QED corrections to hadronic processes in Lattice QCD

Vittorino Lubicz

ROMA TRE
UNIVERSITÀ DEGLI STUDI

Pisa (Italy)
29 June - 3 July 2015
CD2015

8TH INTERNATIONAL WORKSHOP ON CHIRAL DYNAMICS
QED corrections to hadronic processes in Lattice QCD


Outline of the talk

- Phenomenological motivations
- The method
- (Very) preliminary numerical results (not in the paper)
Motivations

Consider the determination of $V_{us}$ and $V_{ud}$ from leptonic and semileptonic $K$ and $\pi$ decays

$$\frac{\Gamma(K^+ \to \ell^+ \nu_\ell(\gamma))}{\Gamma(\pi^+ \to \ell^+ \nu_\ell(\gamma))} = \left( \frac{|V_{us}| f_K}{|V_{ud}| f_\pi} \right)^2 \frac{m_K \left(1 - \frac{m_\ell^2}{m_K^2}\right)^2}{m_\pi \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2} \left(1 + \delta_{EM} + \delta_{SU(2)}\right)$$

$$\Gamma(K \to \pi \ell \nu(\gamma)) = \frac{G_F^2 m_K^5}{192\pi^3} C_{S_{EW}}^2 \left(\frac{V_{us}}{f^0_{K^+ \pi^-}(0)}\right)^2 I_{K\ell} \left(1 + \delta_{EM}^{K\ell} + \delta_{SU(2)}^{K\pi}\right)^2$$

From the experimental measurements of the decay rates

$$\frac{|V_{us}| f_K}{|V_{ud}| f_\pi} = 0.2758(5)$$

$$|V_{us}| f^0_{K^+ \pi^-}(0) = 0.2163(5)$$

The accuracy is at the level of 0.2% for both determinations

M. Antonelli et al., EPJ C69 (2010) 399
Electromagnetic and isospin breaking effects

An important source of uncertainty are long distance electromagnetic and SU(2) breaking corrections

\[
\frac{\Gamma(K^+ \to \ell^+ \nu_\ell(\gamma))}{\Gamma(\pi^+ \to \ell^+ \nu_\ell(\gamma))} = \left( \frac{|V_{us}| f_K}{|V_{ud}| f_\pi} \right)^2 \frac{m_K (1-m_\ell^2/m_K^2)^2}{m_\pi (1-m_\ell^2/m_\pi^2)^2} \left( 1 + \delta_{EM} + \delta_{SU(2)} \right)
\]

\[
\Gamma(K \to \pi \ell \nu(\gamma)) = \frac{G_F^2 m_K^5}{192 \pi^3} C_{K} S_{EW} \left( |V_{us}| f_+^{K_0^0 \pi^-}(0) \right)^2 I_K \left( 1 + \delta_{EM}^{K\ell} + \delta_{SU(2)}^{K\pi} \right)^2
\]

For \(\Gamma_{K_{l2}}/\Gamma_{\pi_{l2}}\)

At leading order in ChPT both \(\delta_{EM}\) and \(\delta_{SU(2)}\) can be expressed in terms of physical quantities (e.m. pion mass splitting, \(f_K/f_\pi\), ...)

\[
\delta_{EM} = -0.0069 (17)
\]

25% of error due to higher orders \(\rightarrow 0.2\%\) on \(\Gamma_{K_{l2}}/\Gamma_{\pi_{l2}}\)

M.Knecht et al., EPJ C12 (2000) 469; V.Cirigliano, H.Neufeld, PLB 700 (2011) 7

\[
\delta_{SU(2)} = \left( \frac{f_{K^+}/f_\pi^+}{f_K/f_\pi} \right)^2 - 1 = -0.0044 (12)
\]

25% of error due to higher orders \(\rightarrow 0.1\%\) on \(\Gamma_{K_{l2}}/\Gamma_{\pi_{l2}}\)


ChPT is not applicable to D and B decays. Estimates are model dependent.
Lattice results for \( f_K/f_\pi \) and \( f_+(0) \)

**\( f_K/f_\pi \)**

- **FLAG2013**
  - Nf=2+1+1: \( 1.194(5) \), 0.4%
  - Nf=2+1: \( 1.192(5) \)

