Subtractive Renormalization and Scaling in Low-Energy Few-Body Physics

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Outline

- 1. Introduction and motivation
  - – renormalized fixed-point Hamiltonian
  - - Motivation: Effective Theories
- 2. Formalism: Subtractive Renormalized Method
  - Results of the approach in first order, with one-subtraction
- 3. Renormalization Group Invariance
- 4. Renormalized Hamiltonian
- 5. Subtracted T-matrix equation with n-subtractions
- 6. Results for NN Phase-shifts and Mixing Parameters
- 7. Hamiltonian for n-subtracted Three-Body Theory
- 8. Final Remarks

INTRODUCTION

The renormalization group program followed by K. Wilson and collaborators is of particular interest as it allows one to parameterize

- the physics of the high momentum states and work with effective degrees of freedom. The main idea is to use an effective renormalized Hamiltonian that, in the interaction between low-momentum states, includes the coupling with high momentum states.
- The renormalized Hamiltonian carries the physical information contained in the quantum system in states of high momentum.

- As an example, in the nuclear physics context, the use of effective interactions containing singularities at short distances is motivated by the development of a chirally symmetric nucleon-nucleon interaction, which contains contact interactions (Dirac-delta and its higher order derivatives)

- Singular contact interactions have also been considered in specific treatments of scaling limits and correlations between low-energy observables of three- and four-body systems (atomic and nuclear)

(See, for example, Amorim et al [PRC56(1997)R2378; PRA60(1999)R9] and Hadizadeh et al [PRL107(2011)135304; PRA85(2012)023610]).

Renormalized fixed-point Hamiltonians

- Renormalized fixed-point Hamiltonians are formulated for systems described by interactions that originally contain point-like singularities (as Dirac-delta and/or its derivatives).
- We consider a renormalization scheme for few-nucleon interactions, relying on a subtracted T-matrix equation.
- The fixed-point Hamiltonian, which is Hermitian, contains the renormalized coefficients/operators that carry the physical informations of the quantum mechanical system, as well as all the necessary counterterms that make finite the scattering amplitude.
- It is also behind the renormalization group invariance of quantum mechanics.
- Renormalization group techniques, Callan-Symanzik equation, scale invariance and universality are discussed in this context.

Nucleon-nucleon (NN) system - Conventional approaches

Standard high-precision NN potentials:

Bonn 2000, CD Bonn, Av18, Nijm I,II, ...

Common features:

- Long-range part due to One-Pion-exchange
- Short-range pieces, modeled phenomenologically,

  Describe the existing NN data
- Have typically 40-50 parameters

Effective Field Theory

- Identify the relevant degrees of freedom and symmetries
- Construct the most general Lagrangian consistent with
- Do standard quantum field theory with this Lagrangian.

S.Weinberg, Physica A96 (79) 327

Effective NN Interaction

We consider the following effective NN interaction:

$$V_{EFT}(p',p) = V_\pi^{reg}(p',p) + \sum_{i,j=0}^{1} \lambda_{ij} p'^{2i} p^{2j}$$

$$= \underbrace{V_\pi^{reg}(p',p) + \lambda_{00}}_{V_{\pi+\delta}} + \underbrace{\lambda_{01}p'^2 + \lambda_{10}p^2 + \lambda_{11}p'^2p^2}_{V_{\delta'}}$$

$V_{\pi+\delta}$: Regular part of OPEP + V_δ

V_δ : Dirac-δ Contact Interaction

$V_{\delta'}$: Derivative Contact Interactions

How to treat the singular interaction?

Subtractive Renormalization Method

From the original T-matrix equation

$$T(E) = V + VG_0^{(+)}(E)T(E)$$

$$= [1 - VG_0^{(+)}(E)]^{-1}V \qquad G_0^{(+)}(E) = \frac{1}{(E + i\varepsilon - H_0)}$$

We simply need to:

→ replace V by $T(-\mu^2)$ and

→ multiply the free propagator by a μ-dependent function such that

$$T(E) = T(-\mu^2) + T(-\mu^2)G_R^{(+)}(E; -\mu^2)T(E)$$

where

$$G_R^{(+)}(E; -\mu^2) \equiv G_0^{(+)}(E) - G_0(-\mu^2)$$

$$= \frac{(\mu^2 + E)}{(\mu^2 + H_0)}G_0^{(+)}(E)$$

Partial Wave Decomposition of OPEP

The one-pion-exchange potential is given by:

$$\langle \vec{p}'|V_\pi|\vec{p}\rangle = -\frac{g_a^2}{4(2\pi)^3 f_\pi^2}\vec{\tau}_1\cdot\vec{\tau}_2\frac{\vec{\sigma}_1\cdot(\vec{p}'-\vec{p})\;\vec{\sigma}_2\cdot(\vec{p}'-\vec{p})}{(\vec{p}'-\vec{p})^2+m_\pi^2}$$

We normalize our basis for the partial wave decomposition expressing the plane-wave as

$$|\vec{p};sm_s;I\rangle = \sqrt{\frac{2}{\pi}}\sum_{lsjm_j;I}|p;ls;jm_j\rangle|I\rangle\left[\mathcal{Y}_{ls}^{jm_j}(\hat{p})\right]^\dagger|sm_s\rangle$$

The partial wave decomposition of OPEP in the $^1S_0$ state is

$$V_\pi(p',p) \quad = \quad \frac{g_a^2}{16\pi f_\pi^2}+V_\pi^{reg}(p',p)$$

with a regular part

$$V_\pi^{reg}(p',p) \quad = \quad -\frac{g_a^2}{32\pi f_\pi^2}\int_{-1}^{1}dx\frac{m_\pi^2}{p^2+p'^2-2pp'x+m_\pi^2}$$

Subtracted Equations with only $V_{\pi+\delta}$

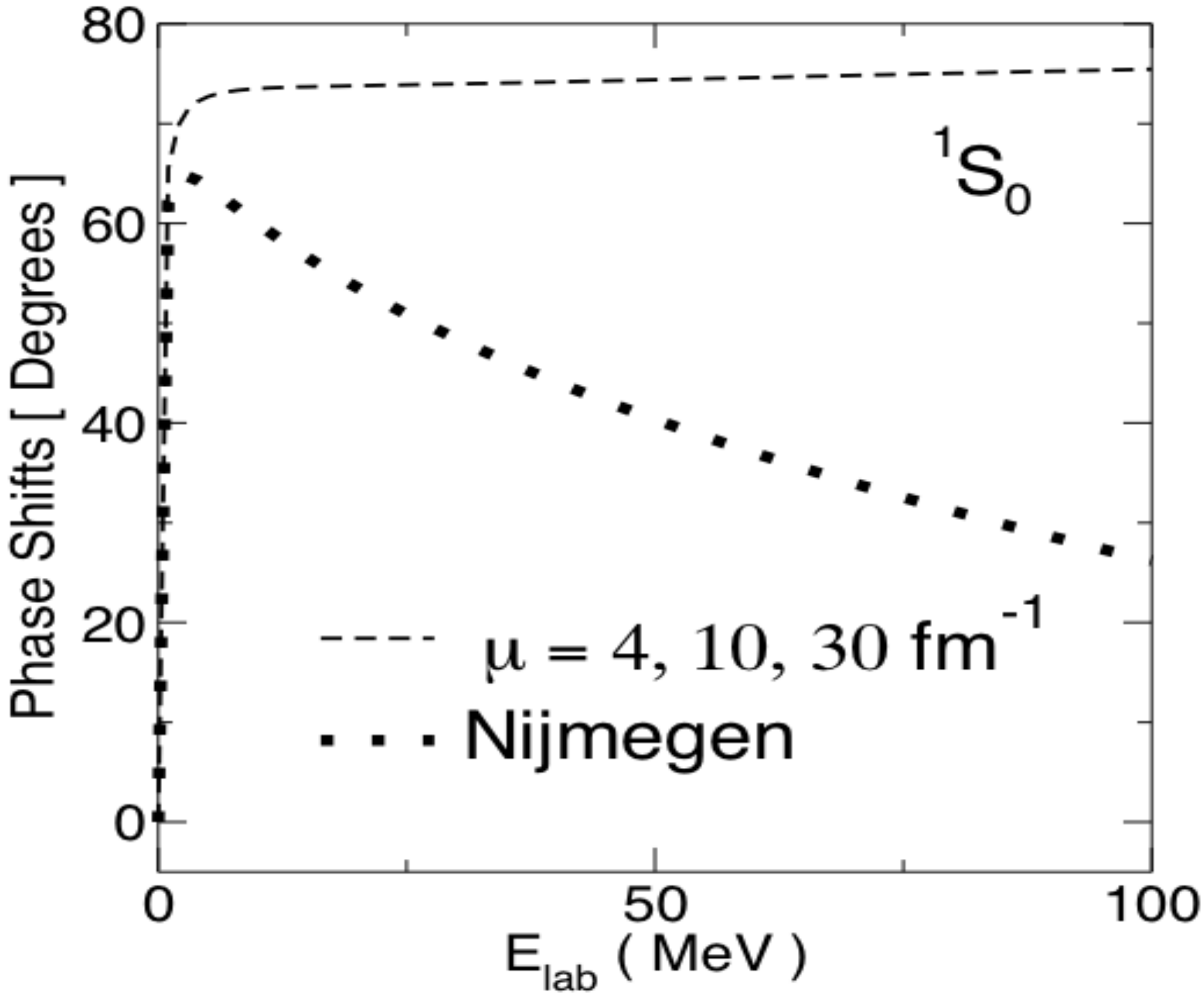
Singlet $^1S_0$:

$$T_s^{(00)}(p',p;k^2) = T_s^{(00)}(p',p;-\mu^2) + \frac{2}{\pi}\int_0^\infty dq q^2 \left(\frac{\mu^2+k^2}{\mu^2+q^2}\right)\frac{T_s^{(00)}(p',q;-\mu^2)}{k^2-q^2+i\epsilon} T_s^{(00)}(p',p;k^2)$$

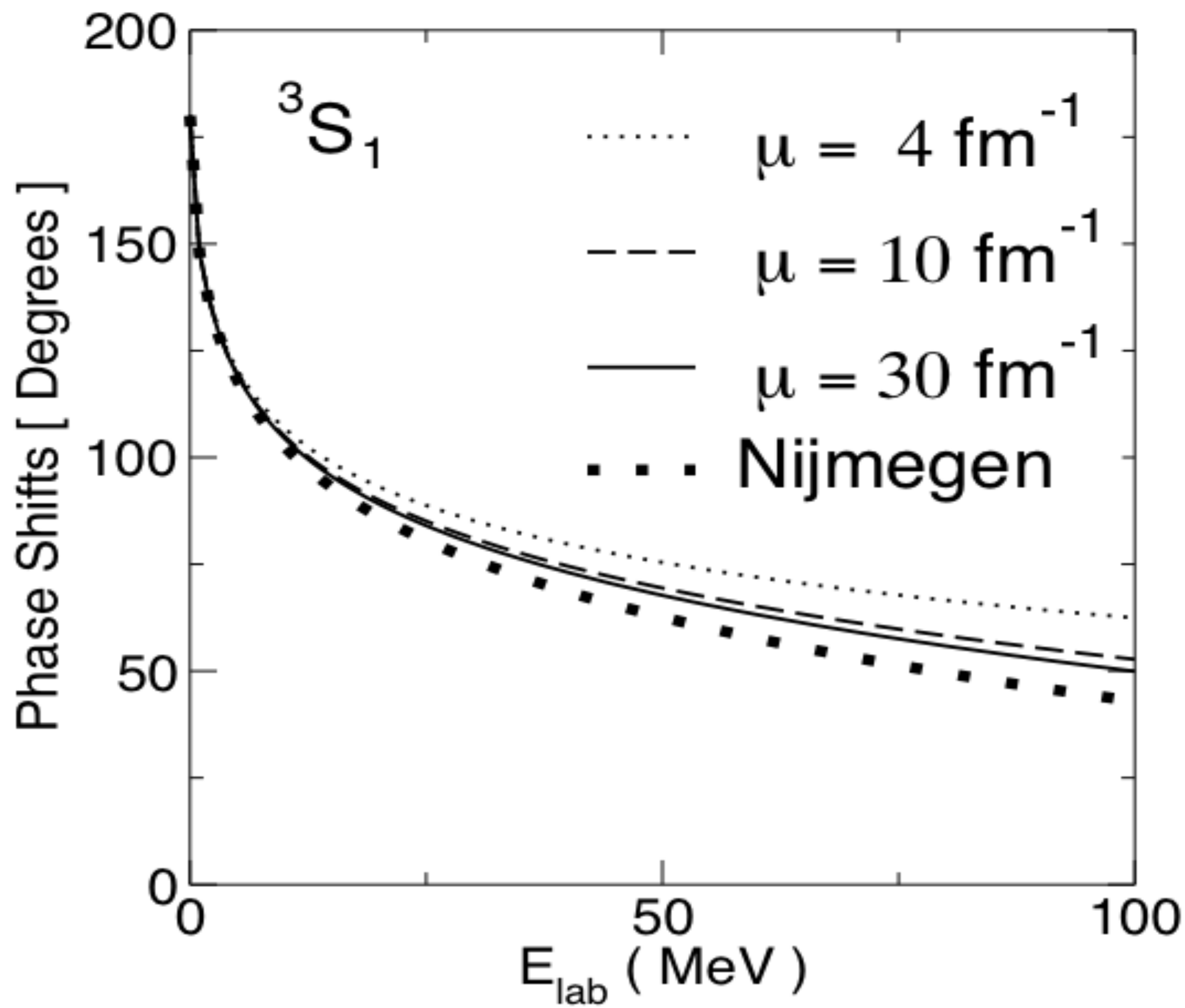
Coupled-channel $^3S_1$-$^3D_1$

$$T_t^{(l_1 l_2)}(p',p;k^2) = T_t^{(l_1 l_2)}(p',p;-\mu^2) + \frac{2}{\pi}\sum_{l_3}\int_0^\infty dq q^2 \left(\frac{\mu^2+k^2}{\mu^2+q^2}\right)\frac{T_t^{(l_1 l_3)}(p',q;-\mu^2)}{k^2-q^2+i\epsilon} T_t^{(l_3 l_2)}(p',p;k^2)$$

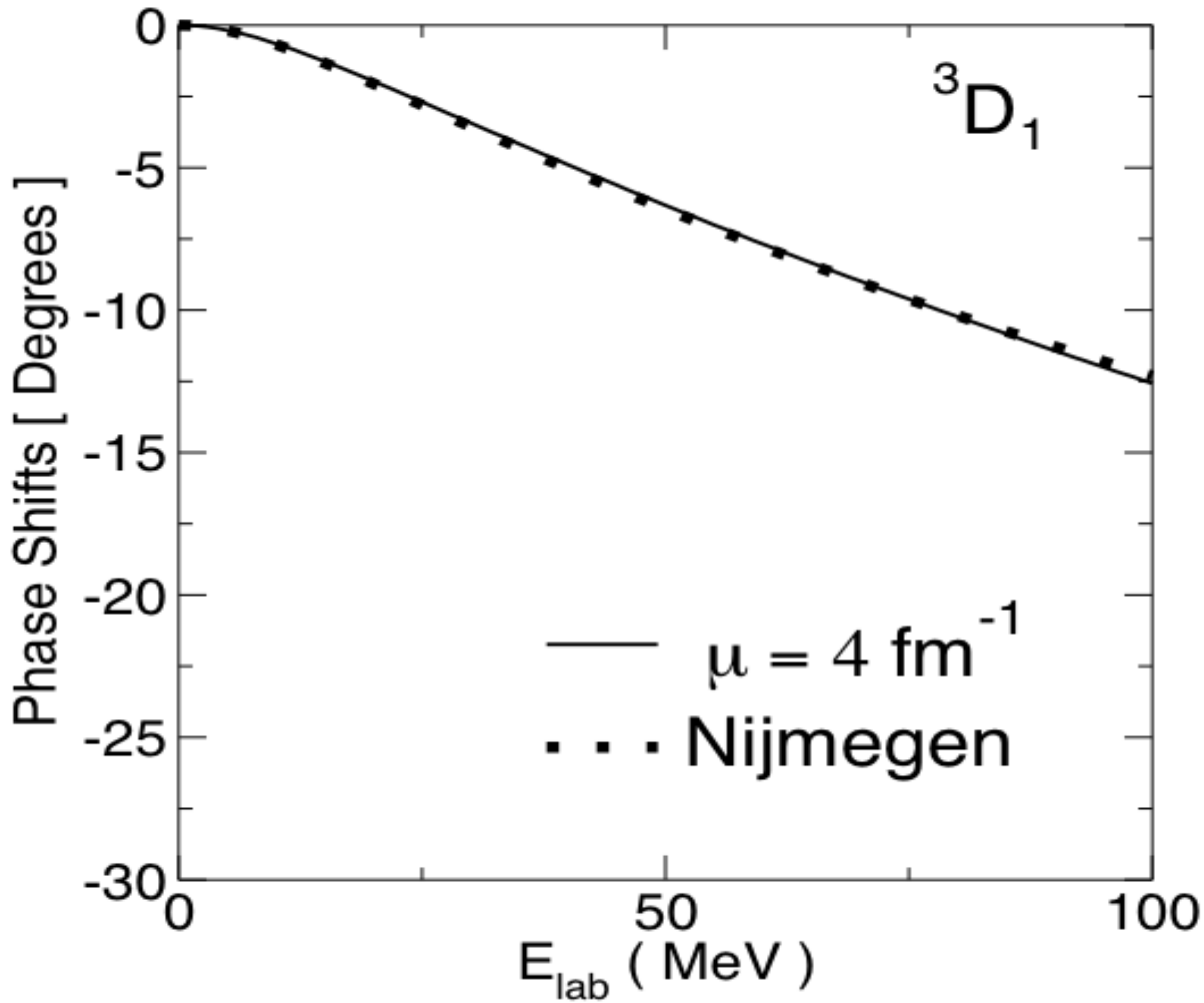
$^1S_0$ phase shifts with only $V_{\pi+\delta}$



