Precise determination of πN scattering and the $\sigma_{\pi N}$

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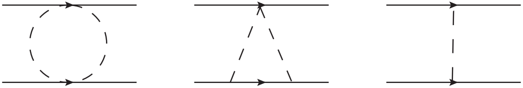
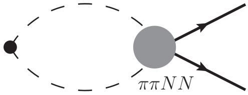
Motivation: Why πN scattering?

- **Low energies:** test chiral dynamics in the baryon sector
  $\Rightarrow$ low-energy theorems e.g. for the scattering lengths
- **Higher energies:** resonances, baryon spectrum
- **Input for NN scattering:** LECs c_i, πNN coupling



- **Crossed channel $\pi\pi \to \bar{N}N$:** nucleon form factors



$\Rightarrow$ probe the structure of the nucleon

- spectral functions of form factors
- vector form factors (P-waves)
- scalar form factors (S-waves)

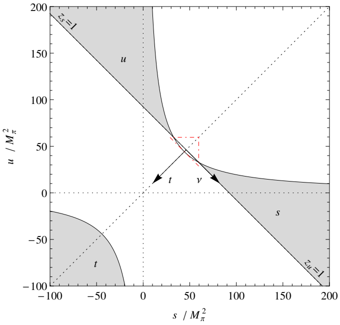
The pion-nucleon σ-term

Scalar form factor of the nucleon:

$\sigma(t) = \langle N(p')|\hat{m}\left(\bar{u}u + \bar{d}d\right)|N(p)\rangle \qquad t = (p'-p)^2 \qquad \sigma_{\pi N} = \sigma(0)$

- $\sigma_{\pi N}$ measures the **light-quark contribution** to the nucleon mass
- Unfortunately, no direct experimental access to it
- Linked to πN via the **Cheng-Dashen** theorem

[Cheng, Dashen 1971]



$$\underbrace{F_\pi^2 \bar{D}^+(\nu=0, t=2M_\pi^2)}_{F_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D} = \underbrace{\sigma(2M_\pi^2)}_{\sigma_{\pi N} + \Delta_\sigma} + \Delta_R$$

$|\Delta_R| \lesssim 2$ MeV [Bernard, Kaiser, Meißner 1996]

$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)$ MeV [Hoferichter et al. 2012]

Phenomenological status

- **Karlsruhe/Helsinki** partial-wave analysis KH80 Höhler et al. 1980s
  $\hookrightarrow$ comprehensive analyticity constraints, old data
- Formalism for the extraction of $\sigma_{\pi N}$ via the **Cheng–Dashen low-energy theorem** Gasser, Leutwyler, Locher, Sainio 1988, Gasser, Leutwyler, Sainio 1991
  $\hookrightarrow$ "canonical value" $\sigma_{\pi N} \sim 45$ MeV, based on KH80 input
- **GWU/SAID** partial-wave analysis Pavan, Strakovsky, Workman, Arndt 2002
  $\hookrightarrow$ much larger value $\sigma_{\pi N} = (64 \pm 8)$ MeV
- More recently: ChPT in different regularizations (w/ and w/o Δ) Alarcón et al. 2012
  $\hookrightarrow$ fit to PWAs, $\sigma_{\pi N} = 59 \pm 7$ MeV

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- More recently: ChPT in different regularizations (w/ and w/o Δ) Alarcón et al. 2012
  $\hookrightarrow$ fit to PWAs, $\sigma_{\pi N} = 59 \pm 7$ MeV
- This talk: two new sources of information on low-energy πN scattering
  - Precision extraction of πN **scattering lengths** from **hadronic atoms** [Baru et al. 2011]
  - **Roy-equation** constraints: analyticity, unitarity, crossing symmetry

Hadronic atoms: constraints for πN

- $\pi H/\pi D$: bound state of π^- and p/d, spectrum sensitive to threshold πN amplitude
- Combined analysis of πH and πD:

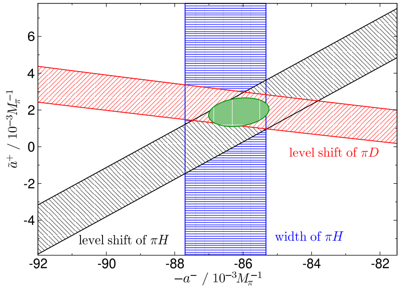
$$a_0^+ \equiv a^+ = (7.5 \pm 3.1) \cdot 10^{-3} M_\pi^{-1}$$
$$a_0^- \equiv a^- = (86.0 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$$

- Large a^+ suggests a large $\sigma_{\pi N}$, but

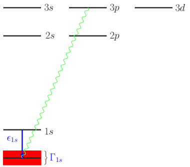
$$\frac{a_{\pi^- p} + a_{\pi^+ p}}{2} = (-1.1 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$$

$\hookrightarrow$ Isospin breaking in $\sigma_{\pi N}$ could be important

- We revisit the Cheng-Dashen low-energy theorem in the presence of isospin breaking



[Baru et al. 2011]



$$\tilde{a}^+ = a^+ + \frac{1}{1 + M_\pi/m_p} \left\{ \frac{M_\pi^2 - M_{\pi^0}^2}{\pi F_\pi^2} c_1 - 2\alpha f_1 \right\}$$

Goldberger-Miyazawa-Oehme sum rule

- Fixed-t dispersion relations at threshold $\hookrightarrow$ **GMO sum rule**

$$\frac{g^2}{4\pi} = \left(\left(\frac{m_p + m_n}{M_\pi}\right)^2 - 1\right)\left\{\left(1 + \frac{M_\pi}{m_p}\right)\frac{M_\pi}{4}\left(a_{\pi^- p} - a_{\pi^+ p}\right) - \frac{M_\pi^2}{2} J^-\right\}$$

$$= 13.76 \pm 0.12 \pm 0.15$$

$$J^- = \frac{1}{4\pi^2}\int_0^\infty dk \frac{\sigma^{\text{tot}}_{\pi^- p}(k) - \sigma^{\text{tot}}_{\pi^+ p}(k)}{\sqrt{M_\pi^2 + k^2}}$$

- J^- known very accurately Ericson et al. 2002, Abaev et al. 2007
- other determinations

	de Swart et al. 97	Arndt et al. 94	Ericson et al. 02	Bugg et al. 73	KH80
method	NN	πN	GM0	πN	πN
$g^2/4\pi$	13.54 ± 0.05	13.75 ± 0.15	14.11 ± 0.20	14.30 ± 0.18	14.28

- With KH80 scattering lengths $g^2/4\pi = 14.28$ MeV is reproduced exactly
  $\hookrightarrow$ discrepancy related to old scattering length values

Motivation: Why Roy-Steiner equations?

Roy(-Steiner) eqs. = Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **Respect all symmetries**: analyticity, unitarity, crossing
- **Model independent** $\Rightarrow$ the actual parametrization of the data is irrelevant once it is used in the integral.
- Framework allows for systematic improvements (subtractions, higher partial waves, ...)
- **PW(H)DRs** help to study processes with high precision:
  - $\pi\pi$-scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - πK-scattering: [Büttiker et al. (2004)]
  - $\gamma\gamma \to \pi\pi$ scattering: [Hoferichter et al. (2011)]

Warm up: Roy-equations for $\pi\pi$

- $\pi\pi \to \pi\pi \Rightarrow$ fully crossing symmetric in Mandelstam variables s, t, and $u = 4M_\pi - s - t$
- Start from twice-subtracted **fixed-t** DRs of the generic form

$$T^I(s,t) = c(t) + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2}\left[\frac{s^2}{(s'-s)} - \frac{u^2}{(s'-u)}\right] \mathrm{Im}T^I(s',t)$$

- Subtraction functions c(t) are determined via crossing symmetry functions of the I=0,2 scattering lengths: a_0^0 and a_0^2
- PW-expansion of these DRs yields the **Roy-equations** [Roy (1971)]

$$t_J^I(s) = ST_J^I(s) + \sum_{J'=0}^{\infty}(2J'+1)\sum_{I'=0,1,2}\int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s',s)\mathrm{Im}\, t_{J'}^{I'}(s')$$

- $K_{JJ'}^{II'}(s',s) \equiv$ kernels $\Rightarrow$ analytically known

Solving Roy-equations: flow information

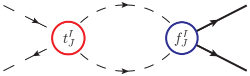
- **Roy-equations** rigorously valid for a finite energy range
  $\Rightarrow$ introduce a matching point s_m
- only partial waves with $J \leq J_{\text{max}}$ are solved
- assume isospin limit
- **Input**
  - High-energy region: $\text{Im} t_{IJ}(s)$ for $s \geq s_m$ and for all J
  - Higher partial waves: $\text{Im} t_{IJ}(s)$ for $J > J_{\text{max}}$ and for all s
- **Output**
  - Self-consistent solution for $\delta_{IJ}(s)$ for $J \leq J_{\text{max}}$ and $s_{\text{th}} \leq s \leq s_m$
  - Constraints on subtraction constants

