Constraints on the $\omega\pi$ form factor from analyticity and unitarity

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Outline

1. Brief introduction to the $\omega\pi$ form factor
2. Experimental status
3. Recent phenomenological analyses
4. Theory of unitarity bounds
5. Results
6. Discussion and Conclusions

- Transition form factors of light mesons is of interest
- Of consequence to hadronic light-by-light scattering contribution to muon $g-2$
- Form factors can be linked to vector-meson transition form factors and measured in, e.g, $\omega(\phi) \to \pi^0(\eta) l^+ l^-$
- $f_{\omega\pi}$ ($\omega\pi$ electromagnetic form factor) from dispersive representations in conflict with experiment above 0.6 GeV
- Dispersion relations use (elastic) unitarity: express the discontinuity in the form factor in terms of the P wave of $\pi\pi \to \omega\pi$
- Elastic region strictly is $4m_\pi^2 \leq t \leq 16m_\pi^2$, but phenomenologically can be taken to be as high as $t_+ \equiv (m_\omega + m_\pi)^2$
- Important early work done by G. Köpp, Physical Review **D 10** (1974) 932, where N/D method was used leading also to a real-analytic form factor
- Important new ingredient in dispersion relation analysis is due to rescattering
- Leads to non-real analytic representation of the form factor
- We present the consequences of studying this form factor using the optimization theory that leads to (stringent) bounds

- Appears in the matrix element

$$\langle \omega(p_a,\lambda)\pi(p_b)|j_\mu(0)|0\rangle = i\epsilon_{\mu\tau\rho\sigma}\epsilon^{\tau *}(p_a,\lambda)p_b^\rho q^\sigma f_{\omega\pi}(t),$$

where j_μ is the isovector part of the e.m. current, λ is the ω polarization, and $q = p_a + p_b$ and $t = q^2$.

- $$\mathrm{disc}\, f_{\omega\pi}(t) = \frac{i\, q_{\pi\pi}^3(t)}{6\pi\sqrt{t}} F_\pi^*(t) f_1(t)\,\theta\big(t - 4m_\pi^2\big),$$

where $q_{\pi\pi}(t) = \sqrt{t/4 - m_\pi^2}$, $F_\pi(t)$ is the pion e.m. form factor, and $f_1(t)$ is the P partial-wave amplitude of

$$\pi^+(q_1)\,\pi^-(q_2) \to \omega(p_a,\lambda)\,\pi^0(p_b).$$

- Rescattering is possible among the pions in the decay

$$\omega(p_a,\lambda) \to \pi^+(q_1)\,\pi^-(q_2)\,\pi^0(-p_b)$$

- Form factor obtained from the once-subtracted dispersion relation:

$$f_{\omega\pi}(t) = f_{\omega\pi}(0) + \frac{t}{2\pi i}\int_{4m_\pi^2}^\infty \frac{\mathrm{disc}\, f_{\omega\pi}(t')}{t'(t-t')}\mathrm{d}t',$$

where $|f_{\omega\pi}(0)|$ is known experimentally from the $\omega \to \pi^0\gamma$ decay rate. PDG gives

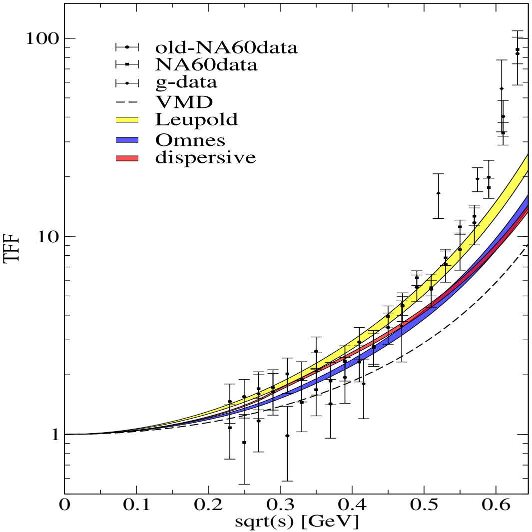
$$|f_{\omega\pi}(0)| = (2.30 \pm 0.04)\,\mathrm{GeV}^{-1}.$$

[S. P. Schneider, B. Kubis and F. Niecknig, Physical Review **D 86** (2012) 054013]

