Exploratory lattice QCD study of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$

Xu Feng (Columbia University)

CD15@Pisa, 07/02/2015
Collaborators

- on behalf of RBC-UKQCD collaboration
- people involved in this project

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- Amarjit Soni (BNL)
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$K \to \pi \nu \bar{\nu}$: FCNC process

$K \to \pi \nu \bar{\nu}$ decays are flavor changing neutral current processes

in SM such decay can happen only through $W-Z$ or $W-W$ exchange

therefore, it is a second-order weak interaction

\[ K^+ \rightarrow \pi^+ \nu \bar{\nu} \]

\[ W-W \]

\[ Z\text{-exchange} \]

SM effects highly suppressed in the second order $\rightarrow$ ideal probes for NP
Past experimental searches

\[ K^+ \to \pi^+ \nu \bar{\nu}, \text{ from 1969 to 2008} \]

<table>
<thead>
<tr>
<th>Year</th>
<th>Event Description</th>
<th>Journal</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1969</td>
<td>first upper limit on $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$ set</td>
<td>PRL 23('69)</td>
<td>326</td>
</tr>
<tr>
<td>1997</td>
<td>1\textsuperscript{st} event observed by E787 at BNL</td>
<td>PRL 79('97)</td>
<td>2204</td>
</tr>
<tr>
<td>2000</td>
<td>no new events, but more constraint on Br.</td>
<td>PRL 84('00)</td>
<td>3768</td>
</tr>
<tr>
<td>2002</td>
<td>2\textsuperscript{nd} event by E787</td>
<td>PRL 88('02)</td>
<td>041803</td>
</tr>
<tr>
<td>2002</td>
<td>3\textsuperscript{rd} event by E787</td>
<td>PLB 537('02)</td>
<td>211</td>
</tr>
<tr>
<td>2004</td>
<td>4\textsuperscript{th} event by E949 at BNL</td>
<td>PRL 93('04)</td>
<td>031801</td>
</tr>
<tr>
<td>2008</td>
<td>another 3 events by E949</td>
<td>PRL 101('08)</td>
<td>191802</td>
</tr>
</tbody>
</table>

a series of PRL publications and 7 events of $K^+ \to \pi^+ \nu \bar{\nu}$ observed

experimental measurement of branching ratio

\[ \text{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\text{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10} \]

search for 1 candidate among every 6 billion events of $K^+$ decays, difficult!
New experiments (plenary talk by A. Ceccucci)

new generation of experiment: NA62 at CERN aims at

- observation of $O(100)$ events in two years
- 10%-precision measurement of $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$

Fig: 09/2014, the final straw-tracker module is lowered into position in NA62

$K_L \to \pi^0 \nu \bar{\nu}$

- even more challenging since $\pi^0$ decays quickly to two photons
- only upper bound was set by KEK E391a in 2010
- new KOTO experiment at J-PARC designed to observe $K_L$ decays
Recent SM prediction

\( K \to \pi \nu \bar{\nu} \) known to be top-quark dominated ⇒ theoretically very clean
- dominated SM uncertainty from CKM matrix \( V_{td}, V_{ts} \leftarrow V_{ub}, V_{cb}, \gamma \)
- two ways to determine \( V_{ub}, V_{cb}, \gamma \)
  - using tree-level measurement of \( b \to c, b \to u \) transitions
    \[
    \text{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\text{SM}} = (8.4 \pm 1.0) \times 10^{-11}
    \]
  - using loop-level observables, \( \varepsilon_K, \Delta M_{d,s}, S_{\psi K_S} \)
    \[
    \text{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\text{SM}} = (9.11 \pm 0.72) \times 10^{-11}
    \]

recall past experimental measurement

\[
\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\text{exp}} = (1.73^{+1.15}_{-1.05}) \times 10^{-10} > 60\% \text{ err}
\]

\( \text{Br}_{\text{exp}} \) is 2 times larger than \( \text{Br}_{\text{SM}} \), but still consistent with > 60\% error
SM error budget for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$

new experiment confronts SM soon ⇒ can we do better?

