pi$ scattering amplitude [dispersive repr. well known]



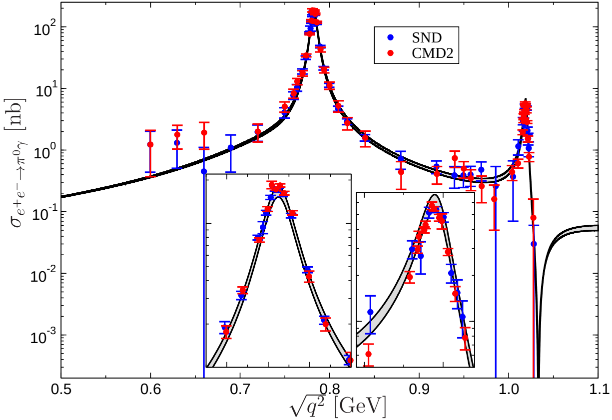


Results for $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$



fit to $\sigma(e^+e^- \to 3\pi)$ Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

Results for $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$



prediction for $\sigma(e^+e^- \to \pi^0\gamma)$ Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

Results for $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$

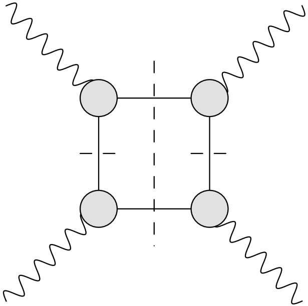
Results for the doubly-virtual pion transition form factor not yet available – data from *e.g.* KLOE on $\phi \to \pi^0 e^+e^-$, or the old, puzzling ones on $\omega \to \pi^0 e^+e^-$ represent useful input

$\to$ talks by B. Ananthanarayan, S. Giovannella

η transition form factor: Hanhart, Kupsc, Meißner, Stollenwerk, Wirzba (2013)

Pion box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



where B_0, C_0 and D_0 are Passarino-Veltman functions.

Pion box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

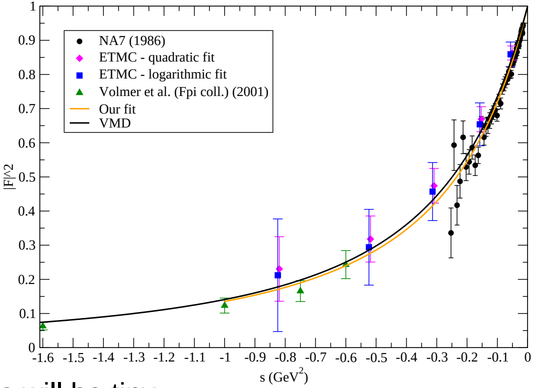
$$\Pi_i^{\text{FsQED}} = F_V^\pi(q_1^2) F_V^\pi(q_2^2) F_V^\pi(q_3^2) \bar{\Pi}_i^{\text{sQED}}(s,t,u)$$

$$\begin{aligned}
\bar{\Pi}_i^{\text{sQED}} = {} & p_i + a_i A_0(M_\pi^2) \\
& + b_i^1 B_0(q_1^2, M_\pi^2, M_\pi^2) + b_i^2 B_0(q_2^2, M_\pi^2, M_\pi^2) + b_i^3 B_0(q_3^2, M_\pi^2, M_\pi^2) + b_i^4 B_0(q_4^2, M_\pi^2, M_\pi^2) \\
& + b_i^s B_0(s, M_\pi^2, M_\pi^2) + b_i^t B_0(t, M_\pi^2, M_\pi^2) + b_i^u B_0(u, M_\pi^2, M_\pi^2) \\
& + c_i^{12} C_0(q_1^2, q_2^2, s, M_\pi^2, M_\pi^2, M_\pi^2) + c_i^{13} C_0(q_1^2, q_3^2, t, M_\pi^2, M_\pi^2, M_\pi^2) + c_i^{14} C_0(q_1^2, q_4^2, u, M_\pi^2, M_\pi^2, M_\pi^2) \\
& + c_i^{34} C_0(q_3^2, q_4^2, s, M_\pi^2, M_\pi^2, M_\pi^2) + c_i^{24} C_0(q_2^2, q_4^2, t, M_\pi^2, M_\pi^2, M_\pi^2) + c_i^{23} C_0(q_2^2, q_3^2, u, M_\pi^2, M_\pi^2, M_\pi^2) \\
& + d_i^{st} D_0(q_1^2, q_2^2, q_4^2, q_3^2, s, t, M_\pi^2, M_\pi^2, M_\pi^2, M_\pi^2) \\
& + d_i^{su} D_0(q_1^2, q_2^2, q_3^2, q_4^2, s, u, M_\pi^2, M_\pi^2, M_\pi^2, M_\pi^2) \\
& + d_i^{tu} D_0(q_1^2, q_3^2, q_2^2, q_4^2, t, u, M_\pi^2, M_\pi^2, M_\pi^2, M_\pi^2),
\end{aligned}$$