Experimental status

- Lepton G experiment from the 1980's (Serpukhov experiment, see R. I. Dzhelyadin, Physics Letters **B 102** (1981) 296
- NA-60 experiment, 2009 and 2011 [R. Amaldi et al., Physics letters **B 677** (2009) 260; G. Usai, Nuclear Physics, **A 855** (2011) 189]
- VMD model [see Schneider, Kubis and Niecknig]
- Chiral model with explicit resonances [C. Terschlüsen, S. Leupold and M. F. M. Lutz, European Physical Journal **A 48** (2012) 190; C. Terschlüsen and S. Leupold, Physics Letters **B 691** (2010) 191]

- Inputs for computing the partial wave
- $\pi\pi$ P-wave comes from recent Roy equation solutions, R. García Martin et al., Physical Review **D 83** (2011) 074004; I. Caprini, G. Colangelo and H. Leutwyler, European Physical Journal **C 72** (2012) 1860]
- Solutions of the Khuri-Treiman equations
- Discontinuity of the form factor comes from left- and right-hand cuts
- Right-hand cut solved using the Omnès solution involving the pion-pion P-wave
- Left-hand cut obtained computing angular averages
- Outcome are form factors that are not real-analytic
- Features are also confirmed by news analysis of Danilikin et al (arXiv:1409.7708), using conformal mapping techniques to extend domain of validity, but does not add anything new in the domain of interest to us, viz., $t < (m_\omega + m_\pi)^2$.



Experimental information on the transition form factor and comparison with theoretical treatments

- What could be the reason for the discrepancy?
- Lepton-G is an old experiment with a small event sample
- NA-60 is a complex experiment, and some discussion may be needed
- The dispersion relation methods assume elastic unitarity
- Need for a method that would avoid such assumptions
- Use of analyticity and unitarity, and use data only where it is well-known without having to extrapolate
- The method of unitarity bounds
- Elegant formalism has been developed by us
- Extended to non-real analytic functions
- Result is (stringent) bounds, rather than predictions but competitive

Method of unitarity bounds

- To find bounds on the form factor at specific points t (in the analyticity domain) knowing the value of an integral

$$\frac{1}{2\pi}\int_{t_+}^{\infty}\rho(t')|f(t')|^2$$

(typically if f is real analytic on the cut)

- Bounds can be improved if, e.g., the phase is known in the elastic region, using Lagrange multiplier
- Can be improved by extending the 'Meiman problem'
- Can be improved if information is known at a set of 'spacelike' points [for a review of these methods, see G. Abbas, B. Ananthanarayan, I. Caprini, I. Sentitemsu Imsong and S. Ramanan, European Physical Journal **A 45** (2010)389]
- A different version arises when Im f is known on part of the cut [I. Caprini, I. Guiasu and E. E. Radescu, Physical Review **D 25** (1982) 1808; I. Caprini, Physical Review **D 27** (1983) 1479]

Unitarity bounds continued

- Using standard techniques

$$\Pi^{\mu\nu}(q) = \int \mathrm{d}x\, e^{iqx} \langle 0| T[j^\mu(x) j^\nu(0)]|0\rangle$$
$$= \left(q^\mu q^\nu - g^{\mu\nu} q^2\right)\Pi(t), \qquad t = q^2,$$

- $\Pi'(t)$ the derivative of the QCD vacuum polarization amplitude $\Pi(t)$ satisfies

$$\Pi'(t) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\Pi(t' + i\epsilon)}{(t'-t)^2}\,\mathrm{d}t',$$

- The spectral function satisfy

$$\left(q^\mu q^\nu - g^{\mu\nu} q^2\right)\mathrm{Im}\Pi(t + i\epsilon) =$$
$$\frac{1}{2}\sum_\Gamma \int \mathrm{d}\rho_\Gamma (2\pi)^4 \delta^{(4)}(q - p_\Gamma)\langle 0|j^\mu(0)|\Gamma\rangle\langle\Gamma|j^\nu(0)^\dagger|0\rangle.$$

Unitarity bounds continued

- - Retaining explicitity the $\pi\pi$ and $\omega\pi$ intermediate states
  - carrying out the two-body phase space integrals
  - using the positivity of the spectral function,
  - we obtain the inequality

$$\Pi'(t) \geq \int_{4m_\pi^2}^{\infty} w_\pi(t',t)|F_\pi(t')|^2 dt' + \int_{t_+}^{\infty} w_{\omega\pi}(t',t)|f_{\omega\pi}(t')|^2 dt',$$

