SFB 1044

THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

Cluster of Excellence

PRISMA

Precision Physics, Fundamental Interactions and Structure of Matter

CHIRAL PERTURBATION THEORY OF HYPERFINE SPLITTING IN MUONIC HYDROGEN

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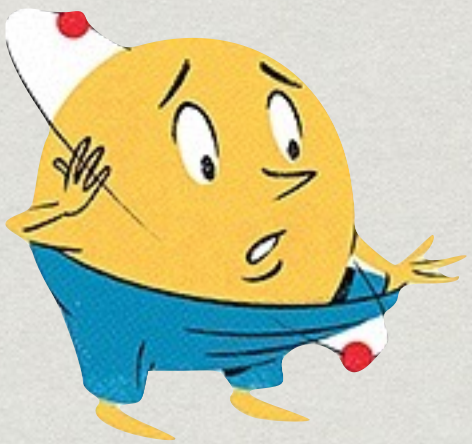
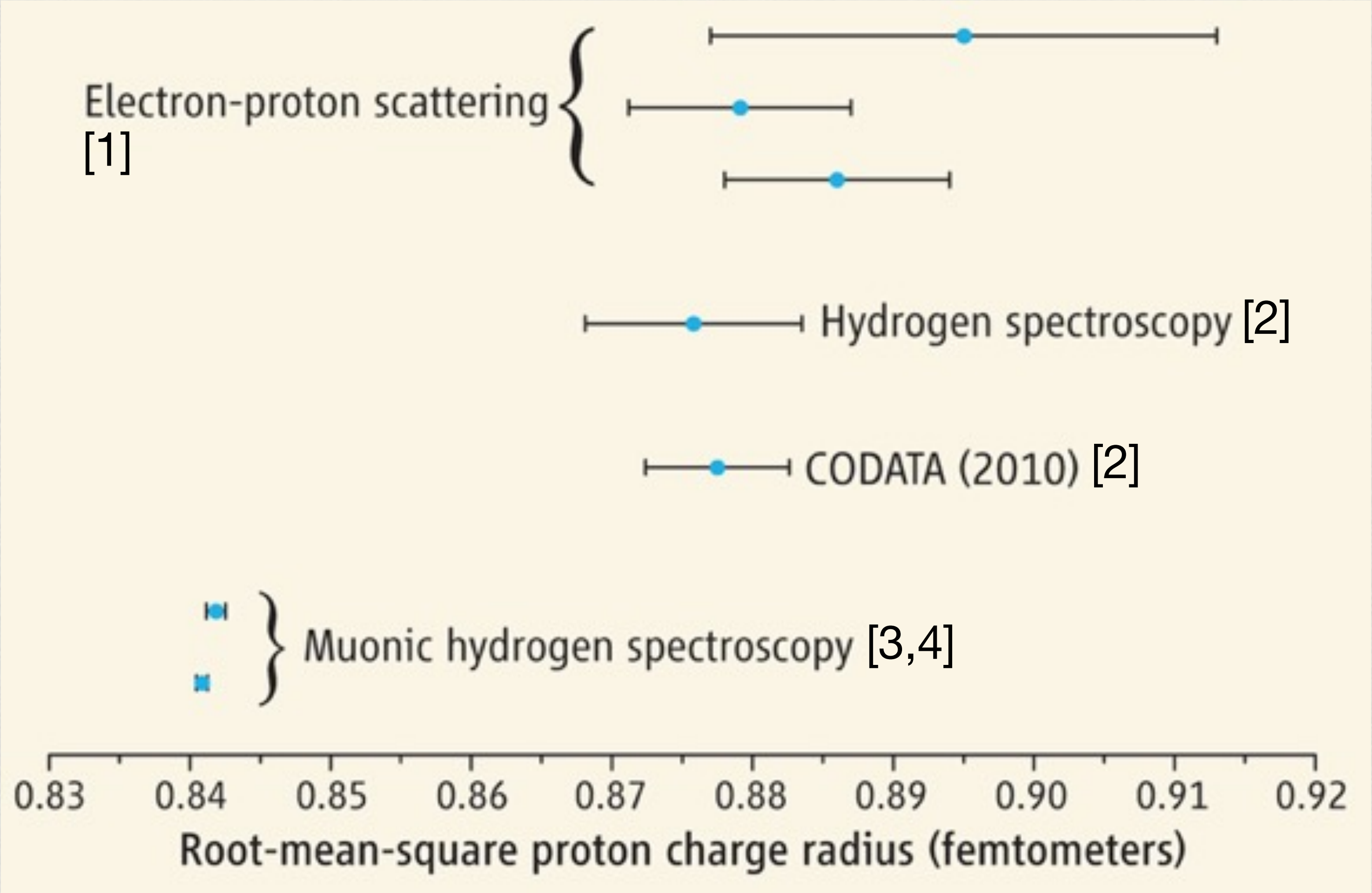
JG|U