Roy-Steiner equations for πN: difficulties

Key difficulties compared to $\pi\pi$ Roy-equations

- Crossing: coupling between $\pi N \to \pi N$ (s-channel) and $\pi\pi \to \bar{N}N$ (t -channel)
  $\Rightarrow$ hyperbolic dispersion relations [Hite, Steiner 1973], [Büttiker et al. 2004]
- Unitarity in t-channel, e.g. in single-channel approximation

$$\mathrm{Im} f_\pm^J(t) = \sigma_t^\pi f_\pm^J(t) t_J^I(t)^*$$

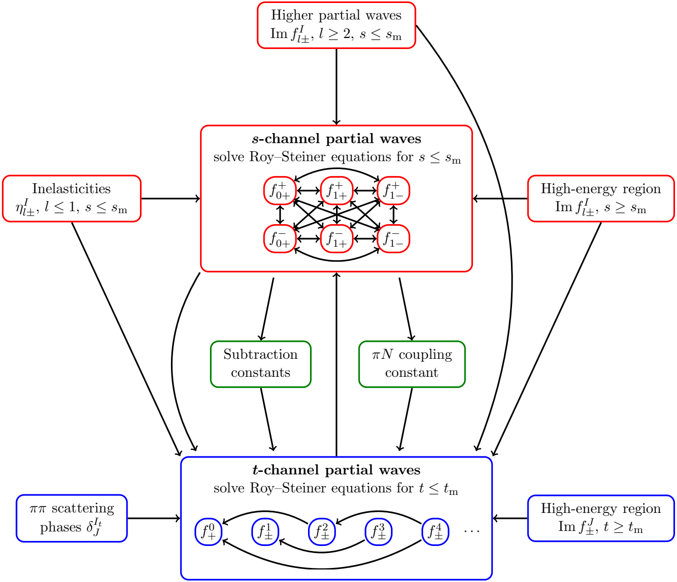


  $\Rightarrow$ **Watson's theorem**: phase of $f_\pm^J(t)$ equals δ_{IJ} [Watson 1954]

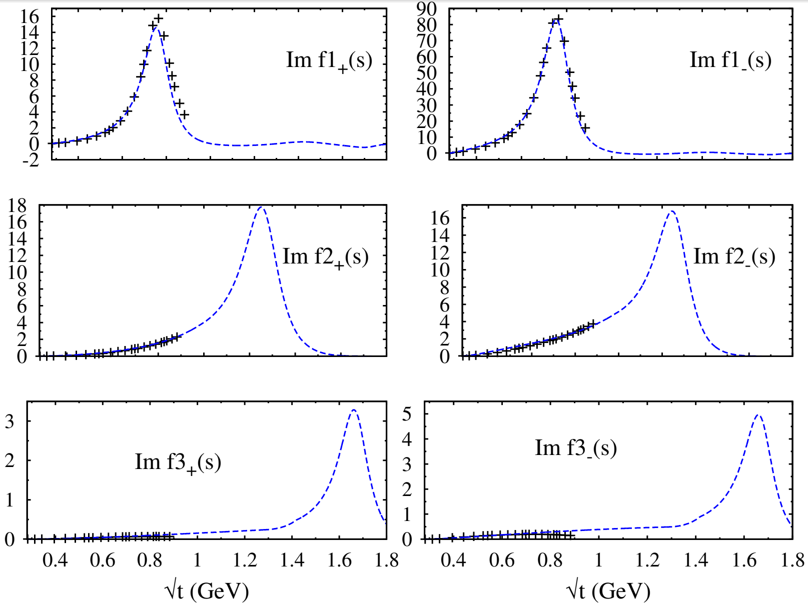
  solve with Muskhelishvili-Omnès techniques [Muskhelishvili 1953, Omnès 1958]

  $\Rightarrow$ Omnès function: $\Omega_J^I(t) = \exp\left\{ \frac{t}{\pi} \int_{t_{th}}^{tm} dt' \frac{\delta_J^I(t)}{t'(t'-t)} \right\}$
- Large pseudo-physical region in t -channel
  $\Rightarrow$ $\bar{K}K$ intermediate states for s-wave in the region of the $f_0(980)$

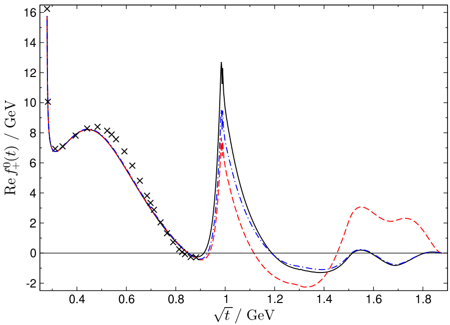
Roy-Steiner equations for πN: flow of information

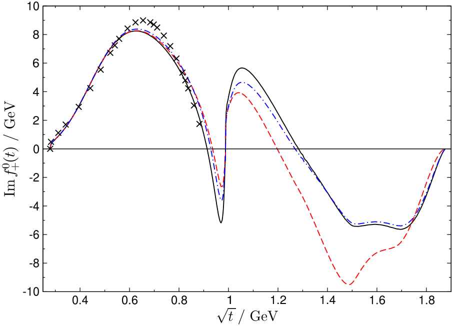


t-channel: P, D and F waves with KH80 parameters



t-channel: S-wave results





MO solutions in general consistent with KH80 results

s-channel: strategy

- Parametrize S and P waves up to $W < W_m$
  - Using SAID partial waves as starting point
- Impose as a **constraint** scattering lengths from a combined analysis of pionic hydrogen and deuterium [Baru et al. 2011]

$$a_{0+}^{1/2} = (169.8 \pm 2.0)10^{-3} M_\pi^{-1} \quad a_{0+}^{3/2} = (-86.3 \pm 1.8)10^{-3} M_\pi^{-1}$$

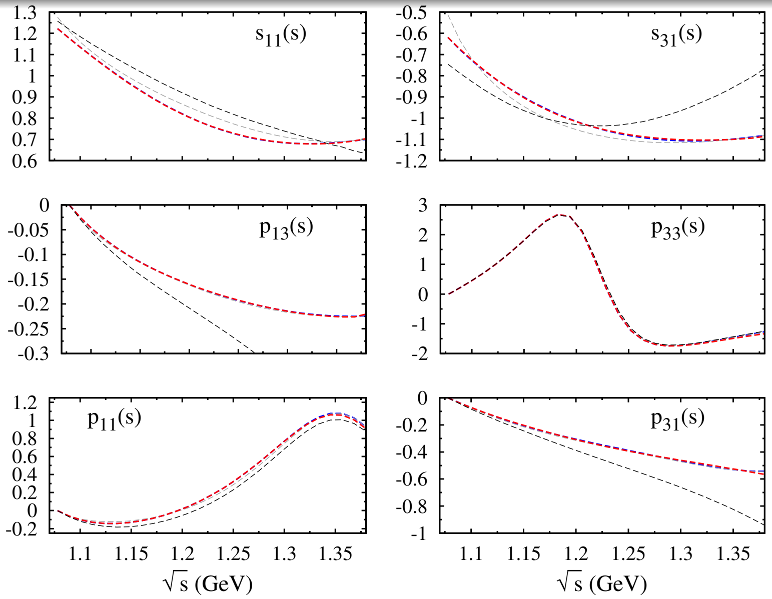
- Introduce as many subtractions as necessary to match d.o.f [Gasser, Wanders 1999]
- Minimize difference between LHS and the RHS on a grid of points W_j

$$\chi^2 = \sum_{l,I_s,\pm} \sum_{j=1}^{N} \left(\frac{\mathrm{Re}\, f_{l\pm}^{I_s}(W_j) - F[f_{l\pm}^{I_s}](W_j)}{\mathrm{Re}\, f_{l\pm}^{I_s}(W_j)} \right)^2$$

$F[f_{l\pm}^{I_s}](W_j) \equiv$ right hand side of RS-equations

- Parametrization and subthreshold parameters are the fitting parameters

s-channel: results



blue/red

⇕

LHS/RHS

after the fit

gray/black

⇕

LHS/RHS

before the fit

Notation: $L_{2I_s 2J}$

Solving the full RS system: strategy

- Full solution: self-consistent, iterative solution of the full RS system
  $\Rightarrow$ consistent set of **s**- and **t**-channel PWs & low-energy parameters
- However:
  - t-channel RS eqs. depend only weakly on s-channel PWs
  - resulting s-channel PW change little from **SAID**

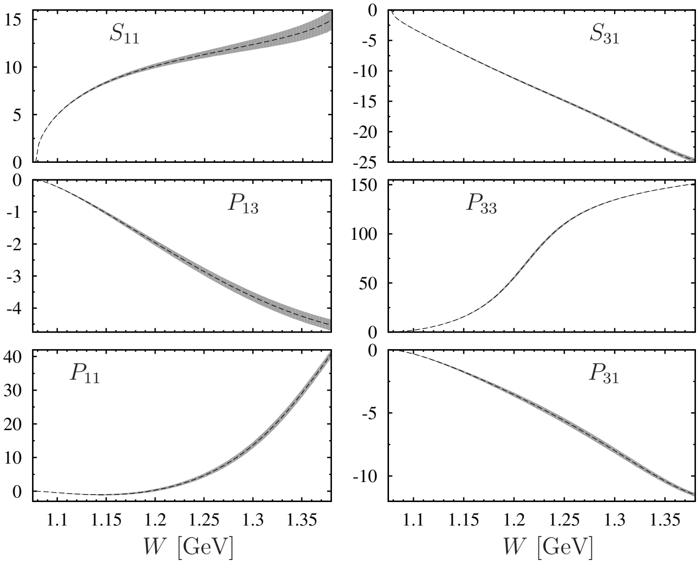
A **full solution** can be achieved including in the s-channel RS eqs. the t-channel dependence on the **subthreshold parameters**

Solving the full RS system: uncertainties

- Statistical errors (at intermediate energies)
  - important correlations between subthreshold parameters
  - shallow fit minima