- where

$$w_\pi(t',t) = \frac{1}{48\pi^2}\frac{1}{(t'-t)^2}\left(1-\frac{4m_\pi^2}{t'}\right)^{3/2},$$
$$w_{\omega\pi}(t',t) = \frac{1}{192\pi^2}\frac{t'}{(t'-t)^2}\left(1-\frac{t_-}{t'}\right)^{3/2}\left(1-\frac{t_+}{t'}\right)^{3/2}.$$

Unitarity bounds continued

- Writing the integral constraint on the modulus of the $\omega\pi$ form factor as

$$\int_{t_+}^{\infty} w_{\omega\pi}(t',t)|f_{\omega\pi}(t')|^2 \mathrm{d}t' \leq I(t),$$

- where

$$I(t) = \Pi'(t) - \int_{4m_\pi^2}^{\infty} w_\pi(t',t)|F_\pi(t')|^2 \mathrm{d}t'.$$

- Can be evaluated for $t \equiv -Q^2 < 0$ using OPE and perturbative QCD and information available on the modulus of the pion form factor (convenient and typical choice is $Q^2 = 2\,\mathrm{GeV}^2$.)
- Perturbative QCD to four loops [P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, Physical Review Letters **101** (2008) 012002]
- This gives

$$\Pi'_{\mathrm{pert}}(-Q^2) = \frac{1}{8\pi^2 Q^2}\Big(1 + 0.318\alpha_s + 0.166\alpha_s^2 + 0.205\alpha_s^3 + 0.504\alpha_s^4\Big),$$

where α_s is the strong coupling at $Q^2 = 2\,\mathrm{GeV}^2$.

Unitarity bounds continued

- Using as input the value $\alpha_s(m_\tau^2) = 0.320 \pm 0.020$, and the coupling's running we obtain $\alpha_s = 0.357 \pm 0.025$. This yields for $\Pi'_{\rm pert}$ the central value $0.0073\,\mathrm{GeV}^{-2}$ with an error of about 1.3%. higher-order term as $0.925\alpha_s^5$ changes $\Pi'_{\rm pert}$ by about 1.2%.
- For the gluon condensate the standard value $\langle \alpha_s G^2 \rangle / \pi = 0.012\,\mathrm{GeV}^4$ [M. A. Shifman, A. I. Vainshtain and V. I. Zakharov, Nuclear Physics **B 147** (1979) 385; *ibid.* 448.] we obtain for its contribution the value $0.0001\,\mathrm{GeV}^{-2}$.
- This leads to $\Pi'(-2\,\mathrm{GeV}^2) = (0.0074 \pm 0.0001)\,\mathrm{GeV}^{-2}$, where the uncertainty includes quadratically the effects of the α_s uncertainty and the truncation error.
- The integral involving the pion electromagnetic form factor can be calculated using
- BaBar experimental data and the bounds on $|F_\pi(t)|$ in the low-energy region, data obtained by BaBar up to 3 GeVand a smooth transition to the $1/t$ decrease predicted by QCD
- Evaluates to $(0.0033 \pm 0.0001)\,\mathrm{GeV}^{-2}$, which leads to

$$I \equiv I(-2\,\mathrm{GeV}^2) = (0.0041 \pm 0.0002)\,\mathrm{GeV}^{-2}.$$

- From the established inequality and the discontinuity adopted in the elastic region of validity $t < t_+$, we obtain bounds on the form factor at points below t_+.

Conformal map

- Using conformal mapping methods, starting with the mapping

$$\tilde{z}(t) = \frac{1 - \sqrt{1 - t/t_+}}{1 + \sqrt{1 - t/t_+}},$$

 such that the $\tilde{z}(0) = 0$.

