Resonances in coupled-channel scattering from lattice QCD

David Wilson

Old Dominion University

Based on work in collaboration with J.J. Dudek, R.G. Edwards and C.E. Thomas.

Chiral Dynamics
Pisa, Italy.
June 29th 2015
Resonances from QCD

\[ \Gamma(s) = \frac{g_R^2}{6\pi} \frac{k_{cm}^3}{E_{cm}^2} \]

\[ m_R = 854.1 \pm 1.1 \text{ MeV} \]
\[ g = 5.80 \pm 0.11 \]
\[ \Gamma_R = \frac{g^2}{6\pi} \frac{p_R^2}{m_R^2} = 12.4 \pm 0.6 \text{ MeV} \]

- \( L = 1.9 \text{ fm} \)
- \( L = 2.4 \text{ fm} \)
- \( L = 2.9 \text{ fm} \)

\( m_{\pi} = 391 \text{ MeV} \)

J. J. Dudek, R. G. Edwards and C. E. Thomas
Phys. Rev. D 87, 034505

David Wilson
Coupled-channel scattering from lattice QCD
Most physical resonances couple to multiple channels.

To understand the physical spectrum, applying coupled-channel methods will be essential.

We consider here $\pi K$, where $\eta K$ can also contribute in $I=1/2$.

The physical amplitudes have resonances in several partial waves:

<table>
<thead>
<tr>
<th>$J^P$</th>
<th>Resonances</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^+$</td>
<td>$\kappa(700), K_0^*(1430), ...$</td>
</tr>
<tr>
<td>$1^-$</td>
<td>$K^*(892), ...$</td>
</tr>
<tr>
<td>$2^+$</td>
<td>$K_2^*(1430), ...$</td>
</tr>
</tbody>
</table>

**Aim:** Obtain the scattering S-matrix from Lattice QCD.
Coupled-channel scattering

Main ingredients to obtain the finite volume spectra:

- Anisotropic lattices - finer temporal spacing.
- Distillation (Peardon et al 2009).
- A large basis of $\bar{q}q$-like constructions.
- Pairs of optimised meson operators.
Coupled-channel scattering

Main ingredients to obtain the finite volume spectra:

- Anisotropic lattices - finer temporal spacing.
- Distillation (Peardon et al 2009).
- A large basis of $\bar{q}q$-like constructions.
- Pairs of optimised meson operators.
Coupled-channel scattering

Main ingredients to obtain the finite volume spectra:

- Anisotropic lattices - finer temporal spacing.
- Distillation (Peardon et al. 2009).
- A large basis of $\bar{q}q$-like constructions.
- Pairs of optimised meson operators.
Many contributors:

Lüscher
Gottlieb & Rummukainen
Christ, Kim, & Yamazaki
Kim, Sachrajda & Sharpe
He, Feng & Liu
Bernard, Lage, Meissner, and Rusetsky
Leskovec & Prelovsek
Briceño & Davoudi
Hansen & Sharpe
Gockeler et al
Guo, Dudek, Edwards & Szczepaniak
Briceño, Davoudi, Luu
+ ...
Coupled-channel extensions of Lüscher’s method

\[
\det \left[ t_{ij}^{-1}(E) + \mathcal{M}_{ij}(E) \right] = 0
\]

\[
S = 1 + 2i \rho t
\]

infinite volume scattering
\(t\)-matrix

known finite-volume functions
diagonal in channels, mixes partial waves

diagonal in partial waves, mixes channels
Coupled-channel extensions of Lüscher’s method

\[
\det \left[ t_{ij}^{-1}(E) + M_{ij}(E) \right] = 0
\]
Coupled-channel scattering

Problem: Three or more unknowns for each energy level, eg:

\[ S_{11} = \eta e^{2i \delta \pi K} \]
\[ S_{22} = \eta e^{2i \delta \eta K} \]

(...and even more with higher partial waves)

2x2 complex matrix (or more) but only one equation.
No one-to-one relation from energy levels to amplitudes
Coupled-channel scattering