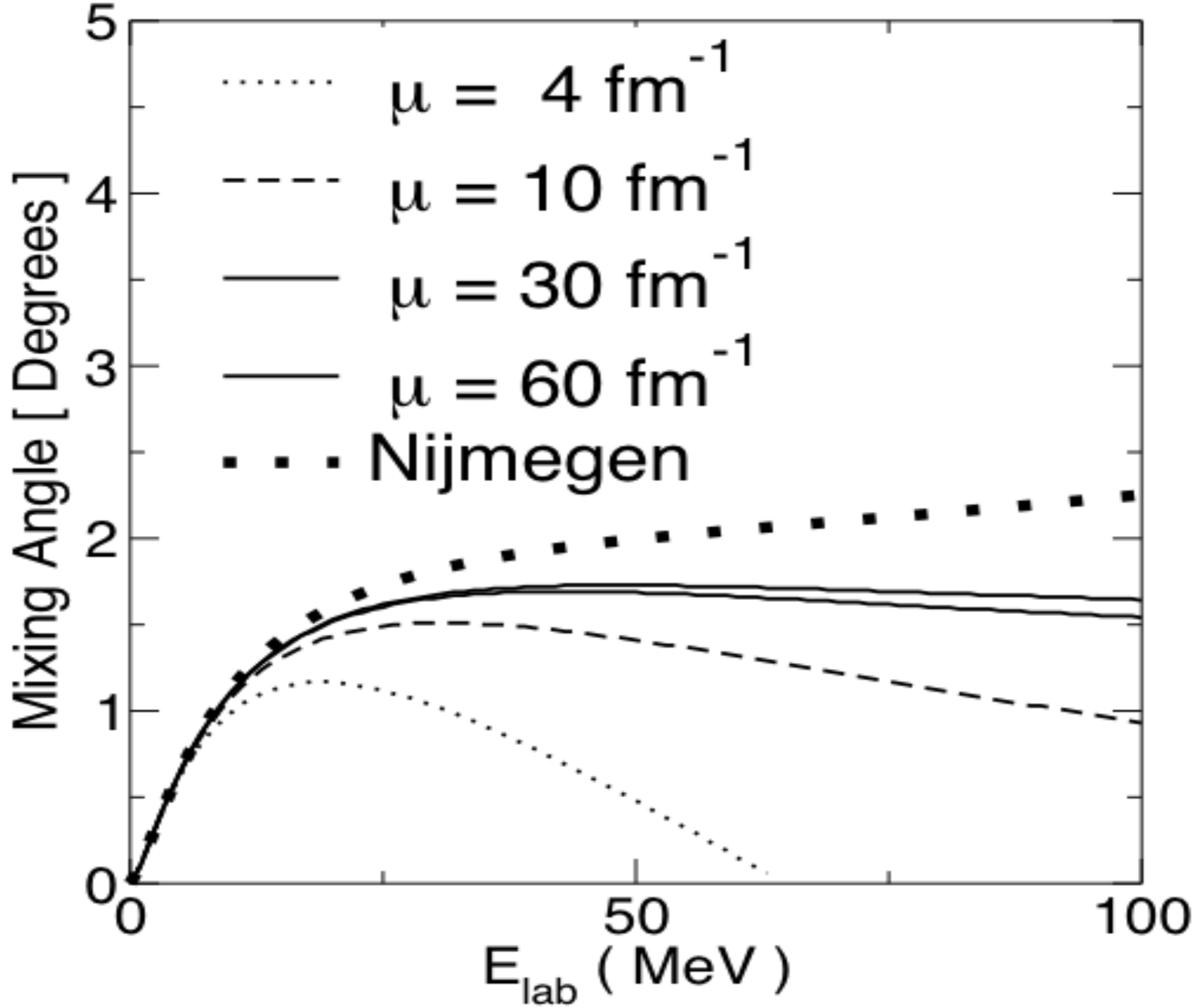
$^3S_1$ phase shifts with only $V_{\pi+\delta}$



$^3D_1$ phase shifts with only $V_{\pi+\delta}$



ε_1 mixing angle with only $V_{\pi+\delta}$



Features of $V_{\pi+\delta}$

- Reasonable agreement for the coupled channel, where the pion dominates.

- Only one subtraction is enough to obtain a finite T-matrix.

- Poor description of the singlet state. Need next order in NN interaction. More subtractions required.

Renormalized Hamiltonian

- The renormalized Hamiltonian is the sum of the free Hamiltonian with the renormalized interaction:

$$H_R = H_0 + V_R$$

$$T_R(E) = V_R + V_R G_0^{(+)}(E)\, T_R(E)$$
$$G_0^{(+)}(E) = (E + i\varepsilon - H_0)^{-1}$$

$$V_R = T_R(-\mu^2) \Sigma_n [-G_0(-\mu^2)\, T_R(-\mu^2)]^n$$
$$= T_R(-\mu^2)/[1+G_0(-\mu^2)\, T_R(-\mu^2)]$$
$$= \{1/[1+T_R(-\mu^2) G_0(-\mu^2)]\}\, T_R(-\mu^2)$$

Subtracted T-matrix Equations

The $n-th$ order subtracted equation is given by:

$$T(E) = V^{(n)}(-\mu^2; E) + V^{(n)}(-\mu^2; E) G_n^{(+)}(E; -\mu^2) T(E)$$

$$V^{(n)} \equiv \left[1 - (-\mu^2 - E)^{n-1} V^{(n-1)} G_0^n(-\mu^2)\right]^{-1} V^{(n-1)}$$

$$G_n^{(+)} \equiv \left[(-\mu^2 - E) G_0(-\mu^2)\right]^n G_0^{(+)}(E)$$

Since we need 3 subtractions, we have

$$T(p', p; k^2) = V^{(3)}_{\pi+\delta+\delta'}(p', p; -\mu^2; k^2)$$

$$+\frac{2}{\pi}\int_0^\infty dq q^2 \left(\frac{\mu^2 + k^2}{\mu^2 + q^2}\right)^3 \frac{V^{(3)}_{\pi+\delta+\delta'}(p', q; -\mu^2; k^2)}{k^2 - q^2 + i\epsilon} T(q, p; k^2)$$

$$V^{(3)}_{\pi+\delta+\delta'} = V^{(3)}_{\pi+\delta} + \lambda_{\mathcal{R}10}(p'^2 + p^2) + \lambda_{\mathcal{R}11} p'^2 p^2$$

Subtracted T-matrix Equations

The integral equations for $V_{\pi+\delta}^{(n)}$ are

$$V_{\pi+\delta}^{(n)} = V_{\pi+\delta}^{(n-1)} + \frac{2}{\pi}\int_0^\infty dq q^2 \left(\frac{\mu^2+k^2}{\mu^2+q^2}\right)^{n-1} \frac{V_{\pi+\delta}^{(n-1)}}{-\mu^2-q^2} V_{\pi+\delta}^{(n)}$$

