Determination of $K\pi$ scattering lengths at physical kinematics

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Scattering phase shifts

For a system with two (pseudo)scalar particles, the S-matrix in centre-of-mass frame can be written in partial wave basis as:

$$\langle E', p', l', m' \mid S \mid E, 0, l, m\rangle = \delta(E' - E)\delta(p)\delta_{ll'}\delta_{mm'}S_l(E)$$

which is required by Lorentz invariance of the S-matrix, specifically $[H, S] = 0$, $[P, S] = 0$, $[J^2, S] = 0$, $[J_3, S] = 0$ and $[J_\pm, S] = 0$. Furthermore, unitarity of the S-matrix implies $S^\dagger S = SS^\dagger = 1$ gives

$$S_l(E) = e^{2i\delta_l(E)}$$

This means that the two (pseudo)scalar particle scattering can be expressed in terms of a single real parameter $\delta_l(E)$ called the *phase shift*.

Scattering length

At low energies (or equivalently momenta, k), phase shifts have the following threshold behaviour:

$$\delta_l(k) \sim k^{l+1}$$

- The dominant contribution will come from the s-wave ($l = 0$).
- We can define a constant called the *scattering length*:

$$(\tan \delta_0(k)/k)^{-1} = \frac{1}{a_0} + \frac{r_{eff}}{2} k^2 + \mathcal{O}(k^4)$$

$K\pi$ scattering

With $m_u = m_d \equiv m_{ud}$ and QCD interactions only, isospin becomes a good quantum number.

Pions have $I = 1$, kaons have $I = 1/2$, so $K\pi$ can be in $I = 3/2$ or $I = 1/2$ state. Specifically:

$$|I = 3/2; I_z = 3/2\rangle = |K^+\pi^+\rangle$$

$$|I = 3/2; I_z = 1/2\rangle = \frac{1}{\sqrt{3}}|K^0\pi^+\rangle + \sqrt{\frac{2}{3}}|K^+\pi^0\rangle$$

$$|I = 3/2; I_z = -1/2\rangle = \frac{1}{\sqrt{3}}|K^+\pi^-\rangle + \sqrt{\frac{2}{3}}|K^0\pi^0\rangle$$

$$|I = 3/2; I_z = -3/2\rangle = |K^0\pi^-\rangle$$

$$|I = 1/2; I_z = 1/2\rangle = \frac{1}{\sqrt{3}}|K^+\pi^0\rangle - \sqrt{\frac{2}{3}}|K^0\pi^+\rangle$$

$$|I = 1/2; I_z = -1/2\rangle = -\frac{1}{\sqrt{3}}|K^0\pi^0\rangle + \sqrt{\frac{2}{3}}|K^+\pi^-\rangle$$

Experimental input

- Experimentally $K\pi$ phase shifts are determined from kaon-nucleon scattering by extrapolating to small transverse momentum
- The experimental input is most accurate at E$>$1 GeV
- Roy-Steiner eqations are used to calculate phase shifts at different energies [1]
- Low energy input (scattering lengths) can help to reduce the uncertainties in the dispersion relations.

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[1] P. Büttiker *et. al.* Eur.Phys.J. C33 (2004) 409-432

Results so far

	$a_0^{3/2} m_\pi$	$a_0^{1/2} m_\pi$
Büttiker et. al.	-0.0448(77)	0.224(22)
$\mathcal{O}(p^4)$ ChPT	-0.05(2)	0.19(2)
NPLQCD [2]	-0.0574(16) $\left(\begin{array}{c} +24 \\ -58 \end{array}\right)$	0.1725(13) $\left(\begin{array}{c} +23 \\ -156 \end{array}\right)$
Fu [3]	-0.0512(18)	0.1819(35)
PACS-CS[4]	-0.0602(31)(26)	0.183(18)(35)

Calculation also done by Lang et. al. [5], but without extrapolation to physical point.

This work: evaluation of scattering length **directly at physical point**.