**\( f_+(0) \)**

- **FLAG2013**
  - Nf=2+1+1: \( 0.970(3) \), 0.3%
  - Nf=2+1: \( 0.966(3) \)
| $|V_{us}| f_+(0)$ | Approx. contrib. to % err from: |
|-----------------|-----------------------------|
|                 | % err | BR  | $\tau$ | $\delta_{SU,EM}$ | Int |
| $K_{Le3}$ 0.2163(5) | 0.26 | 0.09 | 0.20 | 0.11 | 0.05 |
| $K_{L\mu3}$ 0.2166(6) | 0.28 | 0.15 | 0.18 | 0.11 | 0.06 |
| $K_{Se3}$ 0.2155(13) | 0.61 | 0.60 | 0.02 | 0.11 | 0.05 |
| $K^{\pm}e3$ 0.2160(11) | 0.52 | 0.31 | 0.09 | 0.41 | 0.04 |
| $K^{\pm}\mu3$ 0.2158(13) | 0.63 | 0.47 | 0.08 | 0.41 | 0.06 |

Average: $|V_{us}| f_+(0) = 0.2163(5)$ \[\chi^2/\text{ndf} = 0.84/4 \text{ (93\%)}\]
LEADING ISOSPIN BREAKING EFFECTS ON THE LATTICE

These effects are small because:

\[ m_u \neq m_d : \quad O\left(\frac{(m_d - m_u)}{\Lambda_{QCD}}\right) \approx \frac{1}{100} \quad \text{“Strong”} \]

\[ Q_u \neq Q_d : \quad O\left(\alpha_{\text{em}}\right) \approx \frac{1}{100} \quad \text{“Electromagnetic”} \]

The electromagnetic and isospin breaking part of the Lagrangian can be treated as a perturbation.

Expand in:

\[ m_d - m_u + \alpha_{\text{em}} \]

References:

- arXiv:1110.6294
- PHYSICAL REVIEW D 87, 114505 (2013)
  Leading isospin breaking effects on the lattice
  (RM123 Collaboration)
The (md-mu) expansion

G.M.de Divitiis et al., RM123 collaboration, JHEP 04 (2012) 124

Identify the isospin breaking term in the action and expand in \( \Delta m = (m_d - m_u)/2 \)

\[
S_m = \sum_x [m_u \bar{u}u + m_d \bar{d}d] = \sum_x \left[ \frac{1}{2} (m_u + m_d) (\bar{u}u + \bar{d}d) - \frac{1}{2} (m_d - m_u) (\bar{u}u - \bar{d}d) \right] = S_0 - \Delta m \hat{S}
\]

For the kaon decay constant:

\[
C_{K^+K^-} (t) = - \frac{1}{2} \frac{1}{2} + O(\Delta m_{ud})^2
\]

\[\delta_{SU(2)} = -0.0080 \pm (7)\]

which is \( \sim 2.6 \sigma \) larger than

\[\delta_{SU(2)}^{\text{ChPT}} = -0.0044 \pm (12)\]

Lattice - Nf=2
RM123 collab. (2012)

\[\frac{\delta M}{\Delta m} \times 10^3\]
The expansion can be generalized to include the electromagnetic corrections. For the charged - neutral kaon mass splitting:

\[
M_{K^+} - M_{K^0} = (e_u^2 - e_d^2)e_d^2\partial_t - (e_u^2 - e_d^2)e_u^2\partial_t
\]

\[
- 2\Delta m_{ud}(\partial_t - (\Delta m_u^{QCD} - \Delta m_d^{QCD})\partial_t) + (e_u - e_d)e_u^2\sum_f e_f\partial_t
\]

\[
\begin{align*}
\left[ M_{K^+} - M_{K^0} \right]^{QED} &= 2.3(2)(2) \text{ MeV} \quad \text{,} \\
\left[ M_{K^+} - M_{K^0} \right]^{QCD} &= -6.2(2)(2) \text{ MeV}
\end{align*}
\]

\[
(\bar{m}_d - \bar{m}_u) = 2.39(8)(17) \text{ MeV} \quad \quad \quad \quad \quad \bar{m}_u / \bar{m}_d = 0.50(2)(3)
\]

This is an alternative approach to full QCD+QED lattice simulations.

See talk by A. Portelli
QED corrections to hadronic decay rates

In collaboration with:

N. Carrasco, G. Martinelli, C. T. Sachrajda,
N. Tantalo, C. Tarantino, M. Testa

The strategy

- To be specific, we consider the leptonic decay of a charged pion, but the method is general (it can be extended for example to semileptonic decays).

The rate at $O(\alpha^0)$ is:

$$\Gamma_0^\text{tree} (\pi^+ \to \ell^+ \nu_\ell) = \frac{G_F |V_{ud}|^2 f_\pi^2}{8\pi} m_\pi m_\ell^2 \left( 1 - \frac{m_\ell^2}{m_\pi^2} \right)^2$$

In the absence of electromagnetism, the nonperturbative QCD effects are contained in a single number, the decay constant:

$$\langle 0 | \bar{d} \gamma_\mu (1 - \gamma_5) u | \pi^+ (p) \rangle = ip_\mu f_\pi$$

In the presence of electromagnetism, because of the contributions of diagrams like this one, it is not even possible to give a physical definition of $f_\pi$.