- error budget from Buras et. al.
  - $|V_{cb}|$, $\gamma$: CKM inputs for $|V_{td}|$, $|V_{ts}|$
  - $\delta P_{c,u}$: LD contribution
  - $P_c$: $c$-quark contribution (SD part)
  - $X_t$: $t$-quark contribution
  - other: remaining SM parameters

most important thing is to reduce the error from CKM inputs

long-distance contribution yields a sub-dominat uncertainty

phenomenological ansatz involving $\chi$PT and OPE [hep-ph/0503107] yields $\delta P_{c,u} = 0.04 \pm 0.02 \Rightarrow$ branching ratio enhanced by 6%
  - 50% err in $\delta P_{c,u}$ is a guess rather than a controlled error
  - $\delta P_{c,u}$ may be much larger or even smaller

can lattice QCD provide a result with controlled uncertainty?
Lattice methodology
Three-step LQCD development strategy for $K^+ \to \pi^+ \nu \bar{\nu}$

1) this talk, using domain wall fermion configs. generated by RBC-UKQCD

- $16^3 \times 32$, $m_\pi = 420$ MeV, $m_c = 860$ MeV, $a^{-1} = 1.73$ GeV
- set up the calculation at unphysical kinematics

2) USQCD proposal this year (awarded with 27 million BG/Q core hours)

- $32^3 \times 64$, $m_\pi = 170$ MeV, $m_c = 750$ MeV, $a^{-1} = 1.37$ GeV
- further control the unphysical effects from pion mass
- $a^{-1} = 1.37$ GeV $\Rightarrow$ expect large lattice artifacts from $m_c$

3) for the future

- $80^2 \times 96 \times 192$, $m_\pi = 140$ MeV, $m_c = 1.3$ GeV, $a^{-1} = 3$ GeV
- control all systematic effect, produce $\delta P_{c,u}$ with 20% err $\Rightarrow$ 1% in Br
Perturbation theory vs lattice QCD

- perturbation theory (high energy scale above charm quark mass)
  - effective Hamiltonian described by a dim-6 operator \((\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A}\)

\[ \mathcal{H}_{\text{eff}}^{(6)} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \theta_W} \sum_{l=e,\mu,\tau} (\lambda_c X_c^l + \lambda_t X_t) (\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A} \]

- \(X_t\) and \(X_c^l\) are Inami-Lim loop function for top and charm contribution

- lattice QCD (low energy scale at or below charm quark mass)
  - non-local effects given by two local operators \(O_1\) and \(O_2\)

\[ \mathcal{H}_{\text{eff}}^{BL} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \theta_W} \sum_{l=e,\mu,\tau} \lambda_c \left( \frac{\pi^2}{M_W^2} \int d^4x \, O_1(x) \, O_2(0) \right)_{u-c} \]
Bilocal structure [Isidori, Mescia, Smith, hep-ph/0503107]

Z-exchange diagrams:

\[
\int d^4x \left< \pi^+ \nu \bar{\nu} \right| T \{ Q_{1,2}(x) O^Z(0) \} \right| K^+ \rangle
\]

W-W diagrams:

\[
\int d^4x \left< \pi^+ \nu \bar{\nu} \right| T \{ O^{\Delta S=1}(x) O^{\Delta S=0}(0) \} \right| K^+ \rangle
\]
Bilocal LD contribution vs local SD contribution

- compare LD and SD contribution

\[
\langle \pi \nu \bar{\nu} | \left( \frac{\pi^2}{M_W^2} \int d^4x \, O_1(x) \, O_2(0) \right)_{u-c} | K \rangle \quad \Downarrow \quad X_{BL} \\
\langle \pi \nu \bar{\nu} | (\bar{s}d)_{V-A}(\bar{\nu} \nu)_{V-A} | K \rangle \quad \Downarrow \quad X_{C}^\ell
\]