where B_0, C_0 and D_0 are Passarino-Veltman functions.

Pion box contribution

where B_0, C_0 and D_0 are Passarino-Veltman functions.



Uncertainties will be tiny

Preliminary! numbers:

$$a_\mu^{\text{FsQED}} = -15.9 \cdot 10^{-11} \qquad a_\mu^{\text{FsQED,VMD}} = -16.4 \cdot 10^{-11}$$

Pion box contribution

where B_0, C_0 and D_0 are Passarino-Veltman functions.

Table 13

Summary of the most recent results for the various contributions to $a_\mu^{\text{LbL;had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	–	–	–	$2.3 \pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

Uncertainties will be tiny

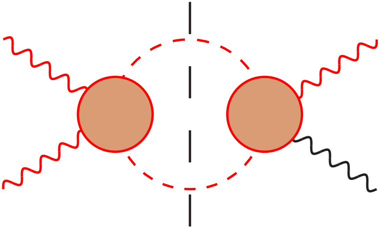
Preliminary! numbers:

$$a_\mu^{\text{FsQED}} = -15.9 \cdot 10^{-11} \qquad a_\mu^{\text{FsQED,VMD}} = -16.4 \cdot 10^{-11}$$

Our dispersive representation of the $\bar{\Pi}^{\mu\nu\lambda\sigma}$ tensor

GC, Hoferichter, Procura, Stoffer (2014)

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Our dispersive representation of the $\bar{\Pi}^{\mu\nu\lambda\sigma}$ tensor

GC, Hoferichter, Procura, Stoffer (2014)

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left(A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u) \right)$$

- the $\Pi_i(s)$ are single-variable functions having only a right-hand cut
- for the discontinuity we keep only the lowest partial wave
- the dispersive integral that gives the $\Pi_i(s)$ in terms of its discontinuity has the required soft-photon zeros
- soft-photon zeros constrain
  the subtraction polynomial to vanish
  (unless one wanted to subtract more, which is unnecessary)

Dispersion relations for the $\Pi_i(s)$

Requiring that the BTT functions be free of singularities determines the kernels, including non-diagonal terms. S-waves:

$$\Pi_1^s = \frac{s - q_3^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s' - q_3^2} \left[K_1 \, \mathrm{Im}\bar{h}^0_{++,++}(s') + \frac{2\xi_1\xi_2}{\lambda'_{12}} \mathrm{Im}\bar{h}^0_{00,++}(s') \right]$$

$$y\Pi_2^s = \frac{s - q_3^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s' - q_3^2} \left[K_1 \, \mathrm{Im}\bar{h}^0_{00,++}(s') + \frac{2q_1^2 q_2^2}{\xi_1\xi_2\lambda'_{12}} \mathrm{Im}\bar{h}^0_{++,++}(s') \right]$$

$$K_1 := \frac{1}{s' - s} - \frac{s' - q_1^2 - q_2^2}{\lambda'_{12}}$$

Remark: $\mathrm{Im}h^0_{++,++}(s)$ and $\mathrm{Im}h^0_{00,++}(s)$ given by S-wave helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$

Once the projection on the BTT basis is done
$\Rightarrow$ use the master formula to calculate the contribution to a_μ