  $\Rightarrow$ Sum rules for subthreshold parameters become essential to reduce the errors
- Input variation (small)
  - small effect for considering s-channel KH80 input
  - very small effects from $L > 5$ s-channel PWs
  - small effect from the different S-wave extrapolation for $t > 1.3$ GeV
  - negligible effect of ρ' and ρ''
  - very significant effects of the D-waves ($f_2(1275)$)
  - negligible effect of the F-waves ($\rho_3(1690)$)
- matching conditions (close to W_m)
- scattering length (SL) errors (on S-waves and subthreshold parameters)
  - very important for the $\sigma_{\pi N}$

Uncertainties: s-channel pw



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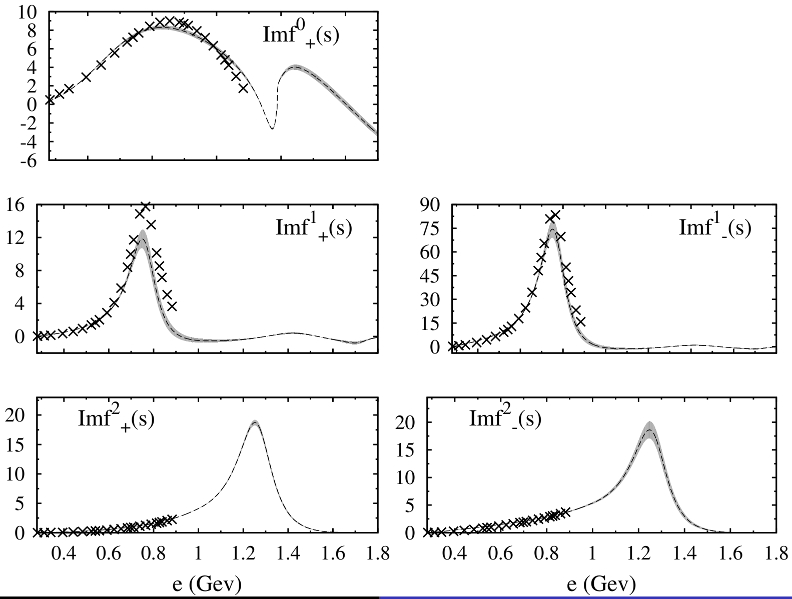


Comparison with KH80

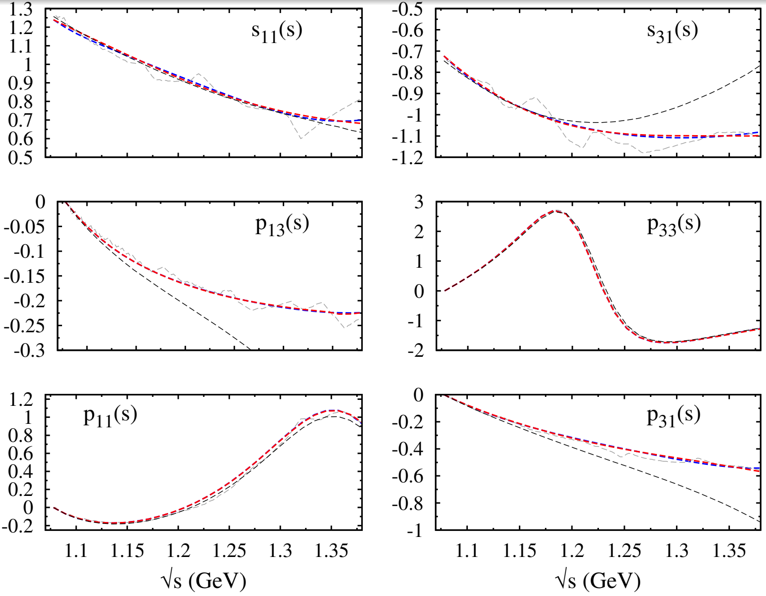
- Karlsruhe-Helsinki analysis **KH80** [Höhler et al. 1980]
  - comprehensive analyticity constraints based on fixed-t dispersion relations
  - old experimental data
- Here, an update of **KH80** results with modern input
  - HDR increase the range of validity of the equations
  - πN scattering length extracted from hadronic atoms $\Rightarrow$ essential for the $\sigma_{\pi N}$
  - Goldberger-Miyazawa-Oehme sum rule:

$$g_{\pi N}^2/4\pi = 13.7 \pm 0.2$$ [Baru et al. 2011]

compare: $$g_{\pi N}^2/4\pi = 14.28$$ [Höhler et al. 1983]

  - s-channel PWs from **SAID**
  - $f_2(1275)$ included $\Rightarrow$ sizable effect
- **KH80** is internally consistent $\Rightarrow$ RS reproduces **KH80** results with **KH80** input

Results: s-channel PWs with KH80 input



blue/red

⇕

LHS/RHS

after the fit

gray/black

⇕

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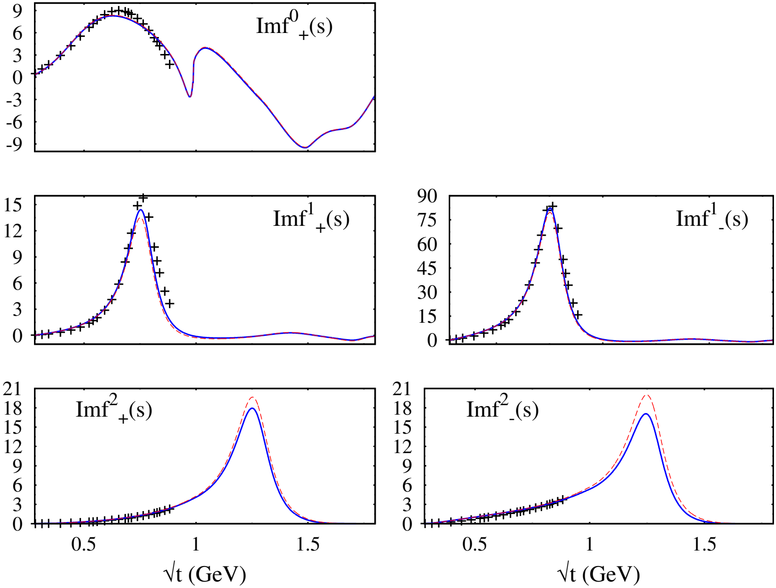

blue

⇳

before the fit

red

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after the fit

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KH80

threshold parameters

- Sum rules for the threshold parameters

	RS	KH80
a^+_{0+}	-0.9 ± 1.4	-9.7 ± 1.7
a^-_{0+}	85.4 ± 0.9	91.3 ± 1.6
a^+_{1+}	131.2 ± 1.7	132.7 ± 1.3
a^-_{1+}	-80.3 ± 1.1	-81.3 ± 1.0
a^+_{1-}	-50.9 ± 1.9	-56.7 ± 1.3
a^-_{1-}	-9.9 ± 1.2	-11.7 ± 1.0
b^+_{0+}	-45.0 ± 1.0	-44.3 ± 6.7
b^-_{0+}	4.9 ± 0.8	13.3 ± 6.0

PRELIMINARY

Results for the $\sigma_{\pi N}$

Result for the $\sigma_{\pi N}$

$$\sigma_{\pi \mathrm{N}} = \Sigma_d - \Delta_R + \Delta_D - \Delta_\sigma$$

$$\Sigma_d = F_\pi^2 \left(d_{00}^+ + 2M_\pi^2 d_{01}^+ \right), \qquad \Delta_D - \Delta_R - \Delta_\sigma = -(1.8 \pm 2.2) \text{ MeV}$$

$\Sigma_d = 57.9 \pm 1.8$ MeV

IV correction$\equiv (3 \pm 2.2)$ MeV

Final results

$\sigma_{\pi \mathrm{N}} = 59.1 \pm 3.5$ MeV

$\sigma_{\pi N}$: comparison with KH80 and SAID

- First: we solve RS eqs. with KH80 input $\hookrightarrow$ $\sigma_{\pi \mathbf{N}} = 46$ MeV
  to be compared with $\sigma_{\pi \mathbf{N}} = 45$ [Gasser, Leutwyler, Socher, Sainio 1988, Gasser, Leutwyler, Sainio 1991]
  $\hookrightarrow$KH80 is internally **consistent** but at odd with the modern SL determinations

How are d_{00}^{+} and d_{01}^{+} extracted in KH80 and SAID?

- Standard approach Gasser, Leutwyler, Locher, Sainio 1988: replace d_{00}^{+} and d_{01}^{+} in favor of threshold parameters: a_{0+}^{+} and a_{1+}^{+}
  $\hookrightarrow$ corrections from PWA via DRs (D^{+} and E^{+})

	Born	a_{0+}^{+}	a_{1+}^{+}	D^{+}	E^{+}	Σ_d
KH80	-133	-7	+352	-91	-72	50
SAID	-127	0	+351	-88	-69	67
diff	+6	7	-1	+3	+3	17

- large weight of a_{1+}^{+}. It has to be known extremely accurately!
  $\hookrightarrow$ the difference 132.7 (SAID)/131.2 (RS) translates in 5 MeV in the $\sigma_{\pi N}$

Matching to LECs

PRELIMINARY

- Express the subthreshold parameters in terms of the LECs up to NNLO

$$d_{00}^{+} = -\frac{2M_\pi^2(2\tilde{c}_1 - \tilde{c}_3)}{F_\pi^2} + \frac{g_a^2(3+8g_a^2)M_\pi^3}{64\pi F_\pi^4} + M_\pi^4\left\{\frac{16\bar{e}_{14}}{F_\pi^2} - \frac{2c_1 - c_3}{16\pi^2 F_\pi^4}\right\}$$