For a discussion on this point based on ChPT see J. Gasser and G.R.S. Zarnauskas, PLB 693 (2010) 122.
At $O(\alpha)$, the rate $\Gamma_0$ contains infrared divergences. One has to consider:

\[ \Gamma(\Delta E) = \Gamma(\pi^+ \rightarrow \ell^+ \nu_\ell) + \Gamma(\pi^+ \rightarrow \ell^+ \nu_\ell \gamma(\Delta E)) \equiv \Gamma_0 + \Gamma_1(\Delta E) \]

with $0 \leq E_\gamma \leq \Delta E$. The sum is infrared finite

F. Bloch and A. Nordsieck, PR 52 (1937) 54

In principle, both $\Gamma_0$ and $\Gamma_1(\Delta E)$ can be evaluated in lattice simulations. But $\Gamma_1(\Delta E)$ is very challenging, due to discretized photon momenta: the integral up to $\Delta E$ replaced by a finite sum, many correlation functions...

We thus propose a different strategy
The strategy

- We propose to consider sufficiently soft photons such that they do not resolve the internal structure of the pion. Then the pointlike approximation can be used to compute $\Gamma_1(\Delta E)$ in perturbation theory.

$$\Delta E \ll \Lambda_{\text{QCD}}$$

A cut-off $\Delta E \sim O(20 \text{ MeV})$ appears to be appropriate, both experimentally and theoretically.

The strategy

The size of the neglected structure-dependent contributions can be estimated, as a function of $\Delta E$, using chiral perturbation theory


$$R^A_1(\Delta E) = \frac{\Gamma^A_1(\Delta E)}{\Gamma^0_0 + \Gamma^pt_1(\Delta E)} \quad , \quad A = \{SD, INT\}$$

$\Delta E \sim O(20 \text{ MeV})$

$\pi \to e\nu(\gamma)$

$K \to e\nu(\gamma)$

$F_V = \frac{m_p}{4\pi^2f_\pi} \quad , \quad F_A = \frac{8m_p}{f_\pi}(L'_9 + L'_{10})$
The strategy

\[ \Gamma(\Delta E) = \Gamma_0 + \Gamma_1^{\text{pt}}(\Delta E) \quad \Delta E \sim O(20 \text{ MeV}) \]

\[ \Gamma(\pi^+ \to \ell^+ \nu_\ell) \]

Monte Carlo simulation
Lattice QCD

\[ \Gamma(\pi^+ \to \ell^+ \nu_\ell \gamma(\Delta E)) \]

Perturbation theory
pointlike pion

In order to ensure the cancellation of IR divergences with good numerical precision, an intermediate step is required. We then rewrite:

\[ \Gamma(\Delta E) = \lim_{V \to \infty} (\Gamma_0 - \Gamma_0^{\text{pt}}) + \lim_{V \to \infty} (\Gamma_0^{\text{pt}} + \Gamma_1^{\text{pt}}(\Delta E)) \]

\[ \Gamma_0^{\text{pt}} = \Gamma(\pi^+ \to \ell^+ \nu_\ell)^{\text{pt}} \]

is an unphysical quantity
The strategy

\[ \Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0 - \Gamma_0^{pt} \right) + \lim_{V \to \infty} \left( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \right) \]

- The second term is calculated in perturbation theory directly in infinite volume. The sum \( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \) is IR finite.

- \( \Gamma_0 - \Gamma_0^{pt} \) in the first term is calculated, in the intermediate step, in the finite volume. The contributions from small virtual photon momenta to \( \Gamma_0 \) and \( \Gamma_0^{pt} \) are the same, and the first term is IR finite.

- IR divergences cancel separately in each of the two terms, and so we can calculate each of these terms separately. We also use different IR regulators: the finite volume for the first term and a photon mass for the second term.

