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$a_0 - f_0$ mixing in the Khuri-Treiman equations for $\eta \to 3\pi$

Bachir Moussallam

work with: *Miguel Albaladejo*

Motivations

- Chiral dynamics important for the η meson ($\Gamma = 1.3$ KeV)
- Three flavour expansion: $\eta \to 3\pi$ modes driven by

$$\frac{m_d - m_u}{2m_s - m_u - m_d}$$

  Small QED contrib.$\Rightarrow$ key process for determination of quark masses .
- Chiral expansion of $\eta \to 3\pi$ amplitude at NLO[Gasser, Leutwyler NP B250(1985)539] is very predictive.
- But: precise measurements of Dalitz plot energy dependence e.g. α parameter in $\eta \to 3\pi^0$[Crystal Ball (2001,2009), WASA(2007,2009), KLOE(2008)]

$$\begin{aligned}\alpha^{exp} &= -(3.15 \pm 0.15) \cdot 10^{-3} \\ \alpha^{NLO} &= +1.41 \cdot 10^{-3}\end{aligned}$$

not reproduced at NLO

- NNLO amplitude computed[Bijnens, Ghorbani JHEP0711(2007)030] but model dependence (C_i couplings)

- Observation: final-state interaction mistreated by ChPT then[Kambor, Wiesendanger, Wyler (1996), Anisovich, Leuwyler (1996)]
  - → Use ChPT in unphysical region (small/no FSI)
  - → Implement FSI via Khuri-Treiman formalism

- In this talk: extend Khuri-Treiman to accomodate inelastic rescattering involving $K\overline{K}$ channel.
  - → Goal: account for "$a_0 - f_0$ mixing" effect (actually double resonance effect)
  - → Study to what extent it influences energy dependence at low energy.

Khuri-Treiman: elastic case

- Khuri and Treiman[Phys. Rev. 119 (1960) 1115]: $\pi\pi$ rescattering in $K \to 3\pi$
  - → Consider $K\pi \to \pi\pi$ instead of $K \to 3\pi$, write fixed t dispersion relations, S-wave rescattering.

- Application to $\eta \to 3\pi$:[Kambor, Wiesendanger, Wyler, NP B465(1996)215, Anisovich, Leutwyler PL B375(1996)335]
  - → Rescattering: S+P waves, "reconstruction theorem"

$$\mathcal{T}^{\eta\pi^0\to\pi^+\pi^-}(s,t,u) = -\epsilon_L \times$$

$$\left[M_0(s) - \frac{2}{3}M_2(s) + (s-u)M_1(t) + M_2(t) + (t \leftrightarrow u)\right]$$

  - → ϵ_L carries information on quark masses

$$\epsilon_L = \frac{m_d^2 - m_u^2}{m_s^2 - m_{ud}^2}\,\frac{m_K^2}{m_\pi^2}\,\frac{m_K^2 - m_\pi^2)}{3\sqrt{3}F_\pi^2}$$

- Khuri-Treiman eqs re-expressed using Omnès-Muskhelishvili technique involving four polynomial parameters ([AL (1996)])

$$M_0(w) = \Omega_0(w) \left[\alpha_0 + w\,\beta_0 + w^2\left(\gamma_0 + \hat{I}_0(w)\right)\right]$$
$$M_1(w) = \Omega_1(w)\, w \left(\beta_1 + \hat{I}_1(w)\right)$$
$$M_2(w) = \Omega_2(w)\, w^2 \hat{I}_2(w)$$

Omnès functions

$$\Omega_I(s) = \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'(s'-s)}\,\delta_I(s')\right]$$

Integrals

$$\hat{I}_I(w) = -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds'\,\frac{\mathrm{Im}\left(1/\Omega_I(s')\right)}{(s')^n(s'-w)}\,\hat{M}_I(s')$$

where $\hat{M}_I(s')$: left-cut parts in $\eta\pi \to (\pi\pi)_I$ partial-waves with $J = 0, 1$ and $I = 0, 1, 2$

$$\begin{aligned}
\mathcal{T}_0^0(s) &= \tfrac{\sqrt{6}\epsilon_L}{32\pi}\left(M_0(s) + \hat{M}_0(s)\right)\\
\mathcal{T}_1^1(s) &= \tfrac{\epsilon_L}{48\pi}\kappa(s)\left(M_1(s) + \hat{M}_1(s)\right)\\
\mathcal{T}_0^2(s) &= -\tfrac{\epsilon_L}{16\pi}\left(M_2(s) + \hat{M}_2(s)\right)
\end{aligned}$$

→ (Elastic) unitarity eqs. for $\mathcal{T}_I^J$ obeyed.