$\hookrightarrow$ expansion in terms of the nucleon mass

- Invert the system of equations

	LO	NLO	NNLO	Krebs et al. 2012 GW	Krebs et al. 2012 KH
c_1 [GeV^{-1}]	-0.74 ± 0.02	-1.07 ± 0.02	-1.11 ± 0.03	-1.13	-0.75
c_2 [GeV^{-1}]	1.81 ± 0.03	3.20 ± 0.03	3.13 ± 0.03	3.69	3.49
c_3 [GeV^{-1}]	-3.61 ± 0.05	-5.32 ± 0.05	-5.61 ± 0.06	-5.51	-4.77
c_4 [GeV^{-1}]	2.17 ± 0.03	3.56 ± 0.03	4.26 ± 0.04	3.71	3.34
$\bar{d}_1 + \bar{d}_2$ [GeV^{-2}]	—	1.04 ± 0.06	7.42 ± 0.08	5.57	6.21
$\bar{d}_3$ [GeV^{-2}]	—	-0.48 ± 0.02	-10.46 ± 0.10	-5.35	-6.83
$\bar{d}_5$ [GeV^{-2}]	—	0.14 ± 0.05	0.59 ± 0.05	0.02	0.78
$\bar{d}_{14} - \bar{d}_{15}$ [GeV^{-2}]	—	-1.90 ± 0.06	-12.18 ± 0.12	-10.26	-12.02
$\bar{e}_{14}$ [GeV^{-3}]	—	—	0.89 ± 0.04	1.75	1.52
$\bar{e}_{15}$ [GeV^{-3}]	—	—	-0.97 ± 0.06	-5.80	-10.41
$\bar{e}_{16}$ [GeV^{-3}]	—	—	-2.61 ± 0.03	1.76	6.08
$\bar{e}_{17}$ [GeV^{-3}]	—	—	0.01 ± 0.06	-0.58	-0.37
$\bar{e}_{18}$ [GeV^{-3}]	—	—	-4.20 ± 0.05	0.96	3.26

- correlation between LECs at each order

Summary

- Derived a closed system of Roy-Steiner equations (PWHDRs) for πN
- Solved the t-channel MO problem for a single- and two-channel approximation
- Numerical solution of the full system of RS eqs.
- KH80 PWA self-consistent, but at odds with hadronic-atom phenomenology
- Complete error analysis
- Precise determination of the $\sigma_{\pi N}$
    - Roy–Steiner formalism reproduces KH80 result with KH80 input
    - With modern input for scattering lengths and coupling constant $\sigma_{\pi N}$ increases
- Precise determination of threshold parameters
- Determination of the LECs from the ChPT subthreshold expansion

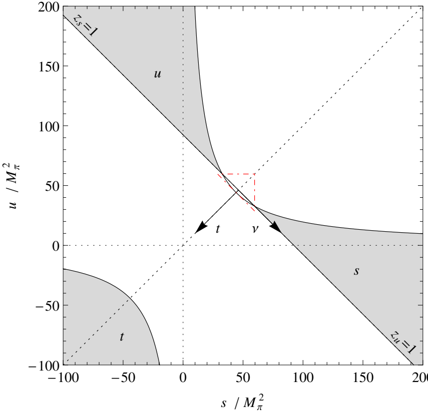
Spare slides

πN-scattering basics

$$\pi^a(q) + N(p) \to \pi^b(q') + N(p')$$

- **Isospin Structure**:
  $T^{ba} = \delta^{ba} T^+ + \epsilon^{ab} T^-$
- **Lorentz Structure**: $I \in \{+,-\}$
  $T^I = \bar{u}(p') \left(A^I + \frac{\not{q} + \not{q}'}{2} B^I \right) u(p)$
  $D^I = A^I + \nu B^I, \quad \nu = \frac{s-u}{4m}$
- **Isospin basis**: $I_s \in \{1/2, 3/2\}$
  $\{T^+, T^-\} \Leftrightarrow T^{1/2}, T^{3/2}$
- PW projection:
  s-channel pw: $f^I_{l\pm}$
  t-channel pw: $f^J_\pm$

Bose symmetry $\Rightarrow$ even/odd $J \Leftrightarrow I = +/-$



πN-scattering basics: partial waves

- **s-channel** projection:

$$f^I_{l\pm}(W) = \frac{1}{16\pi W}\Big\{(E+m)\big[A^I_l(s)+(W-m)B^I_l(s)\big]+(E-m)\big[-A^I_{l\pm1}(s)+(W+m)B^I_{l\pm1}(s)\big]\Big\}$$

$$X^I_l(s) = \int_{-1}^{1} \mathrm{d}z_s\; P_l(z_s)X^I(s,t)\Big|_{t=t(s,z_s)=-2q^2(1-z_s)} \qquad \text{for } X\in\{A,B\} \text{ and } W=\sqrt{s}$$

- **McDowell symmetry**: $f^I_{l+}(W) = -f^I_{(l+1)-}(-W) \qquad \forall\, l\geq 0$
- **t-channel** projection:

$$f^J_+(t) = -\frac{1}{4\pi}\int_0^1 \mathrm{d}z_t\; P_J(z_t)\Big\{\frac{p_t^2}{(p_tq_t)^J}A^I(s,t)\Big|_{s=s(t,z_t)} - \frac{m}{(p_tq_t)^{J-1}}z_tB^I(s,t)\Big|_{s=s(t,z_t)}\Big\} \qquad \forall\, J\geq 0$$

$$f^J_-(t) = \frac{1}{4\pi}\frac{\sqrt{J(J+1)}}{2J+1}\frac{1}{(p_tq_t)^{J-1}}\int_0^1 \mathrm{d}z_t\Big[P_{J-1}(z_t)-P_{J+1}(z_t)\Big]B^I(s,t)\Big|_{s=s(t,z_t)} \qquad \forall\, J\geq 1$$

- **Bose symmetry** $\Rightarrow$ even/odd $J \Leftrightarrow I=+/-$

Roy-Steiner equations for πN: HDR's

- **Hyperbolic DRs**: $(s-a)(u-a) = b = (s'-a)(u'-a)$ with $a, b \in \mathbb{R}$

$$A^+(s,t;a) = \frac{1}{\pi}\int_{s_+}^{\infty} ds' \left[\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right] \mathrm{Im}\, A^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \, \frac{\mathrm{Im}\, A^+(s',t')}{t'-t}$$

- Why **HDR**?
  - Combine all physical regions $\Rightarrow$ crucial for t-channel projection
  - Evade double-spectral regions $\Rightarrow$ the PW decompositions converge
  - No kinematical cuts, manageable kernel functions
- Similar derivation to $\pi\pi$ **Roy equations**.
  - Expand imaginary parts in terms of s- and t-channel partial waves
  - Project onto s- and t-channel partial waves
  - Combine the resulting equations using s- and t-channel PW unitarity relations
- **Validity**: $W \in [W_+ = 1.08\,\text{GeV}, 1.38\,\text{GeV}]$, $\sqrt{t} \in [\sqrt{t_\pi} = 0.28\,\text{GeV}, 2.00\,\text{GeV}]$.
- Subtractions: subthreshold expansion around $\nu = t = 0$

$$\bar{A}^+(\nu,t) = \sum_{m,n=0}^{\infty} a^+_{mn} \nu^{2m} t^n$$

Roy-Steiner equations for πN: HDR's

- **Hyperbolic DRs**: $(s-a)(u-a) = b = (s'-a)(u'-a)$ with $a, b \in \mathbb{R}$

$$A^+(s,t;a) = \frac{1}{\pi}\int_{s_+}^{\infty} ds' \left[\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right] \operatorname{Im} A^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \, \frac{\operatorname{Im} A^+(s',t')}{t'-t}$$

$$B^+(s,t;a) = N^+(s,t) + \frac{1}{\pi}\int_{s_+}^{\infty} ds' \left[\frac{1}{s'-s} - \frac{1}{s'-u}\right] \operatorname{Im} B^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \, \frac{\nu}{\nu'} \frac{\operatorname{Im} B^+(s',t')}{t'-t}$$

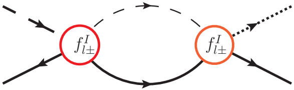
$$N^+(s,t) = g^2\left(\frac{1}{m^2-s} - \frac{1}{m^2-u}\right)$$ similar for A^-, B^- and N^- [Hite/Steiner (1973)]

- Why **HDR**?
  - Combine all physical regions $\Rightarrow$ crucial for t-channel projection
  - Evade double-spectral regions $\Rightarrow$ the PW decompositions converge
  - Range of convergence can be maximized by tuning the free hyperbola parameter a
  - No kinematical cuts, manageable kernel functions

πN-scattering basics: Unitarity relations

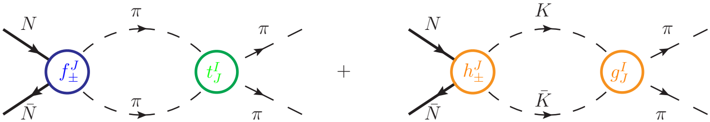
- **s-channel** unitarity relations $(I_s \in \{1/2, 3/2\})$:

$$\mathrm{Im}\, f^{I_s}_{l\pm}(W) = q\left|f^{I_s}_{l\pm}(W)\right|^2 \theta(W-W_+) + \frac{1-\left(\eta^{I_s}_{l\pm}(W)\right)^2}{4q}\theta(W-W_{\mathrm{inel}})$$



- **t-channel** unitarity relations: 2-body intermediate states: $\pi\pi + \bar{K}K + \cdots$

$$\mathrm{Im}\, f^J_{\pm}(t) = \sigma^\pi_t \left(t^{I_t}_J(t)\right)^* f^J_\pm(t)\,\theta(t-t_\pi) + 2c_J\sqrt{2}\,k_t^{2J}\sigma^K_t\left(g^{I_t}_J(t)\right)^* h^J_\pm(t)\,\theta(t-t_K)$$