- we define \( X_{BL} \) as a ratio so that it can be compared to \( X_{C}^\ell \) directly

\[
X_{BL} = \frac{\langle \pi \nu \bar{\nu} | \left( \frac{\pi^2}{M_W^2} \int d^4x \, O_1(x) \, O_2(0) \right)_{u-c} | K \rangle}{\langle \pi \nu \bar{\nu} | (\bar{s}d)_{V-A}(\bar{\nu} \nu)_{V-A} | K \rangle} = \frac{\frac{\pi^2}{M_W^2} F_{BL}(p_K, p_\pi, p_\nu)}{F_{SD}(p_K, p_\pi)}
\]

- \( Z \)-exchange diag: \( \nu \, \bar{\nu} \) are localized \( \Rightarrow \) \( X_{BL} \) depends on \( p_K, p_\pi \) only
- \( W-W \) diag: non-local neutrino structure, \( X_{BL} \) also rely on \( p_\nu, p_\bar{\nu} \)
Dalitz plot

- $S = (p_\nu + p_{\bar{\nu}})^2$, $\Delta = (p_K - p_{\bar{\nu}})^2 - (p_K - p_\nu)^2$
- allowed momentum region highly suppressed at $m_\pi = 420$ MeV
- on-shell massless neutrinos $\rightarrow$ modulus of decay amplitude vanishes at the edge of the Dalitz plot
- away from edge $(\Delta, s) = (0, 0) \Rightarrow \vec{p}_\nu = \vec{p}_{\bar{\nu}}, \vec{p}_\pi = -\vec{p}_\nu - \vec{p}_{\bar{\nu}}$
Preliminary lattice results

(use $W\!-\!W$ diagram as example)
Type 1 diagram, gluon exchanges not drawn

\[ K^+ \rightarrow \pi^+ \nu \]

\[ \bar{d} \rightarrow e, \mu, \tau \]

\[ O_{\Delta S=1} \]

\[ O_{\Delta S=0} \]
Exponential growing contamination
given a non-local matrix element in Minkowski space

\[ T^M = i \int dt \langle f | T[O^{\Delta S=1}(t)O^{\Delta S=0}(0)]|K \rangle \]

\[ = \sum_{n_n} \frac{\langle f | O^{\Delta S=1}| n_n \rangle \langle n_n | O^{\Delta S=0} | K \rangle}{E_{n_n} - E_f + i\epsilon} - \sum_{n} \frac{\langle f | O^{\Delta S=0} | n \rangle \langle n | O^{\Delta S=1} | K \rangle}{E_K - E_n + i\epsilon} \]

in Euclidean space

\[ T^E = \sum_{t=-T_a}^{T_b} \langle f | T[O^{\Delta S=1}(t)O^{\Delta S=0}(0)]|K \rangle \]

\[ = \sum_{n_n} \frac{\langle f | O^{\Delta S=1}| n_n \rangle \langle n_n | O^{\Delta S=0} | K \rangle}{E_{n_n} - E_f} \left( 1 - e^{(E_f - E_{n_n})T_b} \right) \]

\[ - \sum_{n} \frac{\langle f | O^{\Delta S=0} | n \rangle \langle n | O^{\Delta S=1} | K \rangle}{E_K - E_n} \left( 1 - e^{(E_K - E_n)T_a} \right) \]

if \( E_n < E_K \), remove exp growing contamination, \( T^E \Rightarrow T^M \)
Unintegrated matrix element

Unintegrated matrix element

Only muon exp growing

- e mode: no exp growing contamination due to helicity suppression
- \( \tau \) mode: no exp growing contamination since \( \tau \) is heavy
$F^\ell_{BL}$ for type 1 diagram

<table>
<thead>
<tr>
<th>$F^\ell_{BL}$</th>
<th>lattice</th>
<th>model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$3.244(90) \times 10^{-2}$</td>
<td>$3.352(12) \times 10^{-2}$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$3.506(77) \times 10^{-2}$</td>
<td>$3.511(13) \times 10^{-2}$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$-2.871(70) \times 10^{-3}$</td>
<td>$-2.836(10) \times 10^{-3}$</td>
</tr>
</tbody>
</table>