→ $\hat{M}_I$ expressed in terms of M_I ⇒ Self-consistent set of equations for M_I functions.

→ Technical issues (left-hand cut overlaps w. right-hand cut): see [KWW(1996), Stefan Lanz, PHD thesis (2011)]

- Matching with ChPT at NLO:

$$\mathcal{T}^{KT}(s,t,u) - \mathcal{T}^{ChPT}(s,t,u) = O(p^6)$$

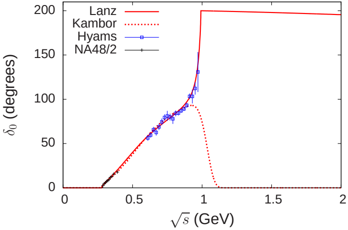
→ Use decomp. theorem and

$$\text{disc}\left[M_I(w) - \bar{M}_I(w)\right] = O(p^6)$$

→ Expand $\mathcal{T}^{KT} - \mathcal{T}^{ChPT}$ as polynomial in s, t, u: four matching equations

$$\begin{aligned}
\alpha_0 &= \quad 9\left(\frac{1}{2}\bar{M}_2'' - \hat{l}_2\right) s_0^2 + 3(\bar{M}_2' - \bar{M}_1)\, s_0 + \bar{M}_0 + \frac{4}{3}\bar{M}_2 \\
\beta_0 &= -9\left(\frac{1}{2}\bar{M}_2'' - \hat{l}_2\right) s_0 + \bar{M}_0' + 3\bar{M}_1 - \frac{5}{3}\bar{M}_2' - \Omega_0'\alpha_0 \\
\beta_1 &= \quad \bar{M}_1' + \frac{1}{2}\bar{M}_2'' - \hat{l}_1 - \hat{l}_2 \\
\gamma_0 &= \quad \frac{1}{2}\bar{M}_0'' + \frac{2}{3}\bar{M}_2'' - \hat{l}_0 - \frac{4}{3}\hat{l}_2 - \frac{1}{2}\Omega_0''\alpha_0 - \Omega_0'\beta_0
\end{aligned}$$

- Formally analogous to [AL(1996)] but: important difference:
  - → Approximation $\gamma_0 \simeq 0$ was used [AL(1996), S. Lanz (2011)]
  - → Approximation valid only if phase-shift $\delta_0^0(s) \to 0$ when $s > 1$ GeV^2



  - → For more general prescriptions (e.g. $\delta_0^0(s) \to \pi$) one must let γ_0 adapt itself.

Khuri-Treiman: $K\overline{K}$ inelastic extension

- Amplitudes to consider:

$$K^+K^-,\ K^0\bar{K}^0,\ K^+\bar{K}^0 \quad \begin{array}{l} \to \pi^+\pi^-,\ \pi^0\pi^0,\ \pi^+\pi^0 \\ \to \eta\pi^0,\ \eta\pi^+ \end{array}$$

- Contain both isospin conserving and isospin violating components (in general)

- Isospin violating amplitudes defined after partial-wave projection ($s+t+u$ depends on $m^2_{K^+}$, $m^2_{K^0}$)

- Consequence

  → No reconstruction theorem for IV amplitudes

  → We will need some modelling of the partial-wave amplitudes

- Consider $\underline{J = 0}$
  - → Isospin conserving amplitudes

$$\underline{\mathbf{T}^{(0)}}: \begin{pmatrix} \pi\pi \to \pi\pi & \pi\pi \to K\overline{K} \\ \pi\pi \to K\overline{K} & K\overline{K} \to K\overline{K} \end{pmatrix}_{I=0}$$

$$\underline{\mathbf{T}^{(1)}}: \begin{pmatrix} \eta\pi \to \eta\pi & \eta\pi \to K\overline{K} \\ \eta\pi \to K\overline{K} & K\overline{K} \to K\overline{K} \end{pmatrix}_{I=1}$$

$$\underline{T^{(2)}}: \pi^+\pi^0 \to \pi^+\pi^0$$

  - → Isospin violating amplitudes

$$I = 0 \to 1 \quad \mathbf{T}^{(01)}: \begin{pmatrix} (\pi\pi)_0 \to \eta\pi & (K\overline{K})_0 \to \eta\pi \\ (\pi\pi)_0 \to (K\overline{K})_1 & (K\overline{K})_0 \to (K\overline{K})_1 \end{pmatrix}$$

$$I = 1 \to 2 \quad \mathbf{T}^{(12)}: \begin{pmatrix} \eta\pi^+ \to \pi^+\pi^0 \\ K^+\bar{K}^0 \to \pi^+\pi^0 \end{pmatrix}$$