**Solution:** Parameterise $t$-matrix, constrain parameters using many energy levels

**E.g.:** $K$-matrix (it’s essential that we preserve unitarity)

$$S_{ij} = \delta_{ij} + 2i (\rho_i \rho_j)^{1/2} t_{ij}$$

$$[S^\dagger S]_{ij} = \delta_{ij}$$

$$\rightarrow \text{Im}[t^{-1}]_{ij} = -\rho_i \delta_{ij}$$

- $K$-matrix contains everything that isn’t constrained by unitarity

$$t_{ij}^{-1}(s) = K_{ij}^{-1}(s) - i\delta_{ij} \rho_i(s)$$

- $K$ must be real for real $s$. One option for two channel scattering:

$$K = \frac{1}{m^2 - s} \left[ \begin{array}{cc} g_{\pi K}^2 & g_{\pi K} g_{\eta K} \\ g_{\pi K} g_{\eta K} & g_{\eta K}^2 \end{array} \right] + \left[ \begin{array}{cc} \gamma_{\pi K, \pi K} & \gamma_{\pi K, \eta K} \\ \gamma_{\pi K, \eta K} & \gamma_{\eta K, \eta K} \end{array} \right]$$

- $m, g, \gamma$ are real free parameters. Simple to add more - more poles, or a polynomial in $s$.

- Simple to generalise to scattering with non-zero angular momentum.

- Can be improved by adding extra physically motivated properties - eg: Chew-Mandelstam phase space.
Coupled-channel scattering

- Describe $t$-matrix using $K$-matrix in $S$-wave only $\rightarrow$ obtain a spectrum.
- Minimise a $\chi^2$ to obtain the best agreement between the $K$-matrix and lattice energies.

$$K = \frac{1}{m^2 - s} \begin{bmatrix}
g_{\pi K}^2 & g_{\pi K} \eta K & g_{\eta K}^2 
g_{\pi K} \eta K & g_{\pi K} \eta K & g_{\eta K}^2 
g_{\pi K} \eta K & g_{\pi K} \eta K & g_{\eta K}^2
\end{bmatrix} + \begin{bmatrix}
\gamma_{\pi K, \pi K} & \gamma_{\pi K, \eta K} 
\gamma_{\pi K, \eta K} & \gamma_{\eta K, \eta K}
\end{bmatrix}.$$
Coupled-channel scattering

- Describe $t$-matrix using $K$-matrix in $S$-wave only $\rightarrow$ obtain a spectrum.
- Minimise a $\chi^2$ to obtain the best agreement between the $K$-matrix and lattice energies.

$$\kappa = \frac{1}{m^2 - s} \begin{bmatrix} g^2_{\pi \kappa} & g_{\pi \kappa} g_{\eta \kappa} \\ g_{\pi \kappa} g_{\eta \kappa} & g^2_{\eta \kappa} \end{bmatrix} + \begin{bmatrix} \gamma_{\pi \kappa, \pi \kappa} & \gamma_{\pi \kappa, \eta \kappa} \\ \gamma_{\pi \kappa, \eta \kappa} & \gamma_{\eta \kappa, \eta \kappa} \end{bmatrix}. $$

$$\chi^2/N_{dof} = \frac{6.40}{15 - 6} = 0.71$$
$S$-wave amplitudes

- Broad resonance in $S$-wave $\pi K$.
- $\eta K$ coupling is small.
- 3 subthreshold points, naturally included in an energy-level fit.

\[
S_{11} = \eta e^{2i \delta_{\pi K}} \\
S_{22} = \eta e^{2i \delta_{\eta K}}
\]

- $m = (0.2466 \pm 0.0020 \pm 0.0009) \cdot a_t^{-1}$
- $g_{\pi K} = (0.165 \pm 0.006 \pm 0.002) \cdot a_t^{-1}$
- $g_{\eta K} = (0.033 \pm 0.010 \pm 0.003) \cdot a_t^{-1}$
- $\gamma_{\pi K, \pi K} = 0.184 \pm 0.054 \pm 0.030$
- $\gamma_{\pi K, \eta K} = -0.52 \pm 0.20 \pm 0.06$
- $\gamma_{\eta K, \eta K} = -0.37 \pm 0.07 \pm 0.05$

\[
\chi^2 / N_{\text{dof}} = \frac{6.40}{15-6} = 0.71.
\]
More energy levels

- Many more energy levels from irreps where the mesons are moving with respect to the lattice.
- More than 100 usable levels.
$S+P$-waves from 80 energy levels