Dispersion relations for the $\Pi_i(s)$

Requiring that the BTT functions be free of singularities determines the kernels, including non-diagonal terms. S-waves:

$$\Pi_1^s = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left[K_1\,\mathrm{Im}\bar{h}^0_{++,++}(s') + \frac{2\xi_1\xi_2}{\lambda'_{12}}\,\mathrm{Im}\bar{h}^0_{00,++}(s')\right]$$

$$y\Pi_2^s = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left[K_1\,\mathrm{Im}\bar{h}^0_{00,++}(s') + \frac{2q_1^2q_2^2}{\xi_1\xi_2\lambda'_{12}}\,\mathrm{Im}\bar{h}^0_{++,++}(s')\right]$$

$$K_1 := \frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda'_{12}}$$

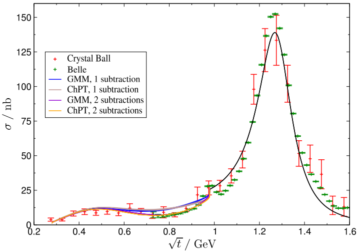
Remark: $\mathrm{Im}h^0_{++,++}(s)$ and $\mathrm{Im}h^0_{00,++}(s)$ given by S-wave helicity amplitudes of $\gamma^*\gamma^*\to\pi\pi$

Extension to D waves is in progress
(diagonal kernels already given explicitly in JHEP (14))

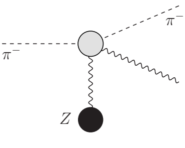
Dispersion relations for $\gamma^*\gamma^* \to \pi\pi$

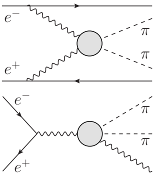
Roy-Steiner eqs. $=$ Dispersion relations $+$ partial-wave expansion $+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- On-shell $\gamma\gamma \to \pi\pi$: prominent D-wave reson. $f_2(1270)$ Moussallam (10) Hoferichter, Phillips, Schat (11)
- $\gamma^*\gamma \to \pi\pi$ Moussallam (13)
- $\gamma^*\gamma^* \to \pi\pi$, new feature: anomalous thresholds Hoferichter, GC, Procura, Stoffer (13)



- Constraints
  - Low energy: pion polar., ChPT
  - Primakoff: $\gamma\pi \to \gamma\pi$ at COMPASS, JLAB
  - Scattering: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - Decays: $\omega, \phi \to \pi\pi\gamma$



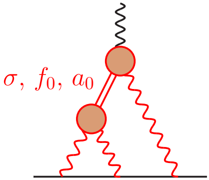


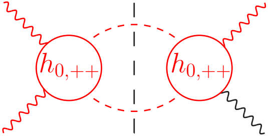
Physics of $\gamma^*\gamma^* \to \pi\pi$

- $\pi\pi$ rescattering $\Leftrightarrow$ resonances, e.g. $f_2(1270)$
- S-wave provides model-independent implementation of the σ
- Analytic continuation with dispersion theory: resonance properties
  - Precise determination of σ-pole from $\pi\pi$ scattering Caprini, GC, Leutwyler 2006

$$M_\sigma = 441^{+16}_{-8}\,\text{MeV} \qquad \Gamma_\sigma = 544^{+18}_{-25}\,\text{MeV}$$

  - Coupling $\sigma \to \gamma\gamma$ from $\gamma\gamma \to \pi\pi$
    Hoferichter, Phillips, Schat 2011





$f_0(500)$ PARTIAL WIDTHS

$\Gamma(\gamma\gamma)$ $\qquad\Gamma_2$

VALUE (keV)	DOCUMENT ID	TECN	COMMENT
• • • We do not use the following data for averages, fits, limits, etc. • • •			
1.7 ± 0.4	[54] HOFERICHTER 11	RVUE	Compilation
3.08 ± 0.82	[55] MENNESSIER 11	RVUE	Compilation
$2.08 \pm 0.2\ ^{+0.07}_{-0.04}$	[56] MOUSSALLAM 11	RVUE	Compilation
2.08	[57] MAO 09	RVUE	Compilation
1.2 ± 0.4	[58] BERNABEU 08	RVUE	

$f_0(500)$ or σ was $f_0(600)$ $\qquad I^G(J^{PC}) = 0^+(0^{++})$

A REVIEW GOES HERE – Check our WWW List of Reviews

$f_0(500)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\,\text{Im}(\sqrt{s_{\text{pole}}})$.