- The two terms are also separately gauge invariant.
\[
\Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0(L) - \Gamma_0^{pt}(L) \right) + \lim_{V \to \infty} \left( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \right)
\]

1. General strategy
2. Calculation of \( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \)
3. Calculation of \( \Gamma_0 \)
   - \( G_F \) and the UV matching
   - Lattice calculation
4. Calculation of \( \Gamma_0^{pt}(L) \)
5. Estimates of structure dependent contributions to \( \Gamma_1(\Delta E) \)
6. Conclusions
**Calculation of $\Gamma_{\text{pt}}(\Delta E)$**

\[
\Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0 - \Gamma_{\text{pt}}^0 \right) + \lim_{V \to \infty} \left( \Gamma_0^\text{pt} + \Gamma_{\text{pt}}^1(\Delta E) \right)
\]

- $\Gamma_{\text{pt}}(\Delta E)$ is calculated in perturbation theory with a pointlike pion

\[
\mathcal{L}_{\pi^-\ell^-\nu_\ell} = i G_F f_\pi V_{ud}^* \left\{ (\partial_\mu - ie A_\mu) \pi \right\} \left\{ \bar{\psi}_{\nu_\ell} \frac{1 + \gamma_5}{2} \gamma^\mu \psi_\ell \right\} + \text{QED for } \pi \text{ and } l^+
\]

- UV divergences in $\Gamma_0^\text{pt}$ are regularized with the “$W$-regularization” (more on this point later)

- IR divergences are regularized with the a photon mass

\[
\frac{1}{k^2} \rightarrow \frac{M_W^2}{M_W^2 - k^2} \frac{1}{k^2}
\]
Calculation of $\Gamma^{pt}(\Delta E)$

\[
\Gamma^{pt}(\Delta E) = \Gamma_0^{\text{rec}} \times \left( 1 + \frac{\alpha}{4\pi} \left\{ 3 \log \left( \frac{m_{\pi}^2}{M_W^2} \right) + \log(r_\ell^2) - 4 \log(r_E^2) + \frac{2-10r_\ell^2}{1-r_\ell^2} \log(r_\ell^2) - 2 \frac{1+r_\ell^2}{1-r_\ell^2} \log(r_E^2) \log(r_\ell^2) \right. \right.
\]
\[
\left. \left. - \frac{4}{1-r_\ell^2} \frac{1+r_\ell^2}{1-r_\ell^2} \text{Li}_2(1-r_\ell^2) - 3 + \left[ 3 + r_E^2 - 6r_\ell^2 + 4r_E(-1+r_\ell^2) \right] \log(1-r_E) + \left. \frac{r_E(4-r_E-4r_\ell^2)}{(1-r_\ell^2)^2} \log(r_E^2) \right]\right\}. \]

CHECK: For $\Delta E = \Delta E_{\text{MAX}}$ we obtain the well known result for total rate as in S. M. Berman, PRL 1 (1958) 468 and T. Kinoshita, PRL 2 (1959) 477.


\[ \Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0(L) - \Gamma_{0,pt}(L) \right) + \lim_{V \to \infty} \left( \Gamma_{0,pt} + \Gamma_{1,pt}(\Delta E) \right) \]

1. General strategy  
2. Calculation of \( \Gamma_{0,pt} + \Gamma_{1,pt}(\Delta E) \)  
3. Calculation of \( \Gamma_0 \)  
   - \( G_F \) and the UV matching  
   - Lattice calculation  
4. Calculation of \( \Gamma_{0,pt}(L) \)  
5. Estimates of structure dependent contributions to \( \Gamma_1(\Delta E) \)  
6. Conclusions
**Calculation of** $\Gamma_0(L)$

$$\Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0 - \Gamma_0^{pt} \right) + \lim_{V \to \infty} \left( \Gamma_0^{pt} + \Gamma_1^{pt} (\Delta E) \right)$$

- $\Gamma_0$ is calculated in the finite volume with a lattice simulation
- At lowest order in electromagnetic (and strong) perturbation theory the amplitude can be rewritten in terms of a four-fermion local interaction
- This replacement is necessary in a lattice calculation, since $1/\alpha \ll M_W$
- When including the $O(\alpha)$ corrections, the UV contributions to the matrix element of the local operator are different from those in the Standard Model: a matching between the two theories is required.
The Fermi constant $G_F$ is conventionally taken from the muon lifetime using

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} \left( 1 - \frac{8m_e^2}{m_\mu^2} \right) \left[ 1 + \frac{\alpha}{2\pi} \left( \frac{25}{4} - \pi^2 \right) \right]$$

$$G_F = 1.16634 \times 10^{-5} \text{ GeV}^{-2}$$

Electromagnetic corrections obtained in the local effective theory: [UV finite]

$$\frac{1}{k^2} = \frac{1}{k^2 - M_W^2} + \frac{M_W^2}{M_W^2 - k^2} \frac{1}{k^2}$$

In the Standard Model, it is convenient to write the (Feynman gauge) photon propagator as:

$\text{UV divergent. Absorbed in the definition of } G_F \text{ together with the other EW corrections}$

A. Sirlin, RMP 50 (1978) 573, PRD 22 (1980) 971

$\text{UV convergent. Equal to the corresponding diagrams in the eff. theory with the W-regularization}$

S.M. Berman, PR 112 (1958) 267; T. Kinoshita and A. Sirlin, PR 113 (1959) 1652
SM electroweak corrections to pion decay

Most of the terms which are absorbed into the definition of $G_F$ are common to other processes, including the leptonic decays of pseudoscalar mesons.