The T-matrix of the OPE plus the δ potential is obtained using Distorted Wave Theory:

$$\begin{aligned} T_{\pi+\delta}(E) &= T_\pi(E) + \left[1 + T_\pi(E) G_0^{(+)}(E)\right] \\ &\times T_\delta(E)\left[1 + G_0^{(+)}(E) T_\pi(E)\right] \end{aligned}$$

with the singular T-matrix being solution of

$$T_\delta(E) = V_\delta + V_\delta G_\pi^{(+)}(E) T_\delta(E)$$

The Green's function for the regular part of OPE is

$$G_\pi^{(+)}(E) = G_0^{(+)}(E) + G_0^{(+)}(E) T_\pi(E) G_0^{(+)}(E)$$

Subtracted T-matrix Equations

Renormalization is also required to obtain $T_\delta(E)$. But in this case only 1 subtraction is enough

$$\begin{aligned} T_\delta(E) &= T_\delta(-\mu^2) \\ &+ T_\delta(-\mu^2)\left[G_\pi^{(+)}(E) - G_\pi(-\mu^2)\right] T_\delta(E) \end{aligned}$$

The renormalized strength of the δ interaction defines $T_\delta(E)$ at the subtraction point

$$T_\delta(-\mu^2) = \lambda_{\mathcal{R}00}$$

The result is

$$\begin{aligned} & T_{\pi+\delta}(p', p\,; -\mu^2) = T_\pi(p', p\,; -\mu^2) \\ + & \left[1 + \frac{2}{\pi}\int_0^\infty dq\ q^2 \frac{T_\pi(p', q; -\mu^2)}{-\mu^2 - q^2}\right] \\ \times & \lambda_{\mathcal{R}00} \\ \times & \left[1 + \frac{2}{\pi}\int_0^\infty dq'\ q'^2 \frac{T_\pi(q', p; -\mu^2)}{-\mu^2 - q'^2}\right] \end{aligned}$$

Renormalization Group Invariance

→ Observables are invariant under the change of the subtracion energy scale μ^2

→ The driving term $V^{(n)}$ has to be modified in order to keep T invariant

The rule to modify $V^{(n)}$ appears in the form of a non-relativistic Callan-Symanzik equation:

$$\frac{\partial V^{(n)}}{\partial \mu^2} = -V^{(n)} \frac{\partial G_n^{(+)}}{\partial \mu^2} V^{(n)}$$

which is derived from

$$\frac{\partial T(E)}{\partial \mu^2} = 0$$

Model results with OPE, NLO and NNLO

$$V_{\rm NLO}(p,p') = V^{\rm NLO}_{\rm TPE}(p,p') + \lambda_1(pp')\delta_{L,1}\delta_{L',1} + (\lambda_2(p^2+p'^2) + \lambda_3(p^2p'^2))\delta_{L,0}\delta_{L',0} + \lambda_4(p^2\delta_{L,2}\delta_{L',0} + p'^2\delta_{L',2}\delta_{L,0}) \,,$$

$$V^{\rm NLO}_{\rm TPE}(\vec{p},\vec{p}') = -\left(\frac{\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{384\pi^2 f_\pi^4}\right)\frac{L(q)}{(2\pi)^3}\left\{4M_\pi^2\left(5g_A^4 - 4g_A^2 - 1\right) + q^2\left(23g_A^4 - 10g_A^2 - 1\right) + \frac{48g_A^4M_\pi^4}{4M_\pi^2+q^2}\right\} - \left(\frac{3g_A^4}{64\pi^2 f_\pi^4}\right)L(q)\{(\vec{\sigma}_1\cdot\vec{q})(\vec{\sigma}_2\cdot\vec{q}) - q^2\,\vec{\sigma}_1\cdot\vec{\sigma}_2\}.$$

For the NNLO chiral potential, we adop a momentum space form, which is explicitly given by Epelbaum in Prog.Part.Nucl.Phys.57(2006)654. See also in PRC 83(2011)064005.

Numerical Results

For each set of $\lambda_{\mathcal{R}10}$, $\lambda_{\mathcal{R}11}$ and μ, we fit the singlet scattering length $a_s = -23.739$ fm through the value of $\lambda_{\mathcal{R}00}$. With $\mu = 214$ MeV, the two parameters left are adjusted to reproduce the Nijgemen data up to the center of mass momentum of $k = 300$ MeV/c .

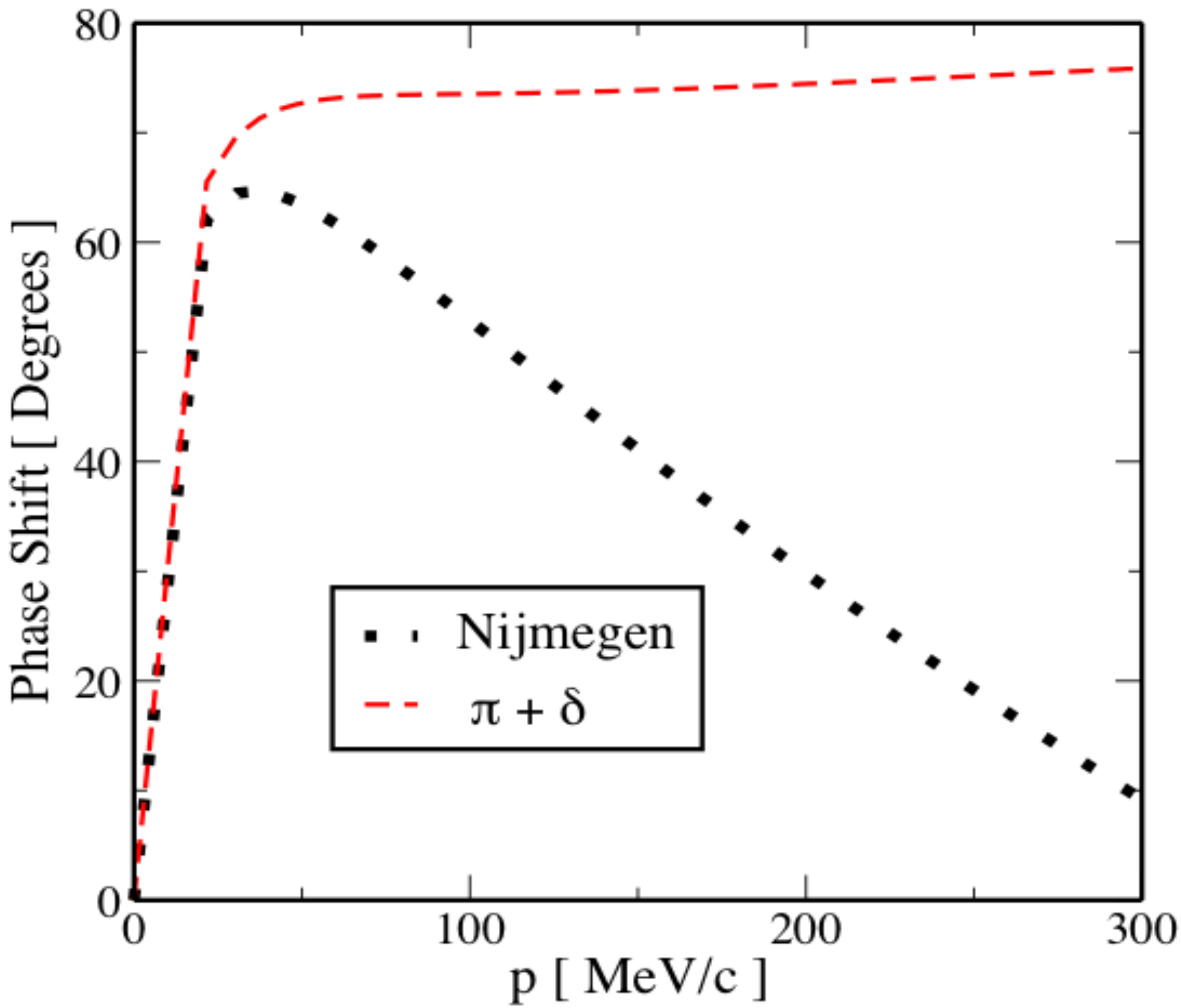
First, as a straightforward check of our method, we obtain the singlet S-wave phase shifts for $V_{\pi+\delta}$ obtained by solving the three-fold equation with $\lambda_{\mathcal{R}10} = \lambda_{\mathcal{R}11} = 0$. The present calculation reproduces the results obtained with the one-subtracted equation.