- In the z-plane the elastic region $4m_\pi^2 \leq t < t_+$ becomes the segment $x_\pi \leq x < 1$ of the real axis, where $x_\pi = \tilde{z}(4m_\pi^2)$, and the upper (lower) edges of the cut $t > t_+$ become the upper (lower) semicircles.
- Outer function $C(z)$, *i.e.* a function analytic and without zeros in $|z| < 1$, its modulus on $|z| = 1$ being equal to $\sqrt{w_{\omega\pi}(\tilde{t}(z), -Q^2)|d\tilde{t}(z)/dz|}$.

$$C(z) = \frac{(1-z)^2(1+z)^{-1/2}}{16\sqrt{6}\pi} \times \frac{(1+\tilde{z}(-Q^2))^2(1 - z\,\tilde{z}(t_-))^{3/2}}{(1 - z\,\tilde{z}(-Q^2))^2(1+\tilde{z}(t_-))^{3/2}}.$$

- Then the inequality written in terms of the new function $h(z)$ defined as

$$h(z) = C(z)\, f_{\omega\pi}(\tilde{t}(z)),$$

- $$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta |h(e^{i\theta})|^2 \leq I,$$

 where $\theta = \arg z$.

Unitarity bounds continued

- Since $C(z)$ is real analytic in $|z| < 1$, $C(x)$ is real for $x_\pi \leq x < 1$, and

$$\mathrm{disc}\, h(x) \equiv \Delta(x) = C(x)\, \mathrm{disc}\, f_{\omega\pi}(\tilde{t}(x)),$$

  where the discontinuity of the form factor is known.
- The function $h(z)$ can be expressed in terms of its discontinuity as

$$h(z) = \frac{1}{2\pi i} \int_{x_\pi}^{1} \frac{\Delta(x)}{x - z} \mathrm{d}x + g(z),$$

  where the function $g(z)$ is analytic in $|z| < 1$, as its discontinuity vanishes:

$$\mathrm{disc}\, g(x) = 0, \quad -1 < x < 1.$$

- Since we consider in general form factors that are not real analytic, the function $g(z)$ is analytic, but its values on the real axis may be complex.
- We now express the available information on the form factor as a number of constraints on the function g.
- We obtain the condition

$$\frac{1}{2\pi} \int_{0}^{2\pi} \mathrm{d}\theta \left| \frac{1}{2\pi i} \int_{x_\pi}^{1} \frac{\Delta(x)}{x - e^{i\theta}} \mathrm{d}x + g(e^{i\theta}) \right|^2 \leq I,$$

- $g(0)$ has the value

$$g(0) = f_{\omega\pi}(0) C(0) - \frac{1}{2\pi i} \int_{x_\pi}^{1} \frac{\Delta(x)}{x} \mathrm{d}x.$$

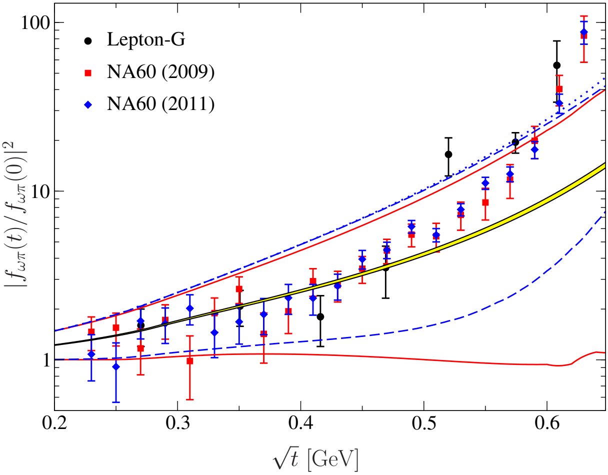
Finding the range

- The problem is to find the maximal allowed range of $|g(z_1)|$ at an arbitrary given point $z_1 = \tilde{z}(t_1)$ in the interval $(x_\pi, 1)$, for functions $g(z)$ analytic in $|z| < 1$ and subject both to the integral constraint and initial value constraint
- We solve the constrained minimum norm problem by the technique of Lagrange multipliers.
- This leads to upper and lower bounds on $|f_{\omega\pi}(t_1)|$:

$$|f_{\omega\pi}(t_1)| \leq \frac{\left| g(0) + \frac{1}{2\pi i} \int_{x_\pi}^{1} \frac{\Delta(x)}{x - z_1} \mathrm{d}x \right| + \frac{z_1 I'}{\sqrt{1 - z_1^2}}}{C(z_1)},$$

$$|f_{\omega\pi}(t_1)| \geq \frac{\left| g(0) + \frac{1}{2\pi i} \int_{x_\pi}^{1} \frac{\Delta(x)}{x - z_1} \mathrm{d}x \right| - \frac{z_1 I'}{\sqrt{1 - z_1^2}}}{C(z_1)}.$$