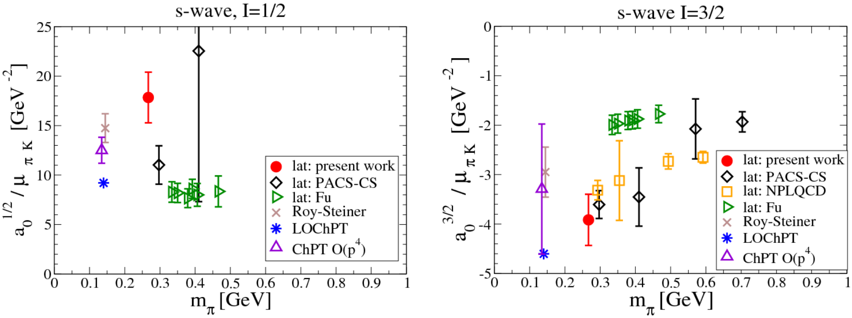
[2] S. R. Beane et al. [NPLQCD Collaboration], Phys. Rev. D 74 (2006) 114503

[3] Z. Fu, Phys. Rev. D 85 (2012) 074501

[4] Kiyoshi Sasaki (Tokyo Inst. Tech.) et al. [PACS-CS Collaboration], Phys.Rev. D89 (2014) 054502

[5] C. B. Lang, L. Leskovec, D. Mohler and S. Prelovsek, Phys . Rev. D 86 (2012) 054508

Chiral extrapolation



Plots taken from Lang *et. al.* Phys.Rev. D86 (2012) 054508.

In infinite volume, the two meson energies can be visualised on the complex energy plane as a branch cut starting at $m_1 + m_2$. In finite volume this is replaced by series of poles.

Position of these poles can are related to the s-wave phase shifts by Lüscher's condition [6]:

$$\delta(k) + \phi(k) = n\pi$$

where $\phi(k)$ is a known function of the momentum k. This formula is valid below inelastic threshold.

[6] M. Lüscher, Nucl. Phys. B354 (1991) 531-578

$$
\begin{aligned}
C'^{ij}_{K\pi}(t) &\equiv \langle O^{i\dagger}_{K_\pi}(t) O^{j}_{K\pi}(0)\rangle \\
&= \mathrm{Tr}\left(e^{-H(T-t)} O^{\dagger i}_{K\pi} e^{-Ht} O^{j}_{K\pi}\right) / \mathrm{Tr}\left(e^{-HT}\right) \\
&\xrightarrow[T\to\infty]{t\to\infty} \langle 0 \mid O^{i\dagger}_{K\pi} \mid K\pi\rangle\langle K\pi \mid O^{j}_{K\pi} \mid 0\rangle e^{-E_{K\pi} t}
\end{aligned}
$$

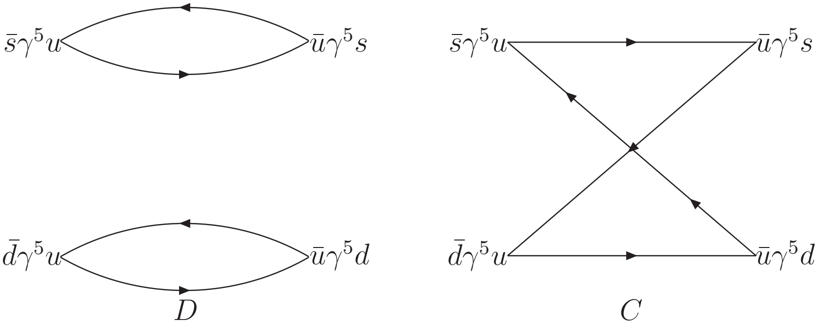
We use:

$$
O^{\pm}_{K\pi}(t) = \left(\bar{s}\gamma^5 l\right)\left(\bar{l}\gamma^5 l\right)(t).
$$

Such operators have good overlap with $K\pi$ states and poor overlap with resonant states (e.g. κ)[7].

[7] Kiyoshi Sasaki (Tokyo Inst. Tech.) et al. [PACS-CS Collaboration], Phys.Rev. D89 (2014) 054502

$K\pi$ I=3/2 contractions

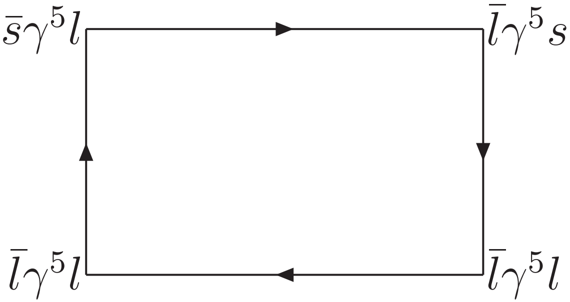


$$D = \mathrm{Tr}\left(S^\dagger(t;0)L(t;0)\right)\mathrm{Tr}\left(L(t;0)^\dagger L(t;0)\right)$$