Some short-distance contributions, however, do depend on the electric charges of the fields in the four-fermion operators. These lead to a correction factor of

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{u d}^* \left( 1 + \frac{3\alpha}{4\pi} \left( 1 + 2\bar{Q} \right) \log \frac{M_Z}{M_W} \right) \left( \bar{d} \gamma^\mu (1 - \gamma^5) u \right) \left( \bar{\nu}_\ell \gamma^\mu (1 - \gamma^5) \ell \right)$$

$$2\bar{Q} = (Q_{\nu_\mu} + Q_{\mu}) = -1$$

$$1 + 2\bar{Q} = 0$$

muon decay

$$2\bar{Q} = (Q_u + Q_d) = 1/3$$

$$1 + 2\bar{Q} = 4/3$$

pion decay

W-regularization

A. Sirlin, NP B196 (1982) 83; E. Braaten & C. S. Li, PRD 42 (1990) 3888
Matching the W and lattice regularizations

The W regularization cannot be implemented directly in present day lattice simulations since $\frac{1}{a} \ll M_W$

The relation between the Fermi effective Hamiltonian in the lattice and W regularizations can be computed in perturbation theory:

W regul. $\left( \frac{M_W^2}{M_W^2 - k^2} \right) \frac{1}{k^2}$

Lattice $\frac{1}{\sum_{\rho} \frac{4}{a^2} \sin^2 \left( \frac{ak_{\rho}}{2} \right)}$

The result, with the lattice Wilson action for both gluons and fermions, is:

$$O_1^{W-reg} = \left( 1 + \frac{\alpha}{4\pi} \left( 2 \log a^2 M_W^2 - 15.539 \right) \right) O_1^{bare}$$

$$+ \frac{\alpha}{4\pi} \left( 0.536 O_2^{bare} + 1.607 O_3^{bare} - 3.214 O_4^{bare} - 0.804 O_5^{bare} \right),$$

$$O_1 = (\bar{d} \gamma^\mu (1 - \gamma^5) u)(\bar{\nu}_\ell \gamma_\mu (1 - \gamma^5) \ell),$$
$$O_2 = (\bar{d} \gamma^\mu (1 + \gamma^5) u)(\bar{\nu}_\ell \gamma_\mu (1 - \gamma^5) \ell),$$
$$O_3 = (\bar{d} (1 - \gamma^5) u)(\bar{\nu}_\ell (1 + \gamma^5) \ell),$$
$$O_4 = (\bar{d} (1 + \gamma^5) u)(\bar{\nu}_\ell (1 + \gamma^5) \ell),$$
$$O_5 = (\bar{d} \sigma^{\mu\nu} (1 + \gamma^5) u)(\bar{\nu}_\ell \sigma_{\mu\nu} (1 + \gamma^5) \ell)$$
\[ \Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0(L) - \Gamma_0^{pt}(L) \right) + \lim_{V \to \infty} \left( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \right) \]

1. General strategy ✓
2. Calculation of \( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \) ✓
3. Calculation of \( \Gamma_0 \) ✓
   - \( G_F \) and the UV matching ✓
   - Lattice calculation
4. Calculation of \( \Gamma_0^{pt}(L) \)
5. Estimates of structure dependent contributions to \( \Gamma_1(\Delta E) \)
6. Conclusions
Lattice calculation of $\Gamma_0(L)$

The lattice calculation at $O(\alpha^0)$, i.e. without electromagnetism, is standard

$$M_0 = \frac{G_F}{\sqrt{2}} V_{ud}^* \langle 0 | \overline{d} \gamma^\mu \gamma^5 u | \pi^+(p_\pi) \rangle \left[ u_{v_i}(p_{v_i}) \gamma_\mu (1 - \gamma^5) v_\ell(p_\ell) \right] =$$

$$= \frac{i G_F}{\sqrt{2}} V_{ud}^* f_\pi P_\pi \left[ u_{v_i}(p_{v_i}) \gamma_\mu (1 - \gamma^5) v_\ell(p_\ell) \right]$$