$\chi^2/N_{\text{dof}} = \frac{49.1}{61-6} = 0.89$

$\chi^2/N_{\text{dof}} = \frac{15.0}{19-4} = 1.00$

- Separate fits and global fits yield consistent results.
- $D$-wave is negligible in this region.
Parameterisation variation

- Separate fits and global fits yield consistent results.
- $D$-wave is negligible in this region.
Narrow $D$-wave resonance

- Many other energy levels containing scattering amplitude information.
- Using only irreps with $\tilde{J}=2$ and higher ($E^+, T_2^+$, [100]$B_{1,2}$) we find a narrow resonance:
  - Fit to energies.
  - In $J \geq 1$ scattering the lowest threshold is $\pi\pi K$ at $a_t E_{cm}=0.235$.
  - Ideally requires 3-body formalism. Although not strictly rigorous, we can apply the $2\rightarrow 2$ formalism anyway.
$S$-matrix poles

$m = \text{Re}\sqrt{s_0} / \text{MeV}$

$\Gamma = 2 \cdot \text{Im}\sqrt{s_0} / \text{MeV}$
Summary

- Coupled-channel scattering amplitudes can be obtained from QCD using lattice methods.
- Using extensions of Lüscher’s method, we were able to connect finite volume energy levels to infinite volume scattering amplitudes.
- There are many exciting possibilities for future calculations using similar methods:
  
  Strongly coupled systems like the $a_0(980)$ and $f_0(980)$ are under investigation.
  
  Investigations into $\pi\gamma \rightarrow \pi\pi$ and similar processes are underway.
  
  Channels involving charm quarks are also under investigation by European collaborators.
- Further in the future: $\pi N \rightarrow \pi N, \gamma N \rightarrow \pi N$. Multiparticle scattering, exotics.
Coming soon: $\pi\eta$-$K\bar{K}$-$\pi\eta'$
$S$-wave amplitudes vs experiment
Backup slides: Lattice
More energy levels

- Many more energy levels from irreps where the mesons are **moving with respect to the lattice**.
- More than 100 usable levels.
Overlaps ~ guide to resonant content
\[ Z_i^n = \langle n | O_i^\dagger | 0 \rangle \]

- Shifted \( \pi K \)-like and \( \eta K \)-like states
- \( J^P=1^- \) state near to \( \pi K \) threshold, \( J^P=2^+ \) state, extra \( J^P=0^+ \).
- Considerable partial-wave mixing.

\[ [011] A_1 \]

- Mostly \( \eta K \leftarrow \rightarrow \) Mostly \( \pi K \)
- \( \sim J^P = 2^+ \)

- Interacting \( \pi K \)'s + single particle overlaps
- \( \sim J^P = 0^+ \) + interactions

- Interacting \( \pi K \)'s + single particle overlaps
- \( \sim J^P = 1^- \)

Coupled-channel scattering from lattice QCD
$P$-wave contributions

Coupled-channel scattering from lattice QCD
Overlaps $\sim$ guide to resonant content

$Z_i^n = \langle n | O_i^\dagger | 0 \rangle$

- Shifted $\pi K$-like and $\eta K$-like states
- $\mathcal{J}^P=1^-$ state near to $\pi K$ threshold, $\mathcal{J}^P=2^+$ state, extra $\mathcal{J}^P=0^+$.
- Considerable partial-wave mixing.

$[011] A_1$ contains $\mathcal{J}^P=0^+$, 1, 2, ...

Mostly $\eta K \leftrightarrow \pi K$

$\sim J^P = 2^+$

$\sim J^P = 0^+$ + interactions

$\sim J^P = 1^-$

Coupled-channel scattering from lattice QCD
**P-wave near-threshold state**

Elastic scattering just above $\pi K$ threshold, no $\eta K$ to consider.

The irreps with $P$-wave overlap:


all have an “extra” level near $\pi K$ threshold.

Fitting the energy levels using an elastic Breit-Wigner in $\pi K$: 

![Graph showing the fitting of energy levels using an elastic Breit-Wigner in $\pi K$.]
**P-wave near-threshold state**

Elastic scattering just above $\pi K$ threshold, no $\eta K$ to consider.