$$C = \mathrm{Tr}\left(S^\dagger(t;0)L(0;0)L^\dagger(t+0;0)L(t;0)\right)$$

$$C_{K\pi}^{I=3/2}(t) = D - C$$

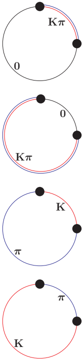
Rectangle graph for I=1/2 correlator



$$C_{K\pi}^{I=1/2}(t) = D + 0.5C - 1.5R$$

Around-the-world effects

$$\mathrm{Tr}\left(e^{-H(T-t_1)}O^{\dagger i}_{K\pi}e^{-H(t_1-t_2)}O^{j}_{K\pi}\right)/\mathrm{Tr}\left(e^{-HT}\right)$$



$$\begin{aligned}
C_{K\pi}(t) = {} & |\langle K\pi \mid O_{K\pi} \mid 0\rangle|^2 e^{-E_{K\pi}t} \\
& + |\langle 0 \mid O_{K\pi} \mid K\pi\rangle|^2 e^{-E_{K\pi}(T-t)} \\
& + |\langle K \mid O_{K\pi} \mid \pi\rangle|^2 e^{-m_\pi(T-t)} e^{-m_K t} \\
& + |\langle \pi \mid O_{K\pi} \mid K\rangle|^2 e^{-m_K(T-t)} e^{-m_\pi t} \\
& + \ldots .
\end{aligned}$$

Physical run parameters

Lattice size	$48^3 \times 96$	$64^3 \times 128$
Gauge action	Iwasaki	Iwasaki
Fermion action	Möbius DWF	Möbius DWF
No. of configs	88	80
a^{-1} [GeV]	1.730(4)	2.359(7)
L [fm]	5.476(12)	5.354(16)
m_π [MeV]	139.2(2)	139.3(3)
m_K [MeV]	499.2(2)	507.9(4)
$m_\pi L$	3.863(6)	3.778(8)

- quark sources every second time slice on 48^3, every fourth on 64^3
- antiperiodic boundary conditions in time direction only

Preliminary results

Continuum extrapolation in a^2:

$a_0 m_\pi$	48^3	64^3	continuum
I=3/2	-0.068(8)	-0.068(7)	-0.07(2)
I=1/2	0.16(1)	0.16(1)	0.16(3)

Improving the statistical error

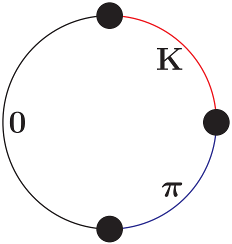
$$
\begin{aligned}
C_{K\pi}(t) = {} & |\langle K\pi \mid O_{K\pi} \mid 0\rangle|^2 e^{-E_{K\pi} t} \\
& + |\langle 0 \mid O_{K\pi} \mid K\pi\rangle|^2 e^{-E_{K\pi}(T-t)} \\
& + |\langle K \mid O_{K\pi} \mid \pi\rangle|^2 e^{-m_\pi (T-t)} e^{-m_K t} \\
& + |\langle \pi \mid O_{K\pi} \mid K\rangle|^2 e^{-m_K (T-t)} e^{-m_\pi t} \\
& + \ldots .
\end{aligned}
$$

3-point functions

$\pi \to K$ and $K \to \pi$ matrix elements can be calculated in an alternative way using the following correlation functions:

$$
\begin{aligned}
C_{K\to\pi}(t) &= \langle \pi(\Delta)\pi(t+\delta)K(t)K(0)\rangle \\
&= \langle 0 \mid O_\pi \mid \pi\rangle\langle \pi \mid O_{K\pi} \mid K\rangle\langle K \mid O_K \mid 0\rangle e^{-m_\pi(\Delta - t)} e^{-m_K t} \\
&+ \langle \pi \mid O_\pi \mid 0\rangle\langle 0 \mid O_{K\pi} \mid K\pi\rangle\langle K\pi \mid K \mid \pi\rangle e^{-m_\pi(T-\Delta)} e^{-E_{K\pi} t} \\
&+ \ldots,
\end{aligned}
$$

$\pi \to K$ matrix element calculated in an analogous way.