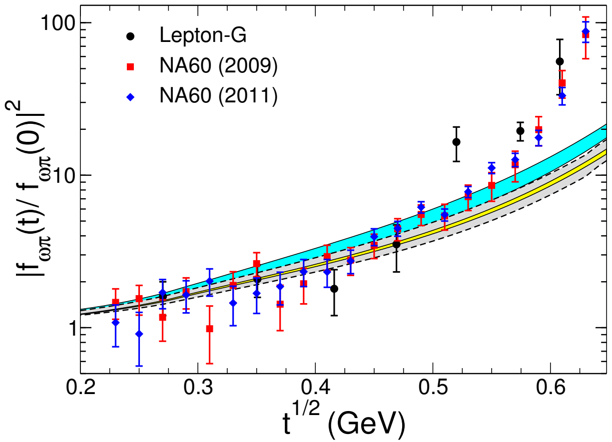
- Analogous expressions for the real-analytic case exist and applicable to the model of Köpp
- We now have the machinery to evaluate the bounds at an energy below the elastic threshold



Upper and lower bounds compared with experimental data on $|f_{\omega\pi}(t)/f_{\omega\pi}(0)|^2$. Solid red line: bounds calculated using the non-analytic form factors from Schneider et al. Dashed blue: optimal bounds calculated with input from Köpp. The dotted blue line is the upper bound calculated with the input from Köpp, but using the nonoptimal expression. The data are from Lepton-G, NA60 (2009), and NA60 (2011). The yellow band is the result of the dispersive calculation performed in Schneider et al..

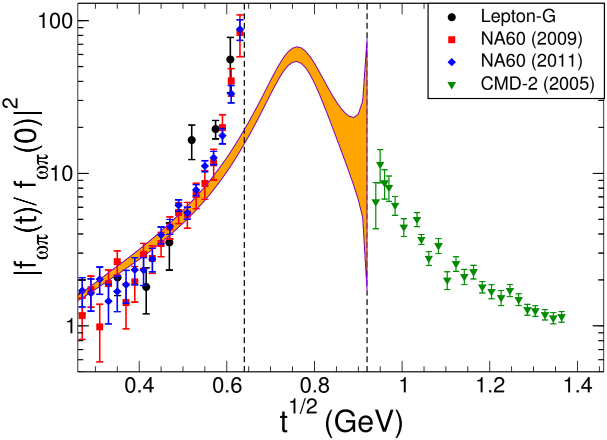
Discussion

- Note that smaller values of I lead to narrower allowed intervals for $|f_{\omega\pi}(t)|$ at $t < t_+$.
- the bounds can be made tighter in principle by taking into account more intermediate states,
- Could possibly be extended to the $\phi - \pi$ form factor
- One may be able to study the impliations to the OZI rule
- The bounds have been made more stringent by extending the formalism and including the data from $e^+e^- \to \pi^0\omega$, and improving the N/D model of Köpp, and other weight functions which can be reliably estimated (I. Caprini, arXiv:1505.05282, Physical Review D, to appear)



Upper and lower bounds compared with experimental data on $|f_{\omega\pi}(t)/f_{\omega\pi}(0)|^2$. Cyan band: bounds calculated using the partial wave amplitude $f_1(t)$ from N/D model. Grey band: bounds calculated using in the dispersive calculation. The yellow band is the result of the dispersive calculation. The data are from Lepton-G, NA60 (2009) and NA60 (2011).

Extended results



Orange band: bounds on $|f_{\omega\pi}(t)/f_{\omega\pi}(0)|^2$ in the whole region $t < t_+$, obtained with the improved N/D model. The data are from Lepton-G, NA60 (2009), NA60 (2011) and CMD-2 (2005).

Conclusions

- Work is motivated by the discrepancy between dispersive analysis and experimental information above 600 MeV
- Dispersive analysis uses elastic unitarity
- Need a framework to avoid this assumption
- Method of unitarity provides this framework
- Downside is that only bounds emerge
- Bounds prove to be competitive and spur the need for new experiments
- Our method required extension of the method to non-real analytic function
- This is a significant theoretical contribution based on general principles of analyticity and unitarity