The irreps with $P$-wave overlap:


all have an “extra” level near $\pi K$ threshold.

Fitting the energy levels using an elastic Breit-Wigner in $\pi K$:

$$t = \frac{1}{\rho(s)} \frac{s^{\frac{3}{2}} \Gamma(s)}{m_R^2 - s - is^{\frac{3}{2}} \Gamma(s)}$$

$$\Gamma(s) = \frac{g_R^2 k_{cm}^3}{6\pi E_{cm}^2}$$

$$k^3 \cot \delta_1 = (m_R^2 - s) \frac{6\pi s^{\frac{1}{2}}}{g_R^2}$$

In $t$ there is a pole on the real axis just below $\pi K$ threshold:

Bound state in $J^P=1^-$.
David Wilson  
Coupled-channel scattering from lattice QCD
Coupled-channel calculation details

- Large basis of operators including:

  **“Single-meson”** like operators, including bilinears and derivatives.

  \[ \mathcal{O}_i = \bar{\psi} \Gamma \nabla \cdots \nabla \psi \]

  **“Meson-meson”** like operators: Made from pairs of projected variationally-optimised single-meson operators at source and sink with definite momentum, e.g.:

  \[ \Omega_{\pi}(\vec{p}_1) \Omega_{K}(\vec{p}_2) \]

  \[ C(t) \nu^n(t) = \lambda_n(t) C(t_0) \nu^n(t) \]

  \[ \Omega_n^\dagger = \sum_{i} \nu_i^n \mathcal{O}^\dagger_i \]

- Include all Wick contractions.

- All relevant irreps with boosts

  \[ p^2 = |\vec{p}_1 + \vec{p}_2|^2 \leq 4 \left( \frac{2\pi}{L} \right)^2 \]
Relative operator overlaps

$$Z_i = \langle n | \mathcal{O}_i | 0 \rangle$$
Operators with overall momentum

Because momentum is quantised, different energies can be accessed by considering operators with an overall momentum

\[
\vec{p} = \frac{2\pi}{\xi L} \vec{n}
\]

\[
E_{\text{lat}}^2 = E_{\text{cm}}^2 + \left(\frac{2\pi}{\xi L} |\vec{n}| \right)^2
\]

Useful to consider systems with \( \vec{n} = (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 0, 0) \)

Overall zero momentum: \( \pi(0, 0, 0)\pi(0, 0, 0), \pi(1, 0, 0)\pi(-1, 0, 0), \ldots \)

One unit: \( \pi(1, 0, 0)\pi(0, 0, 0), \pi(1, 1, 0)\pi(-1, 0, 0), \ldots \)

Less symmetry: More mixing of angular momentum!
Principal correlators

$\lambda_0$

$\chi^2/N_{\text{dof}} = 0.76$

$\alpha_t E_{\text{lat}} = 0.16541(66)$

$\lambda_3$

$\chi^2/N_{\text{dof}} = 0.68$

$\alpha_t E_{\text{lat}} = 0.26802(99)$

$\lambda_6$

$\chi^2/N_{\text{dof}} = 0.62$

$\alpha_t E_{\text{lat}} = 0.3378(54)$

$\lambda_1$

$\chi^2/N_{\text{dof}} = 0.38$

$\alpha_t E_{\text{lat}} = 0.17793(66)$

$\lambda_4$

$\chi^2/N_{\text{dof}} = 0.72$

$\alpha_t E_{\text{lat}} = 0.27666(84)$

$\lambda_7$

$\chi^2/N_{\text{dof}} = 1.02$

$\alpha_t E_{\text{lat}} = 0.3400(113)$

$\lambda_2$

$\chi^2/N_{\text{dof}} = 0.58$

$\alpha_t E_{\text{lat}} = 0.23097(68)$

$\lambda_5$

$\chi^2/N_{\text{dof}} = 1.00$

$\alpha_t E_{\text{lat}} = 0.3364(29)$

$e^{E_n t} \lambda_n(t)$

David Wilson

Coupled-channel scattering from lattice QCD
**Extracting a spectrum**

Getting the ground state is useful, but we want to extract the whole spectrum in a finite volume.