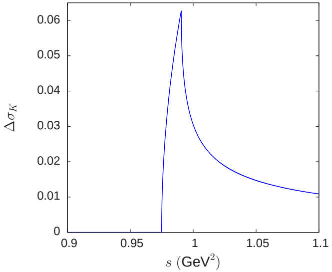
- Unitarity relation (J=0): first order in isospin breaking:
  - → $\mathbf{T}^{(01)}$ relation:

$$\mathrm{Im}\,[\mathbf{T}^{(01)}] = \mathbf{T}^{(0)*}\,\Sigma^0\,\mathbf{T}^{(01)} + \mathbf{T}^{(01)*}\,\Sigma^1\,\mathbf{T}^{(1)} + \mathbf{T}^{(0)*}\begin{pmatrix} 0 & 0 \\ 0 & \Delta\sigma_K \end{pmatrix}\mathbf{T}^{(1)}$$

$\Delta\sigma_K(s) = \frac{1}{2}(\sigma_{K^+}(s) - \sigma_{K^0}(s))$

with $\sigma_P(s) = \sqrt{1 - 4m_K^2/s}$

[Achasov,Devyanin,Shestakov, PL B88(1979)367]



  - → $\mathbf{T}^{(12)}$ relation:

$$\mathrm{Im}\,[\mathbf{T}^{(12)}] = \mathbf{T}^{(1)*}\,\Sigma^1\,\mathbf{T}^{(12)} + \mathbf{T}^{(12)*}\sigma_\pi\,T^{(2)}$$

- Separate PW amplitudes into left-cut and right-cut components

$$\mathbf{T}^{(01)} = \frac{\sqrt{6}\epsilon_L}{32\pi}\left(\mathbf{M}_0 + \hat{\mathbf{M}}_0\right), \quad \mathbf{T}^{(12)} = -\frac{\epsilon_L}{16\pi}\begin{pmatrix} M_2 + \hat{M}_2 \\ G_{12} + \hat{G}_{12} \end{pmatrix}$$

Generalized KT equations:

$$\mathbf{M}_0(w) = \boldsymbol{\Omega}_0(w)\left[\mathbf{P}_0(w) + w^2\left(\hat{\mathbf{I}}_A(w) + \hat{\mathbf{I}}_B(w)\right)\right]{}^t\boldsymbol{\Omega}_1(w)$$

- → $\boldsymbol{\Omega}_I$: Omnès-Muskhelishvili matrices
- → $\mathbf{P}_0$: polynomials, 12 parameters
- → "left-cut" integrals

$$\hat{\mathbf{I}}_A(w) = -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2(s'-w)}\left[\mathrm{Im}\,\boldsymbol{\Omega}_0^{-1}\,\hat{\mathbf{M}}_0\,{}^t\boldsymbol{\Omega}_1^{-1} + \boldsymbol{\Omega}_0^{-1*}\,\hat{\mathbf{M}}_0\mathrm{Im}\,{}^t\boldsymbol{\Omega}_1^{-1}\right]$$

$$\hat{\mathbf{I}}_B(w) = \frac{32}{\sqrt{6}\epsilon_L}\int_{4m_\pi^2}^{\infty}\frac{ds'\,\Delta\sigma_K(s')}{(s')^2(s'-w)}\boldsymbol{\Omega}_0^{-1*}\mathbf{T}^{(0)*}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{T}^{(1)}\,{}^t\boldsymbol{\Omega}_1^{-1}$$

- Note on the numerics:
  Each component of $\mathbf{M}_0$ must satisfy dispersion relation:

$$[\mathbf{M}_0]_{ij}(w) = \alpha_{ij} + \beta_{ij} w + \frac{w^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{(s')^2 (s'-w)} \mathrm{disc}[[\mathbf{M}_0]_{ij}](s')$$

  → We do check that it works !