$$\mu = 214 MeV$$
$$\mu\lambda_{R00} = -8.8395$$
$$\lambda_{R01} = 0$$
$$\lambda_{R11} = 0$$

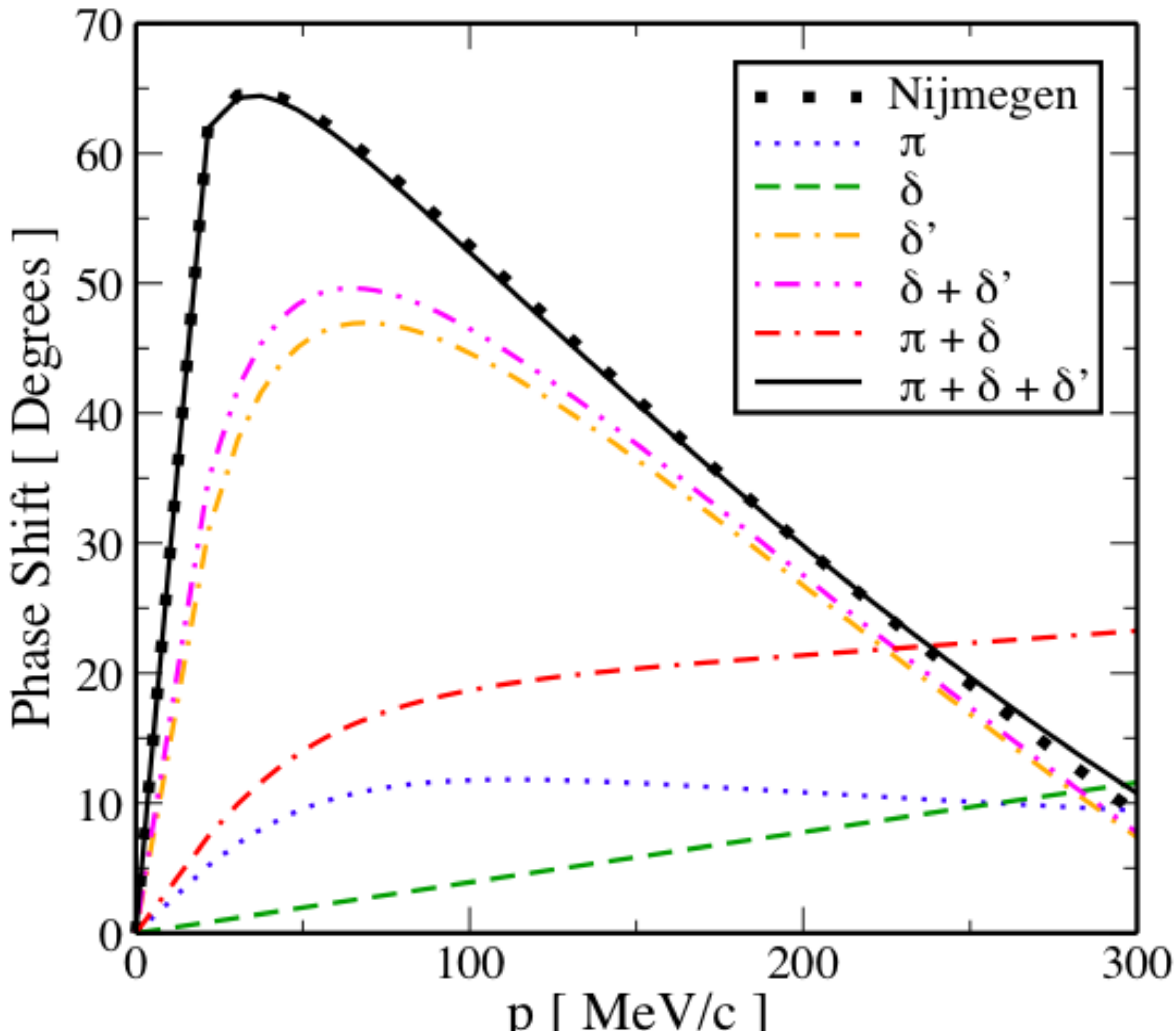
$V_{\pi+\delta}$

$$\mu = 214 MeV$$
$$\mu\lambda_{R00} = -0.1465$$
$$\mu^3\lambda_{R01} = 4.7124$$
$$\mu^5\lambda_{R11} = 5.0265$$

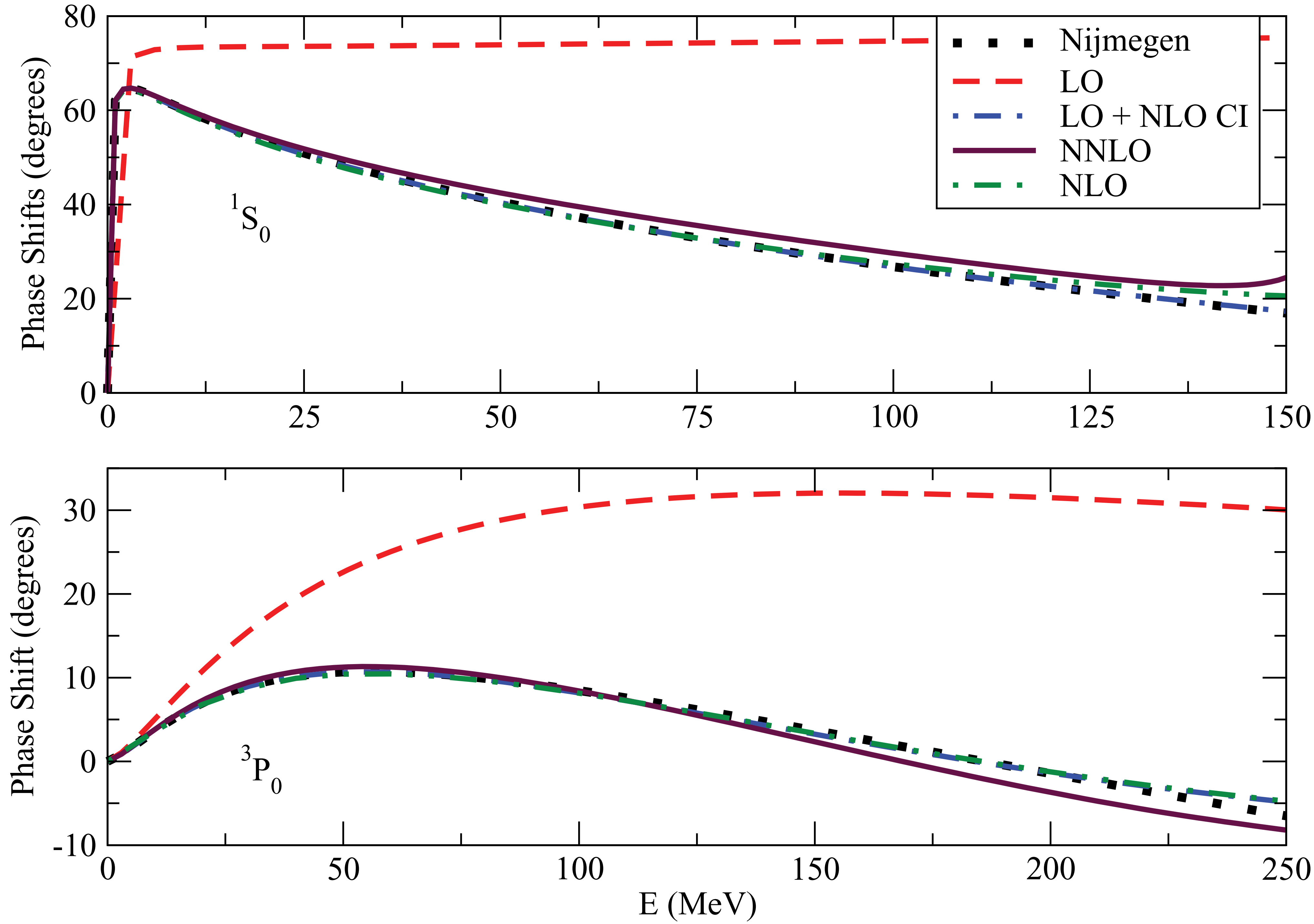
${}^1S_0$ phase shifts with only $V_{\pi+\delta}$