- Only linear in $f^J_\pm(t) \Rightarrow$ less restrictive

Roy-Steiner equations for πN: derivation

- Recipe to derive **Roy-Steiner** equations:
  - Expand imaginary parts in terms of s- and t-channel partial waves
  - Project onto s- and t-channel partial waves
  - Combine the resulting equations using s- and t-channel PW unitarity relations
- Similar structure to $\pi\pi$ **Roy equations**
- **Validity**: assuming Mandelstam analyticity
  - s-channel $\Rightarrow$ optimal for $a = -23.2 M_\pi^2$

$$s \in \left[s_+ = (m + M_\pi)^2, 97.30\, M_\pi^2\right] \quad \Leftrightarrow \quad W \in [W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV}]$$

  - t-channel $\Rightarrow$ optimal for $a = -2.71 M_\pi^2$

$$t \in [t_\pi = 4M_\pi^2, 205.45\, M_\pi^2] \quad \Leftrightarrow \quad \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}] \;.$$

Roy-Steiner equations for πN: subtractions

- **Subtractions** are necessary to ensure the convergence of DR integrals
  $\Rightarrow$ asymptotic behavior
- Can be introduced to lessen the dependence of the low-energy solution on the high-energy behavior
- Parametrize high-energy information in (a priori unknown) subtraction constants
  $\Rightarrow$ matching to ChPT
- Subthreshold expansion around $\nu = t = 0$

$$\bar{A}^+(\nu,t) = \sum_{m,n=0}^{\infty} a^+_{mn} \nu^{2m} t^n \qquad \bar{B}^+(\nu,t) = \sum_{m,n=0}^{\infty} b^+_{mn} \nu^{2m+1} t^n \,,$$

$$\bar{A}^-(\nu,t) = \sum_{m,n=0}^{\infty} a^-_{mn} \nu^{2m+1} t^n \qquad \bar{B}^-(\nu,t) = \sum_{m,n=0}^{\infty} b^-_{mn} \nu^{2m} t^n \,,$$

where

$$\bar{A}^+(s,t) = A^+(s,t) - \frac{g^2}{m} \qquad \bar{B}^+(s,t) = B^+(s,t) - g^2 \left[\frac{1}{m^2 - s} - \frac{1}{m^2 - u}\right] \,,$$

$$\bar{A}^-(s,t) = A^-(s,t) \,, \qquad \bar{B}^-(s,t) = B^-(s,t) - g^2 \left[\frac{1}{m^2 - s} + \frac{1}{m^2 - u}\right] + \frac{g^2}{2m^2} \,,$$

RS-eqs for πN: subthreshold expansion

- Subthreshold expansion around $\nu = t = 0$

$$A^+(\nu,t) = \frac{g^2}{m} + d_{00}^+ + d_{01}^+ t + a_{10}^+ \nu^2 + \mathcal{O}(\nu^2 t, t^2)$$
$$A^-(\nu,t) = \nu a_{00}^- + a_{01}^- \nu t + a_{10}^- \nu^3 + \mathcal{O}(\nu^5, \nu t^2, \nu^3 t)$$
$$B^+(\nu,t) = g^2 \frac{4m\nu}{(m^2-s_0)^2} + \nu b_{00}^+ + \mathcal{O}(\nu^3, \nu t) \ ,$$
$$B^-(\nu,t) = g^2 \left[\frac{2}{m^2-s_0} - \frac{t}{(m^2-s_0)^2}\right] - \frac{g^2}{2m^2} + b_{00}^- + b_{01}^- t + +b_{10}^- \nu^2 + \mathcal{O}(\nu^2, \nu^2 t, t^2)$$

- pseudovector Born terms: $D^I = A^I + \nu B^I$

$$\bar{D}^+ = d_{00}^+ + d_{01}^+ t + d_{10}^+ \nu^2$$
$$d_{mn}^+ = a_{mn}^+ + b_{m-1,n}^+ \ , \qquad d_{mn}^- = a_{mn}^- + b_{mn}^- \ .$$

- Sum rules for subthreshold parameters:

$$d_{00}^+ = -\frac{g^2}{m} + \frac{1}{\pi}\int_{s_+}^{\infty} \mathrm{d}s'\, h_0(s') \left[\mathrm{Im}\, A^+(s', z_s')\right]_{(0,0)} + \frac{1}{\pi}\int_{t_\pi}^{\infty} \frac{\mathrm{d}t'}{t'} \left[\mathrm{Im}\, A^+(t', z_t')\right]_{(0,0)}$$
$$h_0(s') = \frac{2}{s'-s_0} - \frac{1}{s'-a}$$

Roy-Steiner equations for πN: s-channel

s-channel RS equations

$$f^I_{l+}(W) = N^I_{l+}(W) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K^I_{ll'}(W,W')\,\mathrm{Im}\, f^I_{l'+}(W') + K^I_{ll'}(W,-W')\,\mathrm{Im}\, f^I_{(l'+1)-}(W') \right\}$$

$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \sum_J \left\{ G_{lJ}(W,t')\,\mathrm{Im}\, f^J_+(t') + H_{lJ}(W,t')\,\mathrm{Im}\, f^J_-(t') \right\}$$

$$= -f^I_{(l+1)-}(-W) \qquad \forall\, l \geq 0\,, \quad \text{[Hite/Steiner (1973)]}$$

- $K^I_{ll'}(W,W')$, $G_{lJ}(W,t')$ and $H_{lJ}(W,t')$-Kernels: analytically known,
  e.g. $K^I_{ll'}(W,W') = \frac{\delta_{ll'}}{W'-W} + \ldots \quad \forall\, l,l' \geq 0\,,$
- **Validity**: assuming Mandelstam analyticity
  $\Rightarrow$ optimal for $a = -23.2 M_\pi^2$

$$s \in \left[s_+ = (m+M_\pi)^2, 97.30\, M_\pi^2 \right] \quad \Leftrightarrow \quad W \in [W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV}]$$

Roy-Steiner equations for πN: t-channel

t-channel RS equations

$$f_+^J(t) = \tilde{N}_+^J(t) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l=0}^{\infty}\Big\{\tilde{G}_{Jl}(t,W')\,\mathrm{Im} f_{l+}^I(W') + \tilde{G}_{Jl}(t,-W')\,\mathrm{Im} f_{(l+1)-}^I(W')\Big\}$$

$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \sum_{J'}\Big\{\tilde{K}_{JJ'}^1(t,t')\,\mathrm{Im} f_+^{J'}(t') + \tilde{K}_{JJ'}^2(t,t')\,\mathrm{Im} f_-^{J'}(t')\Big\} \qquad \forall\, J \geq 0\,,$$

$$f_-^J(t) = \tilde{N}_-^J(t) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l=0}^{\infty}\Big\{\tilde{H}_{Jl}(t,W')\,\mathrm{Im} f_{l+}^I(W') + \tilde{H}_{Jl}(t,-W')\,\mathrm{Im} f_{(l+1)-}^I(W')\Big\}$$

$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \sum_{J'}\tilde{K}_{JJ'}^3(t,t')\,\mathrm{Im} f_-^{J'}(t') \qquad \forall\, J \geq 1\,,$$

- **Validity**: assuming Mandelstam analyticity
  $\Rightarrow$ optimal for $a = -2.71 M_\pi^2$

$$t \in [t_\pi = 4M_\pi^2,\, 205.45\, M_\pi^2] \quad \Leftrightarrow \quad \sqrt{t} \in [\sqrt{t_\pi} = 0.28\ \mathrm{GeV},\, 2.00\ \mathrm{GeV}]\,.$$

Roy-Steiner equations for πN

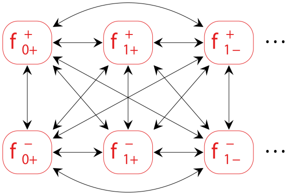
Two different systems of coupled integral equations

- t-channel problem: $\pi\pi \to \bar{N}N$ reaction
  $\hookrightarrow$ solution in terms of the Omnès function: $\Omega(t) = \exp\left\{ \frac{t}{\pi} \int_{t_\pi}^{t_m} \frac{dt'}{t'} \frac{\delta(t')}{t'-t} \right\}$
  - $\pi\pi$ phase-shift
  - subthreshold parameters
  - $\pi N \to \pi N$ phase-shifts
- s-channel problem: $\pi N \to \pi N$ reaction
  $\hookrightarrow$ form of $\pi\pi$ Roy-Equations
  - subthreshold parameters
  - $\pi\pi \to \bar{N}N$ phase-shifts
- Full Solution: self-consistent, iterative solution of the full RS system
  $\hookrightarrow$ consistent set of subthreshold parameters, $\pi N \to \pi N$ and $\pi\pi \to \bar{N}N$ phase shifts

Solving Roy-Steiner equations for πN: Recoupling schemes

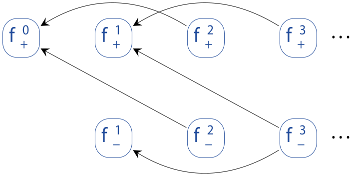
- s-channel subproblem:
  - Kernels are diagonal for $I \in \{+,-\}$, but unitarity relations are diagonal for $I_s \in \{1/2, 3/2\} \Rightarrow$ all partial-waves are interrelated
  - Once the t-channel PWs are known

    $\Rightarrow$ Structure similar to $\pi\pi$ Roy-equations



- t-channel subproblem:
  - Only higher PWs couple to lower ones
  - Only PWs with even or odd J are coupled
  - No contribution from f_+^J to f_-^{J+1}