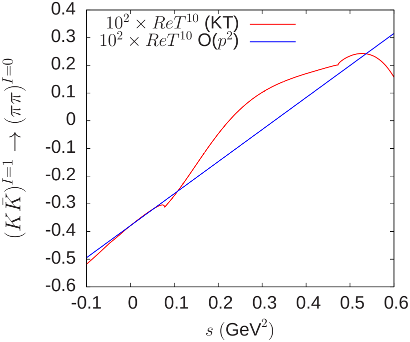
  → Use this for computing angular integrations [w integrals (complex contour) performed analytically]

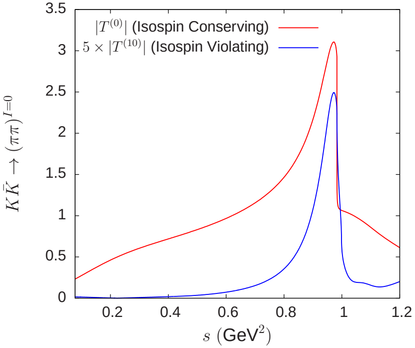
$\eta\pi$ scattering model:

- We have developped a two-channel unitary model for $\mathbf{T}^{(1)}$ [*manuscript with M. Albaladejo to appear soon*]

  1) Reproduces $a_0(980)$, $a_0(1450)$ complex poles
  2) Matches with $\eta\pi/K\overline{K}$ scattering amplitudes from NLO ChPT at low energies
  3) Also allow to compute $I=1$ scalar $\eta\pi$ and $K\overline{K}$ form factors and constrained by ChPT results from scalar radii: $\langle r^2\rangle_S^{\eta\pi}$, $\langle r^2\rangle_S^{K\overline{K}}$.

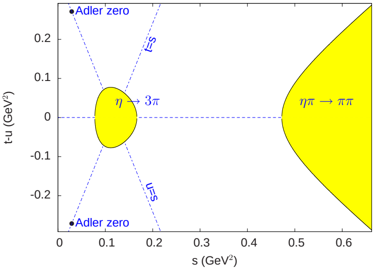
- Simple model for $K\overline{K}$ amplitudes:
  - → set corresponding $\hat{\mathbf{M}}_{ij} = 0$ [only $\hat{\mathbf{M}}_{11} \neq 0$]
  - → corresponding polynomial parameters determined by matching with ChPT at $O(p^2)$
  - → Illustration:



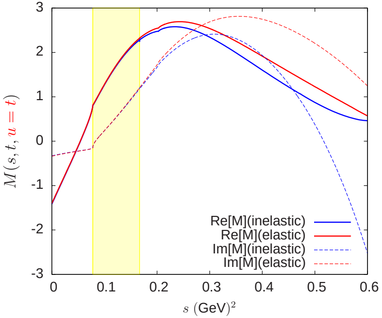


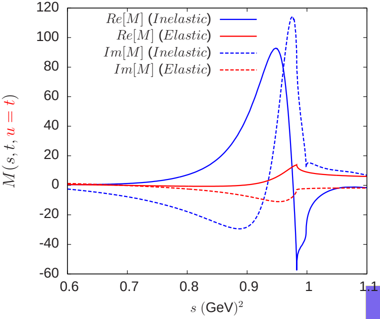
$\eta \to \pi^+\pi^-\pi^0$ and $\eta\pi^0 \to \pi^+\pi^-$ amplitudes

- Physical regions:



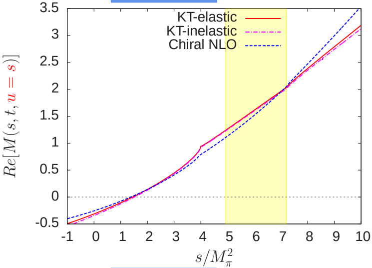
- Amplitude along $t = u$ line:

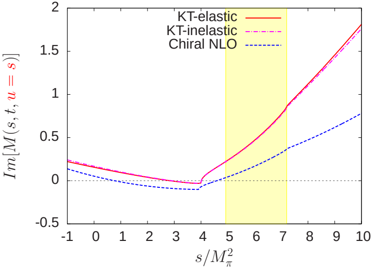




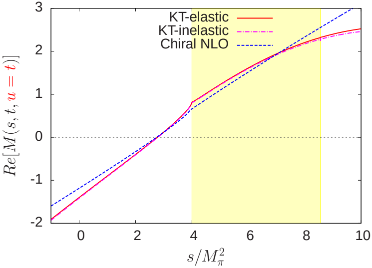
$\eta \to \pi^+\pi^-\pi^0$ at low energy

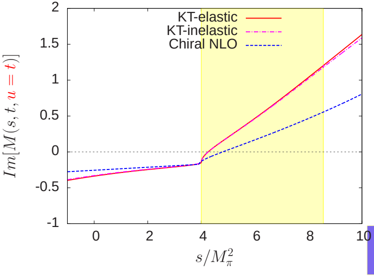
- Along $u = s$ line (Adler zero):