Contributions to the $^1S_0$ phase shifts

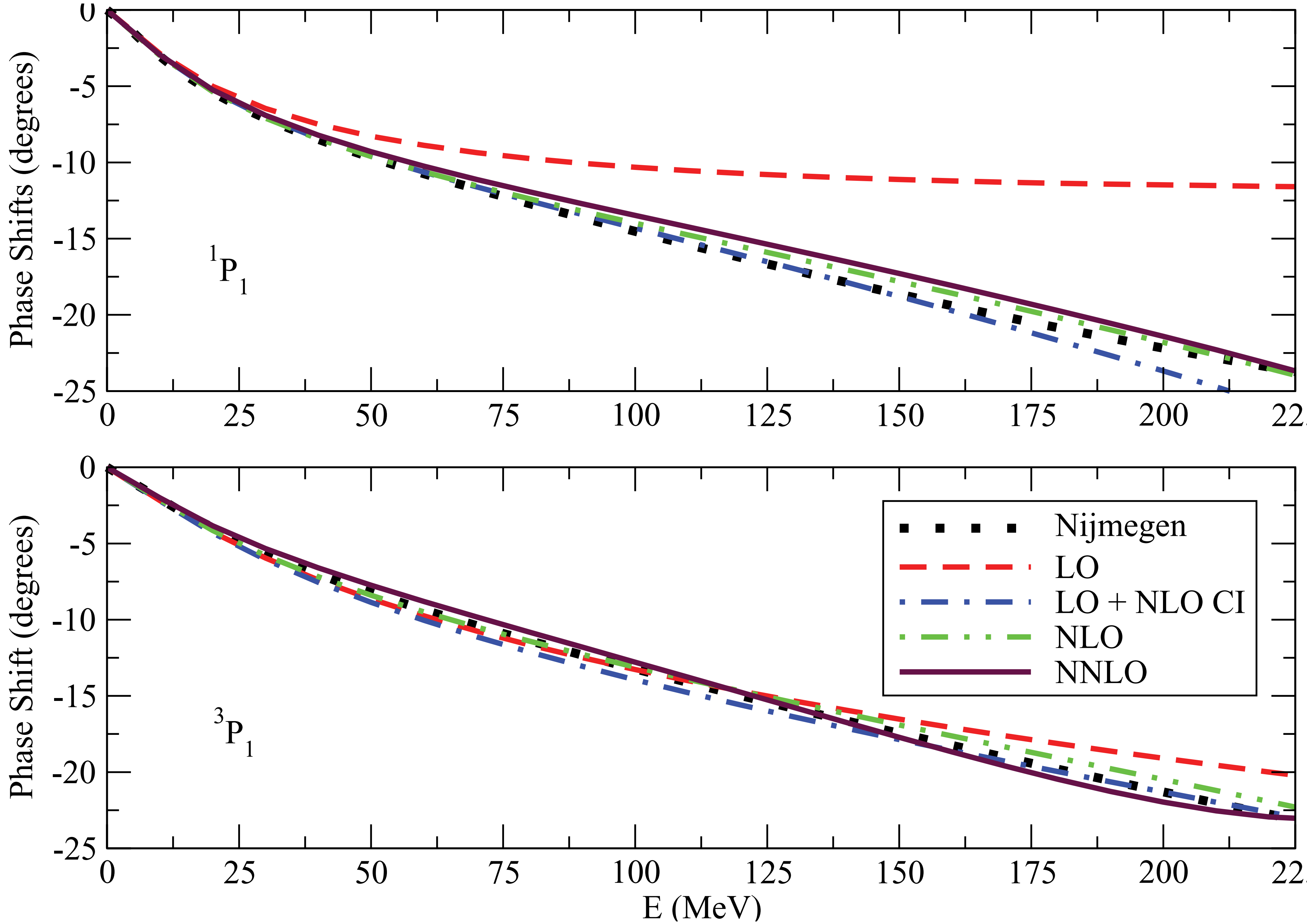


Results for NN Phase-shifts and Mixing Parameters



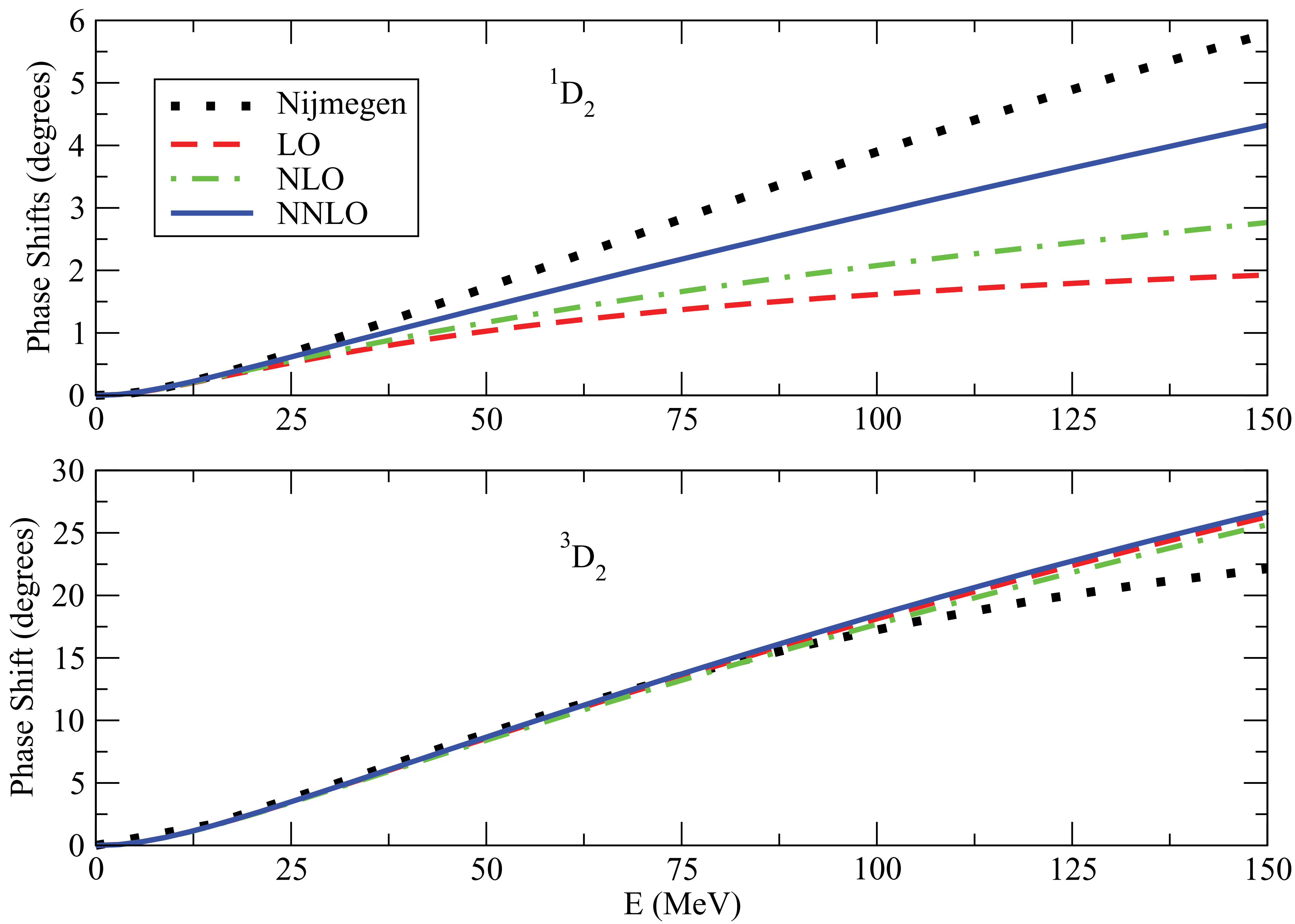
Phase-shifts for the $^1S_0$ wave, with subtracted point at -50 MeV; and, for $^3P_0$ wave with subtraction point at -100 MeV.