    $\Rightarrow$ Leads to Muskhelishvili-Omnès problem



Solving t-channel: single channel

- Elastic-channel approximation: generic form of the integral equation

$$f(t) = \Delta(t) + (a + bt)(t - 4m^2) + \frac{t^2(t-4m^2)}{\pi} \int_{t_\pi}^{\infty} \mathrm{d}t' \frac{\mathrm{Im} f(t')}{t'(t'^2 - 4m^2)(t'-t)}$$

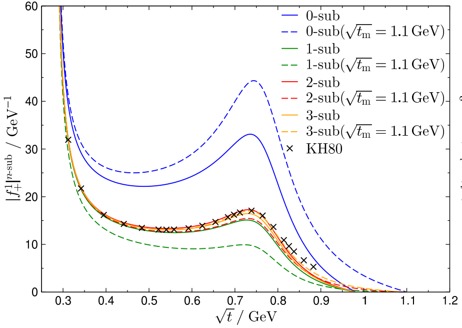
- $\Delta(t)$: Born terms, s-channel integrals, higher t -channel partial waves $\Rightarrow$ left-hand cut
- Introduce subtractions at $\nu = t = 0 \Rightarrow$ subthreshold parameters a, b
- Solution in terms of Omnès function:

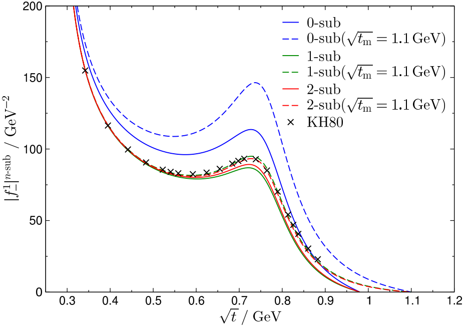
$$f(t) = \Delta(t) + (t-4m^2)\Omega(t)(1 - t\dot{\Omega}(0))a + t(t-4m^2)\Omega(t)b$$
$$- \Omega(t)\frac{t^2(t-4m^2)}{\pi}\left\{ \int_{4M_\pi^2}^{t_\mathrm{m}} \mathrm{d}t' \frac{\Delta(t')\mathrm{Im}\,\Omega(t')^{-1}}{t'^2(t'-4m^2)(t'-t)} + \int_{t_\mathrm{m}}^{\infty} \mathrm{d}t' \frac{\Omega(t')^{-1}\mathrm{Im} f(t')}{t'(t'-4m^2)(t'-t)} \right\}$$

$$\Omega(t) = \exp\left\{ \frac{t}{\pi} \int_{t_\pi}^{t_\mathrm{m}} \frac{\mathrm{d}t'}{t'} \frac{\delta(t')}{t'-t} \right\}$$

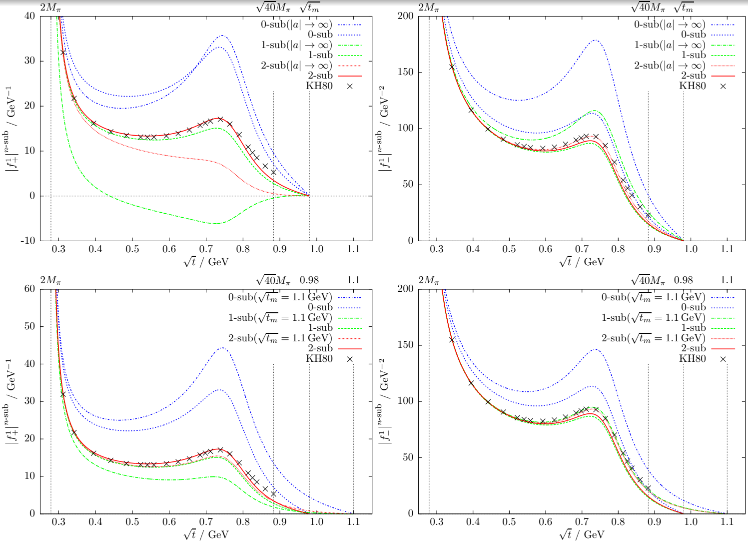
Solving t-channel: input and subtractions

- elastic channel approximation: $\sqrt{t_m} = 0.98 - 1.1$ GeV, for $t > t_m$ $\mathrm{Im} f^J_\pm(t) = 0$
- First step: check **consistency** with KH80 [Höhler 1983]
- Input needed:
  - $\pi\pi$ phase shifts: [Caprini, Colangelo, Leutwyler, (in preparation)], [Madrid group]
  - πN phase shifts: SAID [Arndt et al. 2008], KH80
  - πN at high energies: Regge model [Huang et al. 2010]
  - πN parameters: KH80





Solving t-channel: P-wave results



MO solutions in general consistent with KH80 results

Solving t-channel: coupled channels

- Generic coupled-channel integral equation

$$\mathbf{f}(t) = \boldsymbol{\Delta}(t) + \frac{1}{\pi}\int_{t_\pi}^{t_{\mathrm{m}}} \mathrm{d}t' \frac{T^*(t')\Sigma(t')\mathbf{f}(t')}{t'-t} + \frac{1}{\pi}\int_{t_{\mathrm{m}}}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\,\mathbf{f}(t')}{t'-t}$$

- Formal solution as in the single-channel case (now with Omnès matrix $\Omega(t)$)
  $\Rightarrow$ Two-channel Muskhelishvili-Omnès problem

$$\mathbf{f}(t) = \begin{pmatrix} f_+^0(t) \\ h_+^0(t) \end{pmatrix} \qquad \mathrm{Im}\,\Omega(t) = (T(t))^*\Sigma(t)\Omega(t)$$

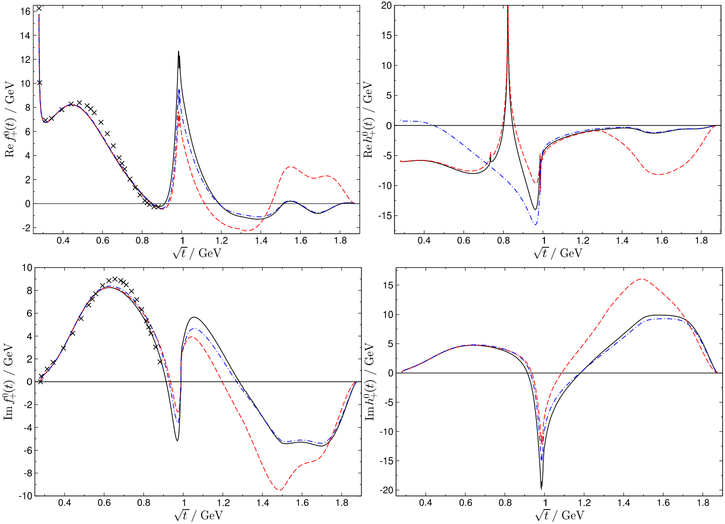
- Two linearly independent solutions Ω_1, Ω_2 [Muskhelishvili 1953]
- In general no analytical solution for the Omnès matrix but for its determinant [Moussallam 2000]

$$\det\Omega(t) = \exp\left\{\frac{t}{\pi}\int_{t_\pi}^{t_{\mathrm{m}}} \mathrm{d}t' \frac{\psi(t')}{t'(t'-t)}\right\}.$$

Solving t-channel S-wave equations: input

- Input needed:
  - $\pi\pi$ s-wave partial waves: [Caprini, Colangelo, Leutwyler, (in preparation)]
  - $K\bar{K}$ s-wave partial waves: [Büttiker. (2004)]
  - πN and KN s-wave pw: SAID [Arndt et al. 2008], KH80
  - πN at high energies: Regge model [Huang et al. 2010]
  - πN parameters: KH80
  - Hyperon couplings from [Jülich model 1989]
  - KN subthreshold parameters neglected
- Two-channel approximation beaks down at $\sqrt{t_0} = 1.3$ GeV $\Rightarrow 4\pi$ channel
- From t_0 to $t = 2$ GeV, different approximations considered

Solving t-channel: S-wave results



Solving s-channel: S-wave results

- General form of the s-channel integral equation

$$f^I_{l+}(W) = \Delta^I_{l+}(W) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K^I_{ll'}(W,W')\,\mathrm{Im} f^I_{l'+}(W') + K^I_{ll'}(W,-W')\,\mathrm{Im} f^I_{(l'+1)-}(W') \right\}$$

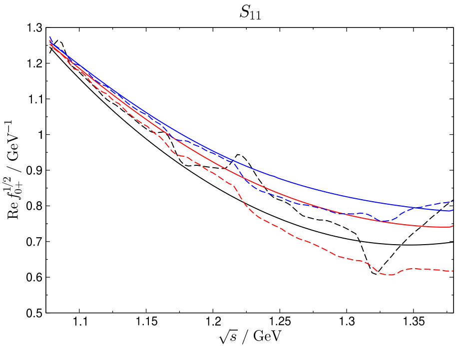
$\Rightarrow$ form of $\pi\pi$ Roy-Equations