Fitting subleading exponentials doesn’t get very far:
With very precise data, sometimes a second state can be found.

A solution: The variational method.

\[ C_{ij}(t)v_i^m = \lambda_n(t)C_{ij}(t_0)v_j^m \]

If more than one operator overlaps onto the same state represented by some eigenvector \( v_i^n \) the generalised eigenvalue problem can be solved and then as many states as operators may be extracted.

\[ \lambda_n(t) \sim e^{-E_n(t-t_0)} \]

... a large basis of operators are needed
\[ C_{ij}(t) \nu_j^n = \lambda_n(t) C_{ij}(t_0) \nu_j^n \]
\[ \lambda_n(t) \sim e^{-E_n(t-t_0)} \]

Use a large basis of operators

\[ \mathcal{O}_i = \bar{\psi} \Gamma \psi \]

\[ \mathcal{O}_i = \bar{\psi} \Gamma \overleftrightarrow{D} \ldots \overleftrightarrow{D} \psi \]

\[ \Gamma_i = \{1, \gamma_0, \gamma_5, \gamma_0 \gamma_5, \gamma_i, \gamma_0 \gamma_i, \gamma_5 \gamma_i, [\gamma_i, \gamma_j]\} \]

Use the variational method with a large correlation matrix
Symmetry on the lattice

The lattice has a cubic symmetry.
It does not have the $O(3)$ symmetry of continuous space.

Eg: 2D QM

Continuous rotational spatial symmetry

\[ e^{i\phi} \rightarrow e^{i\phi + i\alpha} \quad e^{i\phi} \rightarrow e^{i\phi + in\pi / 2} \]

Only symmetric at discrete angles
Symmetry on the lattice

Continuous rotational spatial symmetry

\[ e^{i\phi} \rightarrow e^{i\phi + i\alpha} \]

\[ e^{i\phi} \rightarrow e^{i\phi + in\pi/2} \]

Only symmetric at discrete angles

Cubic symmetry groups mix the continuum angular momentum:

<table>
<thead>
<tr>
<th>Irrep</th>
<th>( J^p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1^+ )</td>
<td>0(^+), 4(^+), ...</td>
</tr>
<tr>
<td>( T_1^- )</td>
<td>1(^-), 3(^-), ...</td>
</tr>
</tbody>
</table>
Backup slides: Finite volume formalism
Coupled-channel extensions of Lüsher’s method

Angular momentum

Channels: eg $\pi K$, $\eta K$

scattering t-matrix

momentum boost vector

lattice irrep

finite volume object - contains generalised Lüscher Zeta functions

\[
S_{ij} = \delta_{ij} + 2i \left( \rho_i \rho_j \right)^{\frac{1}{2}} t_{ij}
\]

\[
\det \left[ \delta_{ij} \delta_{\ell\ell'} \delta_{nn'} + i \rho_i t_{ij}^{(\ell)} \left( \delta_{\ell\ell'} \delta_{nn'} + i \mathcal{M}_{ij, \ell n, \ell' n'} \right) \right] = 0
\]

\[
\mathcal{M} \sim \frac{1}{\gamma} \sum_{\text{CGs}} \sum_{\text{spins}} \frac{r^\ell Y_{\ell m}(\hat{\mathbf{r}})}{r^2 - q^2}
\]

Symmetry of the volume mixes partial waves - $M$ mixes partial waves.

$t$-matrix is diagonal in partial waves, but can couple scattering channels: $\pi K \rightarrow \eta K$
Coupled-channel scattering

\[ \mathcal{M} \sim \frac{1}{\gamma} \sum_{\text{spins}} (\text{CGs}) \sum_{\hat{r}} \frac{r^\ell Y_{\ell m}(\hat{r})}{r^2 - q^2} \]

scattering t-matrix, couples channels, diagonal in \( l \).

\[
\det \left[ \delta_{ij} \delta_{\ell \ell'} \delta_{nn'} + i \rho_i t_{ij}^{(\ell)} \left( \delta_{\ell \ell'} \delta_{nn'} + i \mathcal{M}_{ij, \ell n, \ell' n'}^{d, \wedge} \right) \right] = 0
\]

finite volume object - contains generalised Lüscher Zeta functions mixes partial waves

- Several unknowns at each energy level: Multiple channels, multiple partial waves.
- Problem is unconstrained for a single energy level.
- Solution: Parameterise \( t_{ij} \) using a few free parameters, use many energy levels to constrain them.
Backup slides: Amplitudes
P-wave pole

- Breit-Wigner pole from continuation below threshold.
- Also used a K-matrix below threshold, found almost exactly the same result.
- Poles on physical sheet \( \text{Im}(k_{\text{cm}}) > 0 \).

\[ K^*(892) \]

BW

K-matrix (1 Ch.)