Results for NN Phase-shifts and Mixing Parameters



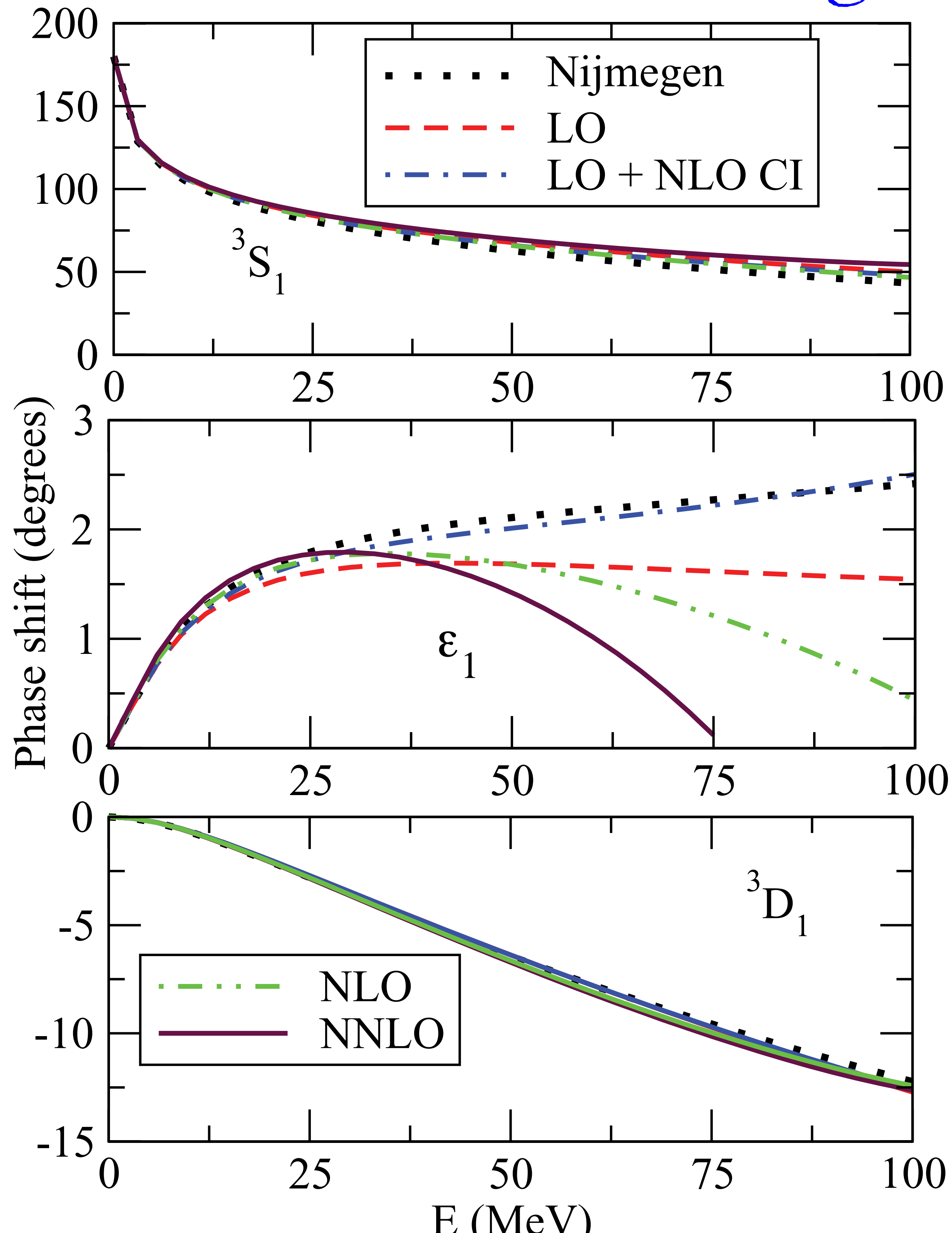
Phase-shifts for the $^1P_1$ and $^3P_1$ waves.

Results for NN Phase-shifts and Mixing Parameters



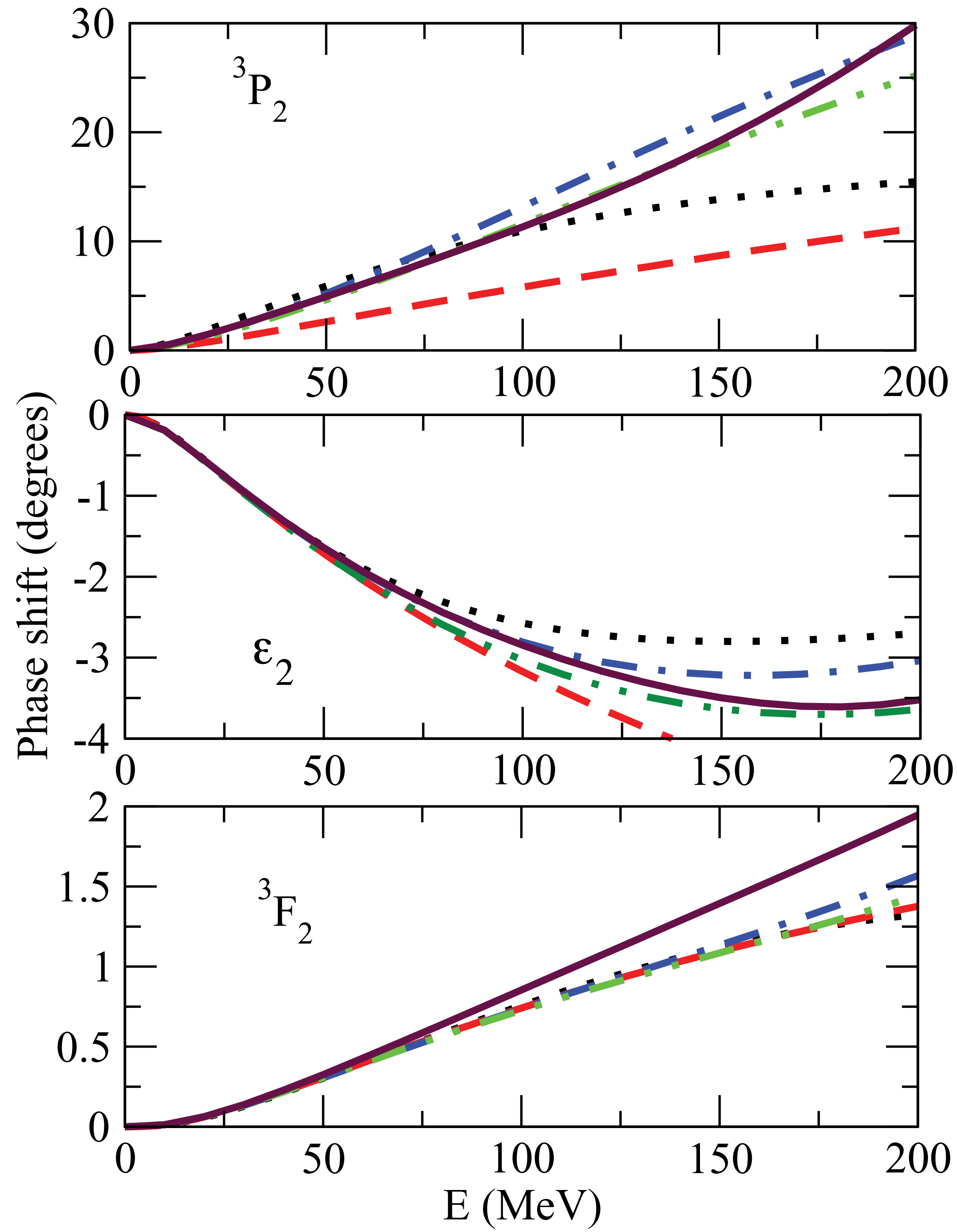
Phase-shifts for the $^1D_2$ and $^3D_2$ waves.

Results for NN Phase-shifts and Mixing Parameters



Phase-shifts and mixing parameter for the $^3S_1$-$^3D_1$ coupled channels.

Results for NN Phase-shifts and Mixing Parameters



Phase-shifts and mixing parameter for the $^3P_2$-$^3F_2$ coupled channels.