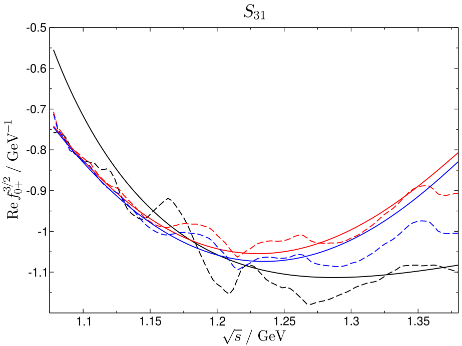
- $\Delta^I_{l+}(W) \equiv$ t-channel contribution and pole term
- valid up to $W_m = 1.38$ GeV
- **Input**:
    - RS t-channel solutions for S and P waves
    - s-channel partial waves for $J > 1$ [SAID analysis]
    - s-channel partial waves for $W_m < W < 2.5$ GeV [SAID analysis]
    - high energy contribution for $W > 2.5$ GeV: Regge model [Huang et al. 2010]
- **Output**:
    - Self-consistent solution for S and P waves for $J \leq J_{\max}$ and $s_{\text{th}} \leq s \leq s_m$
    - Constraints on subtraction constants $\Rightarrow$ subthreshold parameters

Solving s-channel: consistency with KH80

- Consistency with KH80
  - parametrize SAID S and P waves up to $W < W_m$
    Imposing a continuous and differentiable matching point
  - Compare between the input (LHS) and the output (RHS)

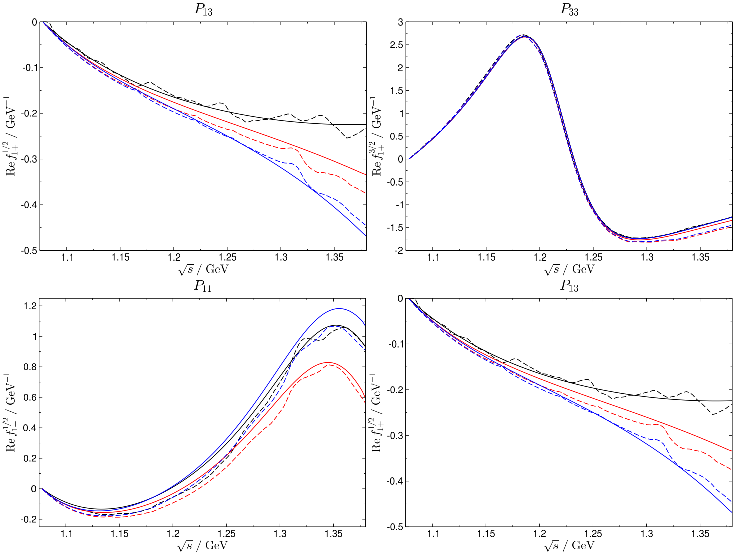
S-WAVES





$\Rightarrow$ important discrepancies

Solving s-channel: consistency with KH80. P-waves



Solving s-channel: subtractions

- Existence and uniqueness of solutions [Gasser, Wanders 1999]

  ⇒ no-cusp condition for each pw + 2 additional constraints are needed
- Take advantage of the precise data for pionic atoms [Gotta et al. 2005, 2010]

  ⇒ Impose as a **constraint** scattering lengths from a combined analysis of pionic hydrogen and deuterium [Baru et al. 2011]

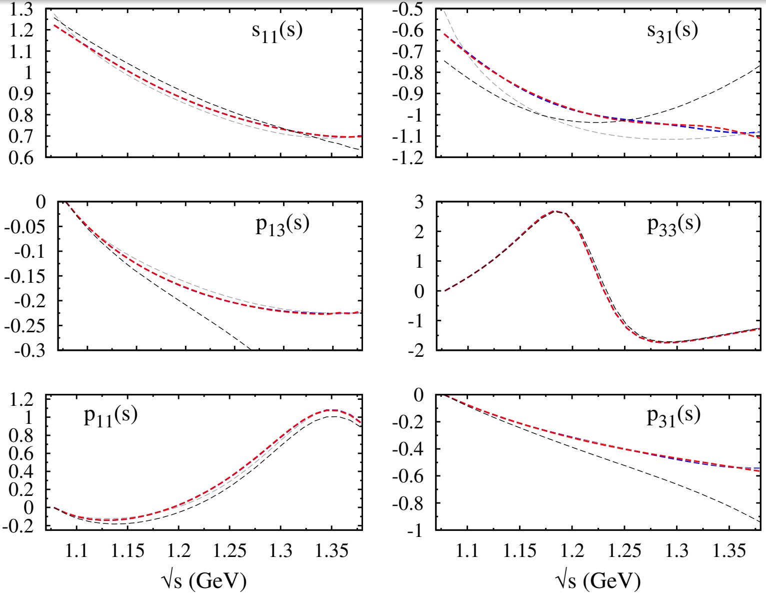
$$a_{0+}^{1/2} = (169.8 \pm 2.0)10^{-3} M_\pi^{-1} \quad a_{0+}^{3/2} = (-86.3 \pm 1.8)10^{-3} M_\pi^{-1}$$

$$\mathrm{Re}\, f_{l\pm}^{I}(s) = \mathbf{q}^{2l} \left(a_{l\pm}^{I} + b_{l\pm}^{I} \mathbf{q}^2 + \cdots \right)$$

10 subthreshold parameters are needed to match **d.o.f**

⇒ **three subtractions**

Results: s-channel PWs

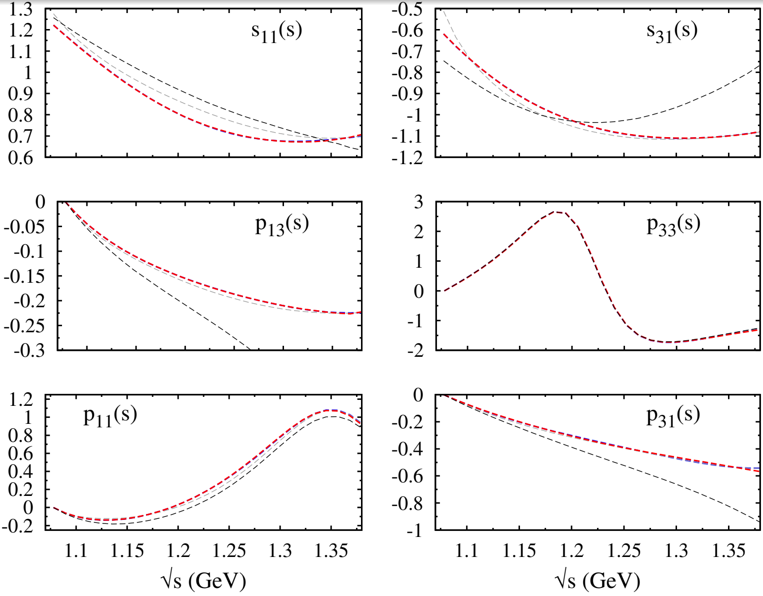


blue/red ⇕ **LHS/RHS** after the fit

gray/black ⇕ **LHS/RHS** before the fit

Notation: $L_{2I_s 2J}$

Results: s-channel PWs



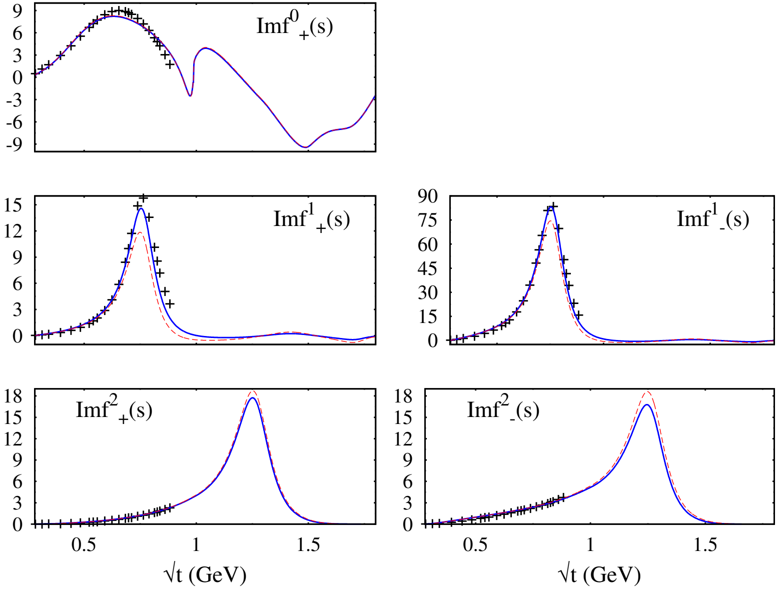
sizable $f_2(1275)$ effect

blue/**red** ⇕ **LHS**/**RHS** after the fit

gray/**black** ⇕ **LHS**/**RHS** before the fit

Notation: $L_{2I_s 2J}$

Results: t-channel PWs



blue

⇕

before the fit

red

⇕

after the fit

+

⇕

KH80

threshold parameters in the isospin basis

$a_{0+}^{1/2}$	169.8 ± 2.0	173 ± 3
$a_{0+}^{3/2}$	-86.3 ± 1.8	-101 ± 4
$a_{1+}^{1/2}$	-29.4 ± 1.0	-30 ± 2
$a_{1+}^{3/2}$	211.5 ± 2.8	214 ± 2
$a_{1-}^{1/2}$	-70.7 ± 4.1	-81 ± 2
$a_{1-}^{3/2}$	-41.0 ± 1.1	-45 ± 2
$b_{0+}^{1/2}$	-35.2 ± 2.1	-18 ± 12
$b_{0+}^{3/2}$	-49.8 ± 1.1	-58 ± 9

RS-eqs for πN: Range of convergence

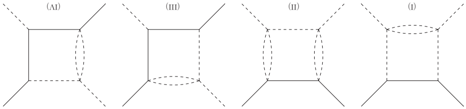
- Assumption: Mandelstam analyticity [Mandelstam (1958,1959)]