VALUE (MeV)	DOCUMENT ID	TECN	COMMENT
$(400–550)-i(200–350)$ OUR ESTIMATE			
• • • We do not use the following data for averages, fits, limits, etc. • • •			
$(445 \pm 25)-i(278^{+22}_{-18})$	[1,2] GARCIA-MAR..11	RVUE	Compilation
$(457^{+14}_{-13})-i(279^{+11}_{-7})$	[1,3] GARCIA-MAR..11	RVUE	Compilation
$(442^{+5}_{-8})-i(274^{+6}_{-5})$	[4] MOUSSALLAM11	RVUE	Compilation
$(452 \pm 13)-i(259 \pm 16)$	[5] MENNESSIER 10	RVUE	Compilation
$(448 \pm 43)-i(266 \pm 43)$	[6] MENNESSIER 10	RVUE	Compilation

Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light (HLbL)
- Status of $(g-2)_\mu$
- Approaches to the calculation of HLbL

The HLbL tensor: gauge invariance and crossing symmetry

A dispersion relation for HLbL
- Master Formula
- Dispersive calculation
- Pion transition form factor
- Pion box contribution
- Pion rescattering contribution

Outlook and Conclusions

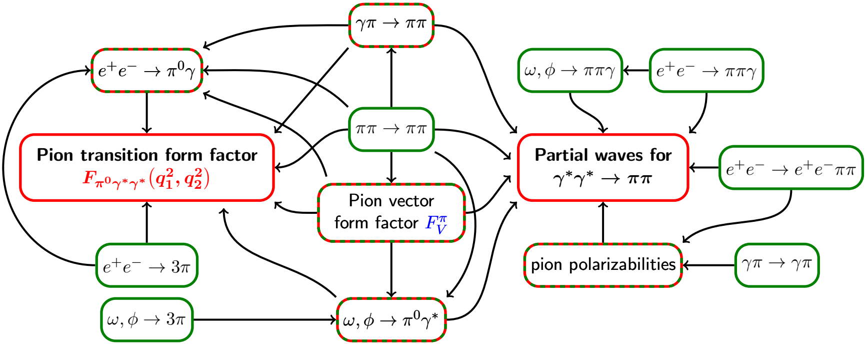
Outlook

Path to a numerical evaluation of HLbL contributions to a_μ:

- take into account experimental constraints on the pion transition form factor to evaluate the pion pole contribution
- using as input a dispersive description of the pion em form factor $\Rightarrow$ evaluate the FsQED contribution
- take into account all experimental constraints on $\gamma^{(*)}\gamma \to \pi\pi$
- estimate the dependence on the q^2 of the second photon (theoretically, there are no data yet on $\gamma^*\gamma^* \to \pi\pi$)
- $\Rightarrow$ solve the dispersion relation for the helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$
- input the outcome into the master formula

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer arXiv:1408.2517 (PLB '14)



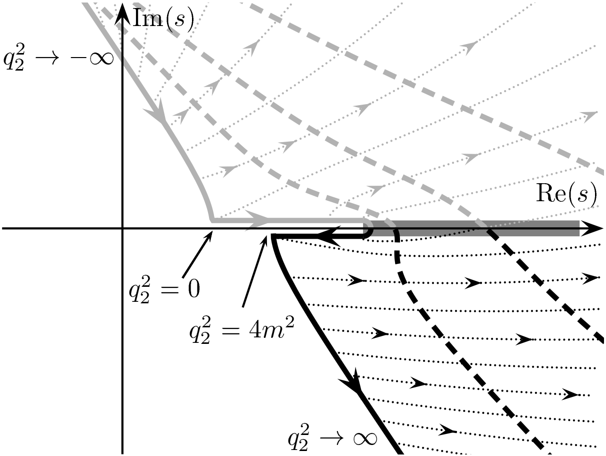
Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among theorists and experimentalists

Conclusions

- I have discussed a dispersive approach to the calculation of the HLbL tensor
- a crucial first step is the derivation of the BTT basis for the HLbL tensor, which I have presented here
- we have derived a master formula which expresses the contributions to a_μ in terms of BTT functions
- we plan to take into account only single- and double-pion intermediate states
  [and all other 1-particle intermediate states (η, $\eta', \ldots$)]
- this is a first step towards a model-independent, data-driven calculation of the HLbL contribution to a_μ

Anomalous cut and Wick rotation



Anomalous cut and Wick rotation