K-matrix (2 Ch.)

\[ \pi K \]

\[ a_t E_{\text{cm}} \]
**S-wave poles**

Multi-sheeted complex plane due to square-root branch cuts at each threshold, in single channel case for now:

\[ k_{cm} = \pm \frac{1}{2} \left( E_{cm}^2 - 4m^2 \right)^{\frac{1}{2}} \]

**Bound state**

**Resonance**

**Virtual Bound state**

poles in \( s \): complex conjugate pairs in \( k \)
S-wave poles

Multi-sheeted complex plane due to square-root branch cuts at each threshold, in single channel case for now:

\[ k_{cm} = \pm \frac{1}{2} \left( E_{cm}^2 - 4m^2 \right)^{\frac{1}{2}} \]

Bound state

Resonance

Virtual Bound state

poles in $s$: complex conjugate pairs in $k$
Poles

Multi-sheeted complex plane due to square-root branch cuts at each threshold, in single channel case for now:

\[ k_{cm} = \pm \frac{1}{2} \left( E_{cm}^2 - 4m^2 \right)^{\frac{1}{2}} \]

Bound state

Coupled-channel scattering from lattice QCD
Poles

Multi-sheeted complex plane due to square-root branch cuts at each threshold, in single channel case for now:

\[ k_{cm} = \pm \frac{1}{2} \left(E_{cm}^2 - 4m^2 \right)^{\frac{1}{2}} \]
Poles

Multi-sheeted complex plane due to square-root branch cuts at each threshold, in single channel case for now:

\[ k_{cm} = \pm \frac{1}{2} \left( E_{cm}^2 - 4m^2 \right)^{\frac{1}{2}} \]
$S$-wave poles

Actual situation: Unequal masses and an extra pair of sheets due to $\eta K$ scattering

$\rightarrow$ Poles and residues on multiple sheets.

$$\kappa_{cm} = \pm \frac{1}{2} \left( E_{cm}^2 - 2 (m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{E_{cm}^2} \right)^{\frac{1}{2}}$$
Virtual bound state $\kappa$

Pelaez and Nebreda using Unitarised SU(3) Chiral Perturbation theory
More on virtual bound state

In an effective range parameterisation, strong interactions near threshold lead to a large $a$

In $S$-wave large $a$ automatically leads to a pole near-threshold.

\[ k_{cm} \cot \delta_0 = \frac{1}{a} + \frac{1}{2} rk_{cm}^2 \]

\[ t = \frac{1}{2} \frac{E_{cm}}{k_{cm} - i} \]

\[ k_{cm} = \pm \frac{i}{a} \]

Arguments appear to hold for constant terms in K-matrix (slightly complicated by Chew-Mandelstam).

Appears to break down for $P$-wave and higher.

![Graph showing the relation between $k_{cm}$ and $\delta_0$ with points indicating strongly attractive, moderately repulsive, and weakly attractive states.](image)
More *K*-matrix details

- *K*-matrix contains everything not constrained by unitarity

\[
t_{ij}^{-1}(s) = K_{ij}^{-1}(s) - i\delta_{ij} \rho_i(s)
\]

\[
K = \frac{1}{m^2 - s} \begin{bmatrix} g_{\pi K}^2 & g_{\pi K} g_{\eta K} \\ g_{\pi K} g_{\eta K} & g_{\eta K}^2 \end{bmatrix} + \begin{bmatrix} \gamma_{\pi K, \pi K} & \gamma_{\pi K, \eta K} \\ \gamma_{\pi K, \eta K} & \gamma_{\eta K, \eta K} \end{bmatrix}
\]

- Chew-Mandelstam phase space -- include also *s*-channel cut along with imaginary part.