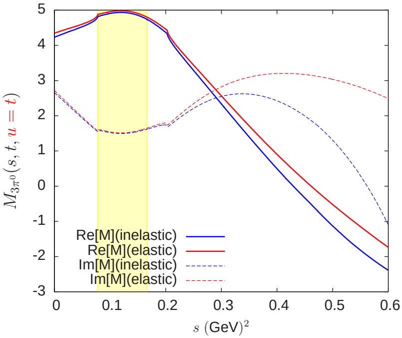
- Along $u = t$ line



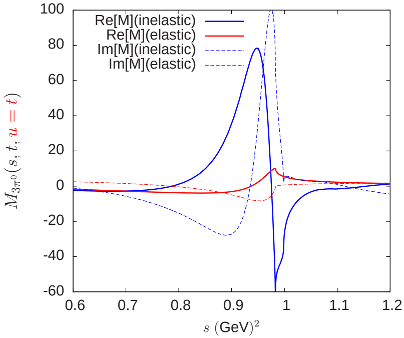


$\eta \to \pi^0\pi^0\pi^0$ and $\eta\pi^0 \to \pi^0\pi^0$ amplitudes

Low energy



Resonance region



→ Influence of inelasticity more visible at low energy

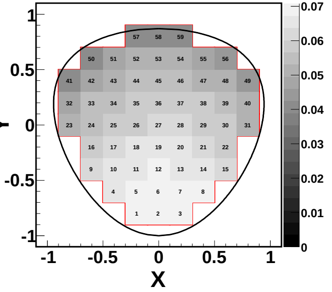
Comparison with experiment

Dalitz plot parameters

- Charged decay: define coordinates X, Y

$$X = \sqrt{3}\frac{E_{\pi^+} - E_{\pi^0}}{Q_c}, \quad Y = 3\frac{E_{\pi^0} - M_{\pi^0}}{Q_c}$$

$$Q_c = M_\eta - 2M_{\pi^+} - M_{\pi^0}$$



Accurate description with 4 parameters

$$|\mathcal{T}_c(X,Y)|^2 = |\mathcal{T}_c(0,0)|^2 \left(1 + a\,Y + b\,Y^2 + d\,X^2 + f\,Y^3\right)$$

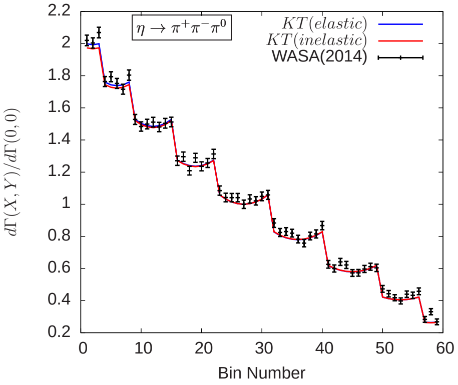
Charge conjugation : $X \to -X$ invariance

- Neutral decay: description with one parameter

$$|\mathcal{T}_n(X,Y)|^2 = |\mathcal{T}_n(0,0)|^2 \left(1 + 2\,\alpha\,(X^2 + Y^2)\right)$$

Comparison of amplitude variation with WASA

- [WASA col.,Phys.Rev.C90 (2014)4, 045207]



Dalitz plot parameters

(Khuri-Treiman equations with polynomial matching to NLO ChPT)

$\eta \to \pi^+\pi^-\pi^0$

Param.	$O(p^4)$	KT-elastic	KT-coupled	WASA	KLOE
a	-1.320	-1.154	-1.146	-1.144(18)	-1.090(14)
b	0.422	0.202	0.181	0.219(19)	0.124(11)
f	0.015	0.107	0.116	0.115(37)	0.140(20)
d	0.083	0.088	0.090	0.086(18)	0.057(17)

$\eta \to \pi^0\pi^0\pi^0$

Param.	$O(p^4)$	KT-elastic	KT-coupled	PDG
α	+0.014	-0.027	-0.031	-0.0315(15)

Conclusions

- Khuri-Treiman formalism for $\eta \to 3\pi$ extended to accomodate inelasticity to $K\overline{K}$ channels. Implementation with simple approximation for left-hand cuts of $K\overline{K}$ amplitudes

- Khuri-Treiman amplitudes + polynomial matching to NLO ChPT (with no approximation) seem compatible with recent experiments

- Influence of $K\overline{K}$ channels on Dalitz plot parameters estimated to be $5 - 10\%$.

- Effect of a_0, f_0 superposition rather large at 1 GeV [$\pi^+\pi^- \to \eta\pi^0$ amplitude measurable ?]