- Vacuum saturation approximation assumes only single-lepton contribution in the intermediate state

\[
f_K f_\pi \bar{u}(p_\nu) \phi_K (1 - \gamma_5) \frac{\phi}{q^2 - m_\ell^2} \phi_\pi (1 - \gamma_5) \nu(p_\bar{\nu})
\]

\[
= f_K f_\pi \frac{2q^2}{q^2 - m_\ell^2} \bar{u}(p_\nu) \phi_K (1 - \gamma_5) \nu(p_\bar{\nu})
\]

with $q = p_K - p_\nu = p_\pi + p_\bar{\nu}$

- In the above table, model results are given by $Z_A^{-2} f_K f_\pi \frac{2q^2}{q^2 - m_\ell^2}$

- Non-local form factor $F^\ell_{BL}$ can be converted to $X^\ell_{BL}$
Type 2 diagram, gluon exchanges not drawn

\[ \bar{d} \bar{s} \quad uu \quad \bar{u}, \bar{c} \quad e, \mu, \tau \quad \nu, \bar{\nu} \]

\[ K^+ \quad O^{\Delta S=1} \quad O^{\Delta S=0} \quad \pi^+ \]

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intermediate state is given by $\ell + \pi^0$, since pion is heavy, we don’t observe significant exponential growing effects.
Short distance divergence

- by dimensional counting the loop integrals are quadratically divergent

- GIM mechanism reduces the divergence to logarithmic

- in the physical world, the SD divergence is cut off by physical $M_W$

- in the lattice calculation it is cut off by an energy scale $\Lambda_{\text{lat}} \sim \frac{1}{a}$

- correction can be made through $A - A_{\text{SD}}^{\text{lat}} + A_{\text{SD}}^{\text{cont}} = \int d^4x \langle f | T \{ O_1(x) O_2(0) \} | K \rangle - \langle f | C^{\text{lat}}(\mu) O_{\text{SD}} | K \rangle + \langle f | C^{\text{cont}}(\mu) O_{\text{SD}} | K \rangle$

- $C^{\text{lat}}(\mu)$ is determined non-perturbatively using RI/SMOM approach

- $C^{\text{cont}}(\mu)$ can be calculated perturbatively, currently in LO
Rome-Southampton method (RI/SMOM approach)

- evaluate off-shell Green’s function with $p_i^2 \gg \Lambda_{QCD}^2$

- energy scale of internal momentum, $\mu^2$, is forced to be larger than $p_i^2$

- at high energy scale $\mu$, mainly SD contribution to off-shell Green’s function

- correctly represented by a SD operator multiplying with Wilson coefficient $C^{\text{lat}}(\mu)$
Pauli-Villars method

- Rome-Southampton

\[ X_{BL}^\ell - Z_{V,A} \frac{\pi^2}{M_W^2} C^{\text{lat}}(\mu^2) + X_c^\ell(\mu^2) \]

- \( \mu^2 \) is the scale of loop momenta
- \( C^{\text{lat}}(\mu^2) \) contains the lattice cutoff effects, but replaced by \( X_c^\ell(\mu^2) \)

- Pauli-Villars

\[ X_{BL}^\ell - X_{BL}^M + X_c^M \]

- we use a heavy lepton \( M \) as a regulator
- \( X_{BL}^M \) contains the lattice cutoff effects, but replaced by correct SD \( X_c^M \)
- we calculated \( X_{BL}^\ell \) with \( \ell = e, \mu, \tau \), thus we use \( M = \tau \) as a regulator
Short-distance matching

\[ X^e_c(\mu^2) \]: LO perturbative result of loop function with \( W \) prop insertion

\[ C^{\text{lat}}(\mu^2) \]: non-perturbative lattice matching coeff at scale \( \mu^2 \)

\[ Z_{V,A} \frac{\pi^2}{M_W^2} C^{\text{lat}}(\mu^2) \]

\[ X^e_c(\mu^2), \text{LO PT} \]

\[ X^e_{\text{BL}} - X^\tau_{\text{BL}} + X^\tau_c \]