\[
t_{ij}^{-1}(s) = K_{ij}^{-1}(s) + \delta_{ij} \ I_i(s)
\]

\[
I_i(s) = I_i(s_{thr i}) - \frac{s - s_{thr i}}{\pi} \int_{s_{thr i}}^{\infty} ds' \frac{\rho_i(s')}{(s' - s)(s' - s_{thr i})}
\]

(Subtract at pole so that \(\text{Re } I(s = m^2) = 0\))

- Threshold factors for \(l > 0\)

\[
t_{ij}^{-1}(s) = \frac{1}{(2k_i)^\ell} K_{ij}^{-1}(s) \frac{1}{(2k_j)^\ell} + \delta_{ij} \ I_i(s)
\]

As used in Guo, Mitchell and Szczepaniak Phys.Rev. D82 (2010) 094002

No modifications were used in \(I(s)\) for higher waves.

Also tested phase space factors instead of \(k_i\) for thresholds.
Virtual bound state $\kappa$

Pelaez and Nebreda using
Unitarised SU(3) Chiral
Perturbation theory

Bound state
Resonance
Virtual Bound state
$S$-matrix poles

\[ m = \text{Re} \sqrt{s_0} / \text{MeV} \]

\[ \Gamma = 2 \cdot \text{Im} \sqrt{s_0} / \text{MeV} \]

\[ \chi^2 / N_{\text{dof}} = 0.89 \]
$S$-matrix poles

\[ m = \text{Re}\sqrt{s_0} / \text{MeV} \]

\[ \Gamma = 2 \cdot \text{Im}\sqrt{s_0} / \text{MeV} \]
$S$-matrix poles

$\Gamma = 2 \cdot \text{Im} \sqrt{s_0} / \text{MeV}$

$m = \text{Re} \sqrt{s_0} / \text{MeV}$

(c) $2^+$

$\chi^2 / N_{\text{dof}} = 0.89$
$S$-matrix $S$-wave poles

\[ t_{ij}(s \sim s_0) \sim \frac{c_i c_j}{S_0 - s} \]

\[ 2 \cdot \text{Im}(a_t \sqrt{s_0}) \]

\[ \text{Re}(a_t \sqrt{s_0}) \]

- $K$-matrix pole + const
- $K$-matrix pole + linear
  - $K^{-1}$ poly $\{1,0,1\}$
  - $K^{-1}$ poly $\{2,0,1\}$
  - $K^{-1}$ poly $\{1,1,1\}$
  - $K^{-1}$ poly $\{1,0,0\}$
  - $K^{-1}$ poly $\{2,0,0\}$
  - $K^{-1}$ poly $\{2,1,0\}$

- elastic scat. len.
- elastic eff. range.
Backup slides: $\rho$ resonance
Extracting the $\rho$ resonance

Several volumes: $L=16, 20, 24$.
Operators in several moving frames, upto $n=(2,0,0)$.

Anisotropic lattices:
temporal spacing 3.5 times finer for better energy resolution.

Combination of single particle and meson-meson operators.

$m_\pi=391$ MeV
Finite volume spectra in $I=1\ J=1$

- $\vec{P} = [000] T_1$
- $\vec{P} = [011] A_1$
- $\vec{P} = [111] A_1$
- $\vec{P} = [001] A_1$
- $\vec{P} = [011] B_1$
- $\vec{P} = [111] E_2$
- $\vec{P} = [001] E_2$
- $\vec{P} = [111] A_1$
- $\vec{P} = [002] A_1$
Resonances from QCD

\[ \Gamma(s) \approx \frac{s^{1/2} \Gamma(s)}{\rho(s) m_r^2 - s - is^{1/2} \Gamma(s)} \]

\[ m_{\pi} = 391 \text{ MeV} \]

\[ m_r = 854.1 \pm 1.1 \text{ MeV} \]

\[ g = 5.80 \pm 0.11 \]

\[ \Gamma_R = \frac{g^2 p_r^3}{6\pi m_r^2} = 12.4 \pm 0.6 \text{ MeV} \]

- \( L = 1.9 \text{ fm} \)
- \( L = 2.4 \text{ fm} \)
- \( L = 2.9 \text{ fm} \)