Proton Size Puzzle





[1] J. C. Bernauer *et al.*, Phys. Rev. Lett. **105**, 242001 (2010).
[2] P. J. Mohr, et al., Rev. Mod. Phys. **84**, 1527 (2012).
[3] R. Pohl, A. Antognini *et al.*, Nature **466**, 213 (2010).
[4] A. Antognini *et al.*, Science **339**, 417 (2013).

seven standard-deviation discrepancy (7σ) !!!

$$[R_E^{\mu\mathrm{H}} = 0.84087(39)\,\mathrm{fm}] \longleftrightarrow [R_E^{\mathrm{CODATA\ 2010}} = 0.8775(51)\,\mathrm{fm}]$$

µH Spectroscopy

$$\Delta E^{\rm th}_{\rm LS} = 206.0336(15) - 5.2275(10)\,(R_E/\text{fm})^2 + \Delta E^{\rm TPE}_{\rm LS}\,, \quad \text{with } \Delta E^{\rm TPE}_{\rm LS} = 0.0332(20),$$

$$\Delta E^{\rm th}_{\rm HFS} = 22.9763(15) - 0.1621(10)\,(R_Z/\text{fm}) + \Delta E^{\rm (pol)}_{\rm HFS}\,, \quad \text{with } \Delta E^{\rm (pol)}_{\rm HFS} = 0.0080(26),$$

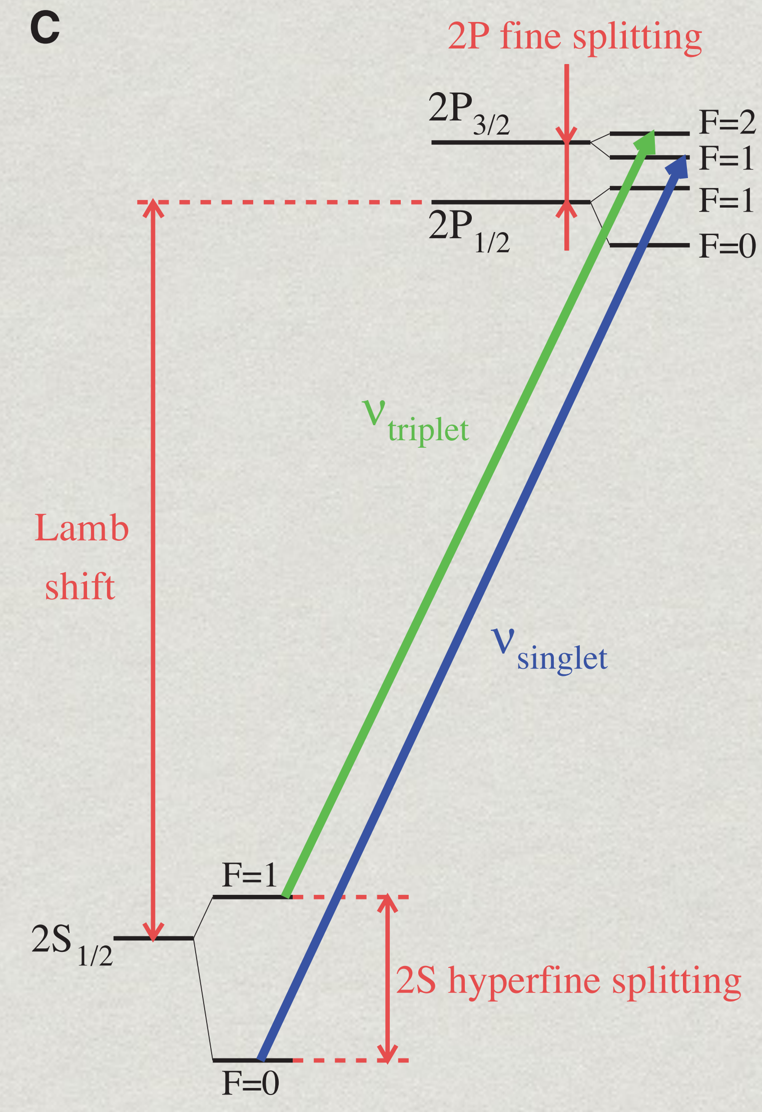
radiative, relativistic and recoil effects