5-7\% correction

\[ \mu^2 \text{ dependence cancel?} \]

loop mom \((p,p,p,p)\)

\((p,0,0,-p)\)
we have established method and shown that the lattice QCD calculation of the LD contribution to rare kaon decay is feasible.

the new allocation of USQCD computer time starts from July, so we will look at $m_\pi = 170$ MeV very soon

we also want to include the physical charm quark mass as soon as $a^{-1} = 3$ GeV ensembles available

considering the huge experimental efforts to search for $K^+ \to \pi^+ \nu \bar{\nu}$, it is important to determine the LD contribution to $K^+ \to \pi^+ \nu \bar{\nu}$ with a controlled uncertainty
Backup slides
Dalitz plot

three Lorentz invariants $s$, $t$, $t'$

$$s = (p_K - p_\pi)^2 = (p_\nu + p_{\bar{\nu}})^2,$$
$$t = (p_K - p_\nu)^2 = (p_{\bar{\nu}} + p_\pi)^2,$$
$$t' = (p_K - p_{\bar{\nu}})^2 = (p_\nu + p_\pi)^2,$$
$$s + t + t' = m_K^2 + m_\pi^2$$

two independent variables: $s$ and $\Delta = t' - t$

Physical Region

$m_\pi = 140$ MeV

mπ=140 MeV

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Exploratory lattice QCD study of $K^+ \to \pi^+ \nu$
Infinite volume vs finite volume

- above two-pion threshold, $\Sigma_n$ and $\Sigma_{ns}$ shall be replaced by $\oint_n$ and $\oint_{ns}$
- for infinite volume, integral is well defined using principal value

\[
\mathcal{I}^\infty = \mathcal{P} \oint_n \langle f | O^{\Delta S=0} | n \rangle^\infty \infty \langle n | O^{\Delta S=1} | K \rangle \frac{E_K - E_n}{E_K - E_n}
\]

- for finite volume, energy states are always discrete, we still have

\[
\mathcal{I}^L = \sum_n \langle f | O^{\Delta S=0} | n \rangle^{LL} \langle n | O^{\Delta S=1} | K \rangle \frac{E_K - E_n}{E_K - E_n}
\]

- finite-volume correction $\mathcal{I}^\infty = \mathcal{I}^L - \delta \mathcal{I}$

\[
\delta \mathcal{I} = \cot(\phi(E) + \delta(E)) (\phi'(E) + \delta'(E)) \langle f | O^{\Delta S=0} | \pi \pi, E \rangle^{LL} \langle \pi \pi, E | O^{\Delta S=1} | K \rangle \bigg|_{E=m_K}
\]

- $\phi(E)$: a known function depending on $L$; $\delta(E)$: $\pi \pi$ scattering phase
- $\cot(\phi(E) + \delta(E))$ is singular at $E = E_n$; it cancels the singularity of $\mathcal{I}^L$

- for a complete derivation of $\delta \mathcal{I}$, see

Evaluation of non-local matrix element

\[ \int dt \langle \pi^+ \nu \bar{\nu} | T \{ O^{\Delta S=1}(t) O^{\Delta S=0}(0) \} | K^+ \rangle \]

- construct 4-point correlator \( \langle \phi_\pi(t_\pi) O^{\Delta S=1}(t_1) O^{\Delta S=0}(t_0) \phi_K^\dagger(t_K) \rangle \)

- perform time translation average → statistical error reduced by \( \sqrt{T} \)
  - propagators generated on all time slices, quite a lot of cost
  - use low-mode deflation w. 100 low-lying eigenvectors to accelerate CG
  - time required to generate light quark propagators is reduced to 10%