The amplitude is obtained from the 2 point correlation function

$$C_0(t) \equiv \sum_{\bar{x}} \langle 0 | \left( \overline{d}(\bar{0}, 0) \gamma^\mu \gamma^5 u(\bar{0}, 0) \right) \phi^\dagger(\bar{x}, -t) | 0 \rangle \approx \frac{Z_0^\phi}{2 m_\pi^0} e^{-m_\pi^0 t} A_0$$

for large $t$

where:

$$Z_0^\phi \equiv \langle \pi^+(\bar{0}) | \phi^\dagger(\bar{0}, 0) | 0 \rangle$$

$$A_0 \equiv \langle 0 | \overline{d} \gamma^\mu \gamma^5 u | \pi^+(\bar{0}) \rangle_0$$

$$C_0^{\phi\phi}(t) \equiv \sum_{\bar{x}} \langle 0 | \phi(\bar{0}, 0) \phi^\dagger(\bar{x}, -t) | 0 \rangle \approx \frac{(Z_0^\phi)^2}{2 m_\pi^0} e^{-m_\pi^0 t}$$
The Feynman diagrams at O(\(\alpha\)) can be divided into 3 classes:

1. The photon connects two quark lines.
2. The photon connects one quark and one charged lepton line.
3. Leptonic wave function renormalization. It cancels in \(\Gamma_0(L) - \Gamma_0^{\text{pt}}(L)\).
Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$ [I]

Let us consider the Feynman diagrams of the 1st class:

The leptonic contribution to the amplitude is factorized and we need to compute:

$$C_1(t) = -\frac{1}{2} \int d^3\vec{x} d^4x_1 d^4x_2 \sum \langle 0 | T \left\{ J_\nu^\gamma(0) j_\mu(x_1) j_\mu(x_2) \phi^\dagger(\vec{x},-t) \right\} | 0 \rangle \Delta(x_1,x_2)$$

For sufficiently large $t$ the correlation function is dominated by the ground state

$$C_0(t) + C_1(t) \approx \frac{Z^\phi}{2m_\pi} e^{-m_\pi t} A \approx \frac{(Z_0^\phi + \delta Z^\phi)}{2(m_\pi^0 + \delta m_\pi)} e^{-m_\pi^0 t} (1 - \delta m_\pi t) (A_0 + \delta A)$$

$$C_1(t) / C_0(t) \approx c_1 t + c_2$$

and similarly

$$C_1^{\phi}(t) / C_0^{\phi}(t) \approx c_1 t + c_2^{\phi}$$

$\delta m_\pi = -c_1$

$\delta A = A_0 \left( c_2 - \frac{c_2^{\phi}}{2} - \frac{c_1}{2m_\pi^0} \right)$

e.m. mass shift

e.m. “correction to $f_\pi$” (unphys.)
The charged-neutral pion mass splitting

Only 2 diagrams contribute to the pion mass splitting

\[ M_{\pi^+} - M_{\pi^0} = \frac{(e_u - e_d)^2}{2} e^2 \partial_t - \text{diagram} \]

- The 1\textsuperscript{st} diagram also contributes to the \( \pi^+ \) decay rate
- The 2\textsuperscript{nd} diagram comes from the neutral pion. It is \( O(\alpha m_{ud}) \) and it has been neglected

From the linear slope in time \( c_1 \) we find

\[ M_{\pi^+} - M_{\pi^0} = 5.33(76) \text{ MeV} \]

G.M. de Divitiis \textit{et al.}, RM123 collaboration, PRD 87 (2013) 114505

\[ M_{\pi^+} - M_{\pi^0} = 4.28(39) \text{ MeV} \]

RM123 collaboration, 2015, preliminary

in good agreement with \( (M_{\pi^+} - M_{\pi^0})_{\text{exp}} = 4.59 \text{ MeV} \)

\[ C_1(t) / C_0(t) \approx c_1 t + c_2 \]
Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$ [II]

For these diagrams the leptonic and hadronic contributions do not factorize

The amplitude is obtained from the following Euclidean space correlation function

$$C_1(t)_{\alpha\beta} = -\int d^3\bar{x} d^4x_1 d^4x_2 \langle 0 | T \left\{ J^\nu_w(0) J_{\mu}(x_1) \phi^*(\bar{x},-t) \right\} | 0 \rangle$$

$$\times \Delta(x_1,x_2) \left( \gamma_\nu (1-\gamma^5) S(0,x_2) \gamma_\mu \right)_{\alpha\beta} e^{E_\ell t_2 - i \not{p}_\ell \cdot \bar{x}_2}$$