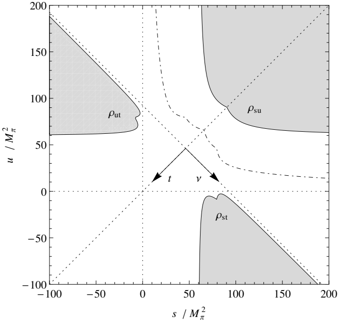
  $\Rightarrow$ T(s,t) can be written in terms double spectral densities: ρ_{st}, ρ_{su}, ρ_{ut}

  $$T(s,t) = \frac{1}{\pi^2}\iint ds'du' \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\iint dt'du' \frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} + \frac{1}{\pi^2}\iint ds'dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)}$$

  integration ranges defined by the support of the double spectral densities ρ

- Boundaries of ρ are given lowest lying intermediate states





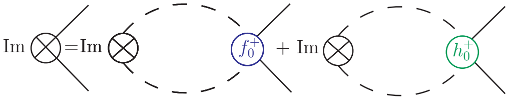
- They limit the range of validity of the HDRS:
  - Pw expansion converge

    $\Rightarrow z = \cos\theta \in$ Lehmann ellipses [Lehmann (1958)]
  - the hyperbolae $(s-a)(u-a) = b$ does not enter any double spectral region

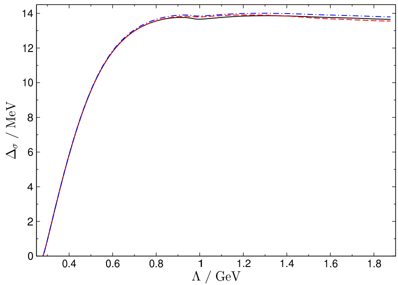
    $\Rightarrow$ for a value of a, constraints on b yield ranges in s & t

Dispersion relation for the scalar form factor of the nucleon

- Unitarity relation: $\mathrm{Im}\,\sigma(t) = \frac{2}{4m^2-t}\left\{\frac{3}{4}\sigma_t^\pi \left(F_\pi^S(t)\right)^* f_+^0(t) + \sigma_t^K \left(F_K^S(t)\right)^* h_+^0(t)\right\}$



- Once subtracted dispersion relation: $\sigma(t) = \sigma_{\pi N} + \frac{t}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\sigma(t')}{t'(t'-t)}$



- $\Delta_\sigma = \sigma(2M_\pi^2) - \sigma_{\pi N}$

Dispersion relation for the πN amplitude

- t-channel expansion of the subtracted pseudo-Born amplitude

$$\bar{D}(\nu=0,t) = 4\pi \left\{ \frac{1}{p_t^2} \bar{f}_0^+(t) + \frac{5}{2} q_t^2 \bar{f}_2^+(t) + \frac{27}{8} p_t^2 q_t^4 \bar{f}_4^+(t) + \frac{56}{16} p_t^4 q_t^6 \bar{f}_6^+(t) + \cdots \right\}$$

- Insert t-channel RS equations for Born-term-subtracted amplitudes $\bar{f}_J^+(t)$

$$\bar{D}(\nu=0,t) = d_{00}^+ + d_{01}^+ t - 16t^2 \int_{t_\pi}^{\infty} dt' \frac{\mathrm{Im}\bar{f}_0^+(t')}{t'^2(t'-4m^2)(t'-t)} + \{J \geq 2\} + \{\text{s-channel integral}\}$$

- $\Delta_D = F_\pi^2 \left(\bar{D}(\nu=0,t) - d_{00}^+ + d_{01}^+ t\right)$ from evaluation at $t = 2M_\pi^2$

Summary: σ-term corrections

- Nucleon scalar form factor

$$\Delta_\sigma = (13.9 \pm 0.3)\,\text{MeV}$$
$$+ Z_1\left(\frac{g^2}{4\pi} - 14.28\right) + Z_2\left(d_{00}^+ M_\pi + 1.46\right) + Z_3\left(d_{01}^+ M_\pi^3 - 1.14\right) + Z_4\left(b_{00}^+ M_\pi^3 + 3.54\right)$$
$$Z_1 = 0.36\,\text{MeV}\,, \qquad Z_2 = 0.57\,\text{MeV}\,, \qquad Z_3 = 12.0\,\text{MeV}\,, \qquad Z_4 = -0.81\,\text{MeV}$$

- πN amplitude

$$\Delta_D = (12.1 \pm 0.3)\,\text{MeV}$$
$$+ \hat{Z}_1\left(\frac{g^2}{4\pi} - 14.28\right) + \hat{Z}_2\left(d_{00}^+ M_\pi + 1.46\right) + \hat{Z}_3\left(d_{01}^+ M_\pi^3 - 1.14\right) + \hat{Z}_4\left(b_{00}^+ M_\pi^3 + 3.54\right)$$
$$\hat{Z}_1 = 0.42\,\text{MeV}\,, \qquad \hat{Z}_2 = 0.67\,\text{MeV}\,, \qquad \hat{Z}_3 = 12.0\,\text{MeV}\,, \qquad \hat{Z}_4 = -0.77\,\text{MeV}$$

$\hookrightarrow$ most of the dependence on the πN parameters cancels in the difference

Full Correction

$$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)\,\text{MeV}$$

Cheng-Dashen theorem in the presence of isospin breaking

- Define as isoscalar as

$$X^+ \to X^p = \frac{1}{2}\left(X_{\pi^+p\to\pi^+p} + X_{\pi^-p\to\pi^-p}\right), \qquad X \in \{D, d_{00}, d_{01}, a_{0+}\ldots\}$$

  and "isospin limit" by proton and charged pion
- Assume virtual photons to be removed
  $\hookrightarrow$ scenario closest to actual πN PWA
- Calculate IV corrections in SU(2) ChPT, mainly due to $\Delta_\pi = M_\pi^2 - M_{\pi^0}^2$
  - For the σ term no differences at $\mathcal{O}(p^3)$

$$\sigma_{\pi N} = \sigma_p = \sigma_N = -4c_1 M_{\pi^0}^2 - \frac{3g_A^2 M_{\pi^0}^2}{64\pi F_\pi^2}(2M_\pi + M_{\pi^0}) + \mathcal{O}(M_\pi^4)$$

  - Slope of the scalar form factor

$$\Delta_\sigma^p = \sigma_p(2M_\pi^2) - \sigma_p = \frac{3g_A^2 M_\pi^3}{64\pi F_\pi^2} + \frac{g_A^2 M_\pi \Delta_\pi}{128\pi F_\pi^2}\left(-7 + \sqrt{2}\log(3+2\sqrt{2})\right) + \mathcal{O}(M_\pi^4)$$

  - Similarly for Δ_D^p

$$\Delta_D^p = F_\pi^2\left\{\bar{D}_p(0, 2M_\pi^2) - d_{00}^p - 2M_\pi^2 d_{01}^p\right\} = \frac{23g_a^2 M_\pi^3}{384\pi F_\pi^2} + \frac{g_a^2 M_\pi \Delta_\pi}{256\pi F_\pi^2}\left(3 + 4\sqrt{2}\log\left(1+\sqrt{2}\right)\right) + \mathcal{O}(M_\pi^4)$$

Cheng-Dashen theorem in the presence of isospin breaking

- Taking everything together

$$\sigma_{\pi N} = F_\pi^2 \left(d_{00}^p + 2M_\pi^2 d_{01}^p\right) - \Delta_R + \Delta_D - \Delta_\sigma + (\Delta_D^p - \Delta_D) - (\Delta_\sigma^p - \Delta_\sigma) + \sigma_p(2M_\pi^2) + F_\pi^2 \bar{D}(0, 2M_\pi^2)$$

$$= F_\pi^2 \left(d_{00}^p + 2M_\pi^2 d_{01}^p\right) - \underbrace{\Delta_R}_{\lesssim 2\text{MeV}} + \underbrace{\Delta_D - \Delta_\sigma}_{(-1.8\pm0.2)\text{MeV}} + \underbrace{\frac{81 g_a^2 M_\pi \Delta_\pi}{256\pi F_\pi^2}}_{3.4\text{MeV}} + \underbrace{\frac{e^2}{2} F_\pi^2 \left(4f_1 + f_2\right)}_{(-0.4\pm2.2)\text{MeV}}$$

$\hookrightarrow$ sizable corrections from Δ_π increasing the value of the $\sigma_{\pi N}$

Spin-independent WIMP–nucleon scattering

- Effective Lagrangian

$$\mathcal{L} = C_{qq}^{SS}\frac{m_q}{\Lambda^3}\,\bar{\chi}\chi\,\bar{q}q + C_{qq}^{VV}\frac{m_q}{\Lambda^2}\,\bar{\chi}\gamma^\mu\chi\,\bar{q}\gamma_\mu q + \bar{C}_{gg}^{S}\frac{\alpha_S}{\Lambda^3}\bar{\chi}\chi G_{\mu\nu}G^{\mu\nu}$$

- WIMP χ **Dirac fermion** and **SM singlet**
- Spin–independent cross section at vanishing momentum transfer

$$\sigma_N^{SI} = \frac{\mu_\chi^2}{\Lambda^4}\left|\left(\frac{m_N}{\Lambda}C_{qq}^{SS}f_q^N - 12\pi C_{gg}^{S}f_Q^N\right) + C_{qq}^{VV}f_V^N\right|^2$$

$$\mu_\chi = \frac{m_\chi m_N}{m_\chi + m_N} \quad f_q^N = 2 \quad f_q^N = \frac{\sigma_{\pi\mathbf{N}}(1-\xi)}{m_N} + \Delta f_q^N$$

- nucleon-matrix elements dominated by $\sigma_{\pi\mathbf{N}}$