- use overlap fermion for lepton propagator
  - time extent for lepton is infinite

\[ \text{QCD} \quad \text{QCD} \quad \text{QCD} \]

\[ \text{lepton field} \]
Extract the form factor

\[ T = \int dt \langle \pi^+ \nu \bar{\nu} | T \{ O^{\Delta S=1}(t) O^{\Delta S=0}(0) \} | K^+ \rangle \]

when operators \( O^{\Delta S=1}, O^{\Delta S=0} \) act on \( \langle \nu \bar{\nu} \rangle \), it produces \( \bar{u}(p_\nu), \nu(p_{\bar{\nu}}) \)

\[ T = \bar{u}(p_\nu) \left( \int dt \langle \pi^+ | T \{ O^{\Delta S=1}(t) O^{\Delta S=0}(0) \} | K^+ \rangle \right) \nu(p_{\bar{\nu}}) \]

due to the chiral property of \( \bar{u}(p_\nu) \) and \( \nu(p_{\bar{\nu}}) \), \( T \) can be written as

\[ T = F^\ell(p_K, p_\nu, p_{\bar{\nu}}) \bar{u}(p_\nu) \phi_K (1 - \gamma_5) \nu(p_{\bar{\nu}}) \]

we therefore extract the form factor \( F^\ell(p_K, p_\nu, p_{\bar{\nu}}), \ell = e, \mu, \tau \)

final result does not have explicit dependence on spinors
perform the double integration to gain a better precision

\[
\sum_{t_1=t_a}^{t_b} \sum_{t_2=t_a}^{t_b} \langle f | T[O^\Delta S=1(t_2)O^\Delta S=0(t_1)]|K\rangle e^{m_k t_1} e^{-m_f t_1}
\]

\[
= \sum_{n_s} \langle f | O^\Delta S=1|n_s\rangle \langle n_s | O^\Delta S=0|K\rangle \left( T_{\text{box}} \frac{1 - e^{(E_f-E_{n_s}) T_{\text{box}}}}{E_{n_s} - E_f} \right)
\]

\[- \sum_{n} \langle f | O^\Delta S=0|n\rangle \langle n | O^\Delta S=1|K\rangle \left( T_{\text{box}} + \frac{1 - e^{(E_K-E_n) T_{\text{box}}}}{E_K - E_n} \right) \]

here \( T_{\text{box}} = t_b - t_a + 1 \) is defined as size of the integral window

remove the exponential growing contamination, and fit with \( a + b T_{\text{box}} \), the slope \( b \) is what we want
right figure: the slope of the curve gives $F^\ell(p_K, p_\nu, p_{\bar{\nu}})$
Preliminary results for $Z$-exchange diagrams
Evaluation of non-local matrix element

\[ T^Z_{\mu} = \int dt \langle \pi^+ | T\{ Q_{1,2}(t) J^Z_\mu (0) \} | K^+ \rangle \]

- \( Z \)-exchange diagrams do not require on-shell neutrinos
  - we use \( \bar{p}_K = \bar{p}_\pi = 0, \ J^Z_\mu, \mu = t \)

- hadronic current \( J^Z_\mu \) has vector and axial vector component
  - for the vector current, according to Ward identity (WI), we have
    \[ T^{Z,V}_\mu = F^{Z,V}(q^2) \left( q^2(p_K + p_\pi)_\mu - (m_K^2 - m_\pi^2)q_\mu \right), \quad q = p_K - p_\pi \]
  - with \( \bar{p}_K = \bar{p}_\pi = 0 \) \( \Rightarrow \) \( q^2(p_K + p_\pi)_\mu - (m_K^2 - m_\pi^2)q_\mu = 0 \)
  - WI suggests \( T^{Z,V}_\mu = 0 \), this is confirmed by our numerical calculation

- in the following, I will present the results for axial vector current
Summary of Z-exchange diagrams

four classes: Type 1 ($Q_1$), Type 1 ($Q_2$), Type 2 ($Q_1$), Type 2 ($Q_2$)

- connected diagrams, $J^Z_\mu$ can be inserted into all the possible quark line

- disconnected diagrams (usually excluded in lattice calculation since they are noisy and difficult to calculate)
although noisy, clear signal from disc. diagram
scale of y-axis in disc. plot is 4 times smaller than that in conn. plot
Integrating matrix element

disc. diag.

conn. diag.

- The disc. diag. is relatively noisy, but its contribution is small. Adding the disc. part does not affect the conn. part significantly.