We need to ensure that the $t_2$ integration converges as $t_2 \to \infty$. The large $t_2$ behavior is given by the factor $\exp \left[ \left( E_\ell - \omega_\ell - \omega_\gamma \right) t_2 \right]$

$$E_\ell = \sqrt{\not{p}_\ell^2 + m_\ell^2} \quad \omega_\ell = \sqrt{k_\ell^2 + m_\ell^2} \quad \omega_\gamma = \sqrt{k_\gamma^2 + m_\gamma^2} \quad \tilde{k}_\ell + \tilde{k}_\gamma = \tilde{p}_\ell$$

The integral is convergent and the continuation from Minkowski to Euclidean space can be performed
Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$ [II]

$\Gamma_0^{\text{exch}} \propto M^1(M^0)^*$

Preliminary

$(V-A) \times (V-A) = VV + AA - VA - AV$

$V_0V_0 < 1 \%$

$V_iV_i < 1 \%$

$A_0A_0 < 1 \%$

$A_iA_i < 1 \%$

$28 \%$

$2.7 \%$
\[
\Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0(L) - \Gamma_0^{\text{pt}}(L) \right) + \lim_{V \to \infty} \left( \Gamma_0^{\text{pt}} + \Gamma_1^{\text{pt}}(\Delta E) \right)
\]

1. General strategy ✓
2. Calculation of \( \Gamma_0^{\text{pt}} + \Gamma_1^{\text{pt}}(\Delta E) \) ✓
3. Calculation of \( \Gamma_0 \) ✓
   - \( G_F \) and the UV matching ✓
   - Lattice calculation ✓
4. Calculation of \( \Gamma_0^{\text{pt}}(L) \)
5. Estimates of structure dependent contributions to \( \Gamma_1(\Delta E) \)
6. Conclusions
Calculation of $\Gamma^\text{pt}_0(L)$

$$\Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0 - \Gamma_0^\text{pt} \right) + \lim_{V \to \infty} \left( \Gamma_0^\text{pt} + \Gamma_1^\text{pt}(\Delta E) \right)$$

- $\Gamma_0^\text{pt}(L)$ is calculated in perturbation theory with a pointlike pion.
- UV divergences are regularized with the W-regularization.
- IR divergences are regularized by the finite volume (same of $\Gamma_0(L)$).

For the pion self energy, the result is:

$$\frac{1}{\pi} \left( \frac{2\pi}{L} \right)^3 \sum_{\bar{q}} \frac{1}{(M_w^4 - 4m^2E^2_{W,\bar{q}})^2} \left[ 16m^4_\pi \left( \frac{\bar{q}^2}{E_{W,\bar{q}}} + \frac{M^2_w}{E_{W,\bar{q}}} + \frac{M^2_w}{E_{\pi,\bar{q}}} \right) + M^4_w \left( \frac{4\bar{q}^2}{E_{W,\bar{q}}} - \frac{4\bar{q}^2}{E_{\pi,\bar{q}}} + \frac{M^2_w}{E_{W,\bar{q}}} + \frac{M^2_w}{E_{\pi,\bar{q}}} \right) \right] - \left( M_w \to 0 \right)$$

$$-4M^2_w m^2_\pi \left( \frac{3\bar{q}^2}{E_{W,\bar{q}}} - \frac{3\bar{q}^2}{E_{\pi,\bar{q}}} + \frac{2M^2_w}{E_{W,\bar{q}}} + \frac{2M^2_w}{E_{\pi,\bar{q}}} \right) - \left( M_w \to 0 \right)$$

$$\bar{q} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$E_{X,\bar{q}} = \sqrt{M^2_x + \bar{q}^2}$$
\[ \Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0(L) - \Gamma_0^{pt}(L) \right) + \lim_{V \to \infty} \left( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \right) \]

1. General strategy ✓
2. Calculation of \( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \) ✓
3. Calculation of \( \Gamma_0 \) ✓
   - \( G_F \) and the UV matching ✓
   - Lattice calculation ✓
4. Calculation of \( \Gamma_0^{pt}(L) \) ✓
5. Estimates of structure dependent contributions to \( \Gamma_1(\Delta E) \) X
6. Conclusions
Conclusions and outlook

- We have presented, for the first time, a method to compute electromagnetic effects in hadronic processes with lattice QCD.

- The implementation of the method is challenging but within reach of present lattice technology. Preliminary numerical results have been already obtained.

- Since the effects we are calculating are of $O(1\%)$, computing the electromagnetic corrections to a precision of 20\% or so would already be more than sufficient.