Continuum extrapolation

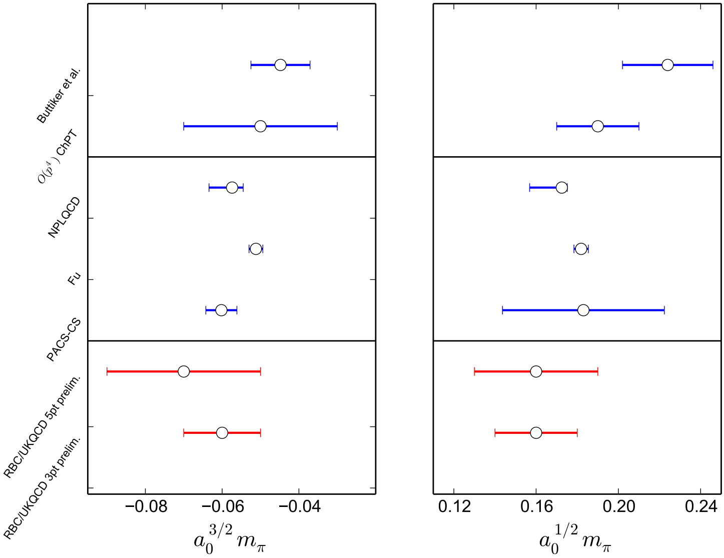
Continuum extrapolation revisited:

$a_0 m_\pi$	48^3	64^3	continuum
I=3/2	-0.068(8)	-0.068(7)	-0.07(2)
I=1/2	0.16(1)	0.16(1)	0.16(3)
I=3/2	-0.063(8)	-0.059(5)	-0.06(1)
I=1/2	0.178(9)	0.170(9)	0.16(2)

Comparison

	$a_0^{3/2} m_\pi$	$a_0^{1/2} m_\pi$
Büttiker et. al.	-0.0448(77)	0.224(22)
$\mathcal{O}(p^4)$ ChPT	-0.05(2)	0.19(2)
NPLQCD	-0.0574(16) $\left(\begin{array}{c} +24 \\ -58 \end{array}\right)$	0.1725(13) $\left(\begin{array}{c} +23 \\ -156 \end{array}\right)$
Fu	-0.0512(18)	0.1819(35)
PACS-CS	-0.0602(31)(26)	0.183(18)(35)
RBC-UKQCD (5p) (**preliminary**)	-0.07(2)	0.16(3)
RBC-UKQCD (3p) (**preliminary**)	-0.06(1)	0.16(2)

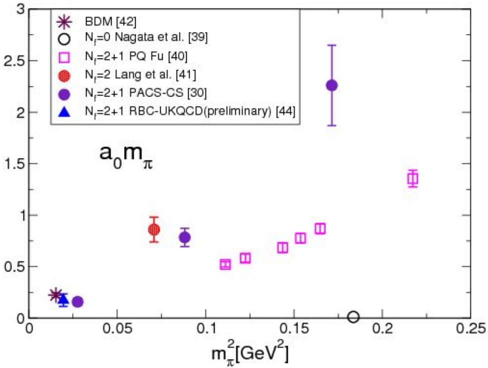
Comparison


Buttiker et al.
$O(p^4)$ ChPT
NPLQCD
Fu
PACS-CS
RBC/UKQCD 5pt prelim.
RBC/UKQCD 3pt prelim.
−0.08
−0.06
−0.04
$a_0^{3/2} m_\pi$
0.12
0.16
0.20
0.24
$a_0^{1/2} m_\pi$

Conclusions

- We are able to generate ensembles with physical pion and kaon masses.
- Calculation of $K\pi$ energies at low values of $m_\pi T$ and $m_K T$ suffers from significant around-the world effects.
- Around-the-world effects can be treated reliably using a 5-parameter fit...
- ...and even more reliably by including $K \to \pi$ and $\pi \to K$ matrix elements.
- First calculation of scattering lengths that does not rely on chiral perturbation theory.
- Although low statistics prevent us from obtaining an accurate $I = 3/2$ result, we can get a good estimate for $I = 1/2$, which has been dominated by χPT errors in previous calculations.

Thank you for your attention!



BDM [42]
N_f=0 Nagata et al. [39]
N_f=2+1 PQ Fu [40]
N_f=2 Lang et al. [41]
N_f=2+1 PACS-CS [30]
N_f=2+1 RBC-UKQCD(preliminary) [44]
a_0m_π
m_π^2[GeV^2]
0
0.05
0.1
0.15
0.2
0.25
0.5
1
1.5
2
2.5
3