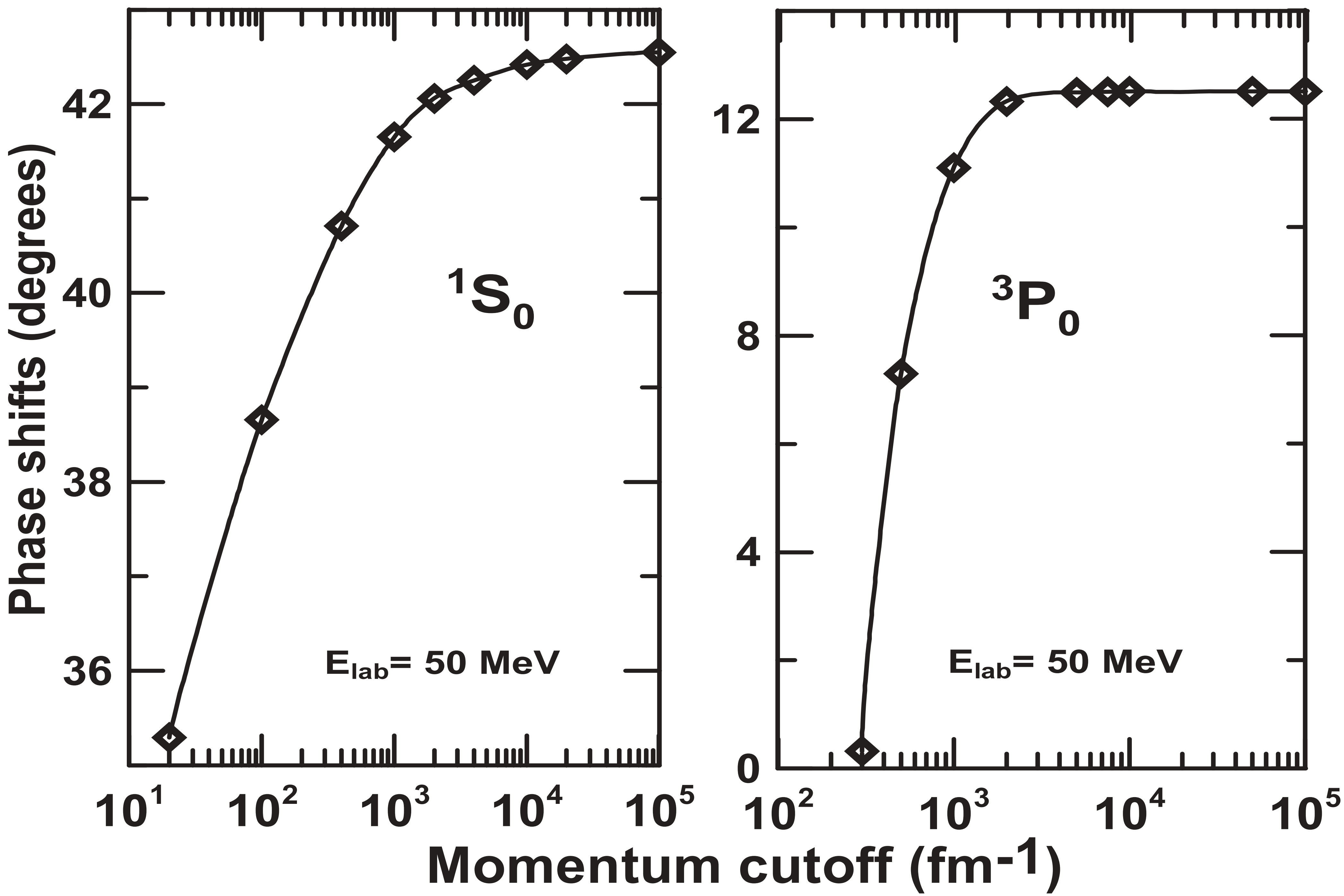


Illustration of the cut-off independence of the method, for two singular cases, by considering NNLO potential with n=4 subtracted scattering equation. Our results were obtained for infinite cut-offs, within the subtractive renormalization approach.

[PRC83 (2011) 064005]

Hamiltonian for Subtracted 3B equations

n-subtracted T-matrix equation (for Dirac-delta n=1)

$$T(E) = V^{(n)}(E, -\mu^2) + (-1)^n (E + \mu^2)^n V^{(n)}(E, -\mu^2) G_0^{(+)}(E) G_0^n(-\mu^2) T(E)$$

Invariance of T-matrix by dislocations of the subtraction point:

$$\frac{\partial V^{(n)}}{\partial \mu^2} = -V^{(n)} \frac{\partial G_n^{(+)}(E; -\mu^2)}{\partial \mu^2} V^{(n)}$$

Renormalized Hamiltonian:

$$H_{\mathcal{R}} = H_0 + V_{\mathcal{R}}$$

$$V_{\mathcal{R}} = [1 + V^{(n)} G_0^{(+)}(E)\,(1 - (-1)^n (\mu^2 + E)^n G_0^n(-\mu^2))]^{-1} V^{(n)}$$

$$\frac{\partial V_{\mathcal{R}}}{\partial \mu^2} = 0 \quad \text{and} \quad \frac{\partial H_{\mathcal{R}}}{\partial \mu^2} = 0$$

Subtracted-Faddeev equations 3B:

$$T_k(E) = t_{(ij)}\left(E - \frac{q_k^2}{2m_{ij,k}}\right)[1 + (G_0^{(+)}(E) - G_0(-\mu_3^2))\,(T_i(E) + T_j(E))]$$

$$H_{\mathcal{R}I}^{(3B)} = \sum_{(ij)} V_{\mathcal{R}(ij)}^{(2B)} + V_{\mathcal{R}}^{(3B)}.$$

Frederico, Delfino, Tomio, Yamashita PPNP 67, 939 (2012)

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Final Remarks

- The scheme works very well, with a T-matrix and Hamiltonian formalism, which doesn't depend on the subtraction point μ^2.
- V_δ is the component of the effective interaction which dominates in the $^1S_0$ channel.
- Next orders are included in the effective interaction (more subtractions may be required).
- The calculations can be extended to higher partial waves.
- The singlet and triplet scattering lengths are given to fix the renormalized strengths of the contact interactions.
- Very good agreement with neutron-proton data, particularly for the triplet. Mixing parameter for $^3S_1$-$^3D_1$ is shown to be the most sensible observable related to the renormalization scale.
- **Rule to modify the driving term follows a non-relativistic Callan-Symanzik equation (Group invariance).**

My thanks to my collaborators

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Antonio Delfino (UFF, Niterói)

Varese S. Timóteo (Unicamp, Limeira)

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Thanks to the organizers, for the invitation;
and
All of you to attend!

Backup slides

Parameters LO and NLO

TABLE I. Strengths of the LO contact interactions, which reproduce the scattering lengths for the S waves. The values of $\lambda_0^{^1S_0}$ and $\lambda_0^{^3S_1}$, in units of fm, are given at the energy scale $-\bar{\mu}^2$, with $\bar{\mu} = 30\ \mathrm{fm}^{-1}$ ($\bar{\mu}^2 = 41.47 \times 900$ MeV).

Strengths	$^1S_0$	$^3S_1$
λ_0 (fm)	−0.0203	−0.24142

TABLE II. Strengths of the contact interactions for the fits with the LO potential plus the NLO contact interactions. The values of $\lambda_0^{^1S_0}$ and $\lambda_0^{^3S_1}$ are given at the same energy scale as in Table I ($-\bar{\mu}^2 = -41.47 \times 900$ MeV); with $\lambda_2^{^1S_0}$ and $\lambda_3^{^1S_0}$ at $-\mu^2 = -50$ MeV. The other strengths are given at $-\mu^2 = -100$ MeV.

Strengths	$^1S_0$	$^3P_0$	$^3S_1$	$^1P_1$	$^3P_1$	$^3P_2$	ϵ_1
λ_0 (fm)	−0.0165	–	−0.2480	–	–	–	–
λ_1 (fm^3)	–	0.25	–	0.04	0.007	−0.07	–
λ_2 (fm^3)	2.2660	–	0.1	–	–	–	–
λ_3 (fm^5)	2.0047	–	–	–	–	–	–
λ_4 (fm^3)	–	–	–	–	–	–	0.001

TABLE III. Strengths of the contact interactions for the fits with the full NLO potential. The values of the λ's are given for $\bar{\mu}^2$ and μ^2 as in Table II.

Strengths	$^1S_0$	$^3P_0$	$^3S_1$	$^1P_1$	$^3P_1$	$^3P_2$	ϵ_1
λ_0	−0.0190	–	−0.1602	–	–	–	–
λ_1 (fm^3)	–	0.37	–	0.063	−0.078	−0.04	–
λ_2 (fm^3)	2.2660	–	0.1	–	–	–	–
λ_3 (fm^5)	2.0047	–	–	–	–	–	–
λ_4 (fm^3)	–	–	–	–	–	–	0.17