Physical results expected soon !
Supplementary slides
\[ \Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0(L) - \Gamma_0^{pt}(L) \right) + \lim_{V \to \infty} \left( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \right) \]

1. General strategy ✓
2. Calculation of \( \Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E) \) ✓
3. Calculation of \( \Gamma_0 \) ✓
   - \( G_F \) and the UV matching ✓
   - Lattice calculation ✓
4. Calculation of \( \Gamma_0^{pt}(L) \) ✓
5. Estimates of structure dependent contributions to \( \Gamma_1(\Delta E) \)
6. Conclusions
Estimates of structure dependent contributions to $\Gamma_1(\Delta E)$

We estimate the size of the neglected structure-dependent contributions to the decay $K^+ / \pi^+ \rightarrow \ell \nu \ell \gamma$ using chiral perturbation theory at $O(p^4)$.


Start with the decomposition in terms of Lorenz invariant form factors of the hadronic matrix element

$$H^{\mu\nu}(k, p_\pi) = \int d^4x e^{ikx} \langle 0 \left| T \left( j^\mu(x) J_W^\nu(0) \right) \right| \pi(p_\pi) \rangle$$

and separate the contribution corresponding to the approximation of a pointlike pion $H^{\mu\nu}_{pt}$ from the structure dependent part $H^{\mu\nu}_{SD}$

$$H^{\mu\nu} = H^{\mu\nu}_{pt} + H^{\mu\nu}_{SD}$$

$H^{\mu\nu}_{pt}$ is simply given by:

$$H^{\mu\nu}_{pt} = f_\pi \left[ g^{\mu\nu} - \frac{(2p_\pi - k)^\mu (p_\pi - k)^\nu}{(p_\pi - k)^2 - m_\pi^2} \right] \left( k_\mu H^{\mu\nu}_{pt} = f_\pi p_\pi^\nu \right)$$
Estimates of structure dependent contributions to $\Gamma_1(\Delta E)$

The structure dependent component $H^\mu\nu_{SD}$ can be parametrized by four independent invariant form factors which we define as

$$H^\mu\nu_{SD} = H_1 \left[ k^2 g^\mu\nu - k^\mu k^\nu \right] + H_2 \left[ (k \cdot p_\pi) k^\mu - k^2 p_\pi^\mu \right] (p_\pi - k)^\nu$$

$$-i \frac{F_V}{m_\pi} \epsilon^{\mu\nu\alpha\beta} k_\alpha p_\beta + \frac{F_A}{m_\pi} \left[ (k \cdot p_\pi - k^2) g^\mu\nu - (p_\pi - k)^\mu k^\nu \right]$$

$$\left( k_\mu H^\mu\nu_{SD} = 0 \right)$$

For the decay into a real photon, only $F_V$ and $F_A$ contribute

At $O(p^4)$ in chiral perturbation theory $F_V$ and $F_A$ are constant:

$$F_V = \frac{m_P}{4\pi^2 f_\pi}$$

$$F_A = \frac{8m_P}{f_\pi} \left( L_9^r + L_{10}^r \right)$$

J. Bijnens, G. Ecker, J. Gasser, NPB 396 (1993) 81

For our estimates we use:

**Direct measurement**

PDG 2014

- $F_V^{(\pi)} = 0.0254$
- $F_A^{(\pi)} = 0.0119$

**ChPT**

- $F_V^{(K)} = 0.096$
- $F_A^{(K)} = 0.042$
\[ R_i^A(\Delta E) = \frac{\Gamma_i^A(\Delta E)}{\Gamma_0^{\alpha,pt} + \Gamma_1^{\alpha,pt}(\Delta E)} \quad , \quad A = \{SD, INT\} \]

SD = structure dependent
INT = interference

- Interference contributions are negligible in all the decays
- Structure-dependent contributions can be sizable for \( K \rightarrow e\nu(\gamma) \) but they are negligible for \( \Delta E < 20 \text{ MeV} \) (which is experimentally accessible)
Structure dependent contributions to decays of D and B mesons

For the studies of D and B mesons decays we cannot apply ChPT.

For B mesons in particular we have another small scale, \( m_{B^*} - m_B \approx 45 \text{ MeV} \)

the radiation of a soft photon may still induce sizeable SD effects.

A phenomenological analysis based on a simple pole model for \( F_V \) and \( F_A \)

confirms this picture.

\[
F_V \approx \frac{\tilde{C}_V}{1 - \left( p_B - k \right)^2 / m_{B^*}^2},
\]

\[
F_A \approx \frac{\tilde{C}_A}{1 - \left( p_B - k \right)^2 / m_{B^*_l}^2}
\]

Under this assumption the SD contributions to \( B \to e\nu(\gamma) \)

for \( E_\gamma \approx 20 \text{ MeV} \) can be very large, but are small for

\( B \to \mu\nu(\gamma) \) and \( B \to \tau\nu(\gamma) \).

A lattice calculation of \( F_V \) and \( F_A \) would be very useful.