finite size effects: proton structure

two-photon exchange (TPE) effects, including the proton polarizability

$$\Delta E^{\rm exp}_{\rm LS} = 202.3706(23)\ \text{meV},$$
$$\Delta E^{\rm exp}_{\rm HFS} = 22.8089(51)\ \text{meV}$$

c



A. Antognini, et al., Annals Phys. **331** (2013) 127–145.

Finite Size Effects

- HFS:

$$\Delta E_{nS}(\mathrm{LO}+\mathrm{NLO}) = E_F\,[1 - 2\,Z\alpha m_r\, R_Z]$$

Fermi - Energy:

$$E_F(nS) = \frac{8}{3}\frac{Z\alpha}{a^3}\frac{1+\kappa}{mM}\frac{1}{n^3}$$

with Bohr radius $a = 1/(Z\alpha m_r)$

NLO becomes appreciable in μH

- Lamb shift:

$$\Delta E_{nl}(LO) = \delta_{l0}\,\frac{2\pi Z\alpha}{3}\,\frac{1}{\pi(an)^3}\,R_E^2$$

wave function at the origin

$$\Delta E_{nS}(\mathrm{NLO}) = -\frac{Z\alpha}{3n^3a^4}R^3_{E(2)}$$

J. L. Friar, Annals Phys. **122** (1979) 151.

$$R_E^2 = -6\lim_{Q^2\to 0}\frac{\mathrm{d}}{\mathrm{d}Q^2}G_E(Q^2)$$

$$R^3_{E(2)} = \frac{48}{\pi}\int_0^\infty \frac{\mathrm{d}Q}{Q^4}\left\{G_E^2(Q^2) - 1 + \frac{1}{3}\langle r^2\rangle_E\,Q^2\right\}$$

RAPID COMMUNICATIONS

PHYSICAL REVIEW A **91**, 040502(R) (2015)

Breakdown of the expansion of finite-size corrections to the hydrogen Lamb shift in moments of charge distribution

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(Received 13 February 2015; published 20 April 2015)

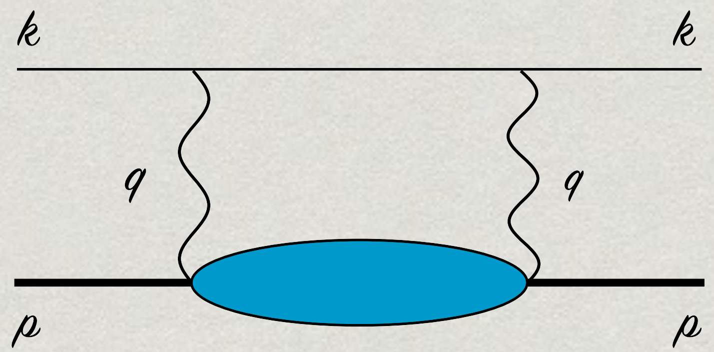
- the finite-size effects are not always expandable in the moments of charge distribution
  - a tiny non-smoothness of the electric form factor $G_E(Q^2)$ at scales comparable to the inverse Bohr radius can break down this expansion

$$E^{\mathrm{FF}(1)}_{2P-2S} = \int_0^\infty \mathrm{d}Q\, w(Q)\, G_E(Q^2) \quad \text{with} \quad w(Q) = -\frac{4}{\pi}(Z\alpha)^5 m_r^4\, Q^2 \frac{(Z\alpha m_r)^2 - Q^2}{[(Z\alpha m_r)^2 + Q^2]^4}$$

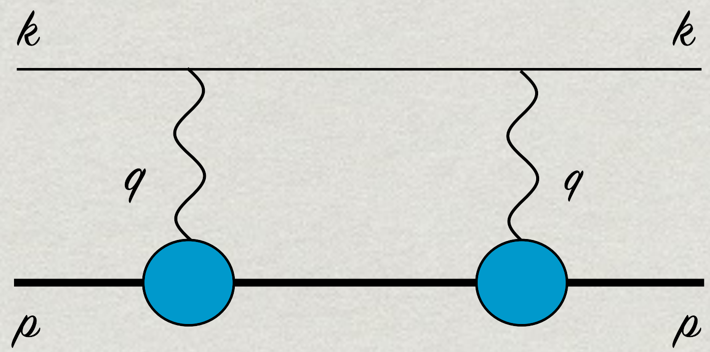
- one needs to know all the "soft" (below several MeV) contributions to proton electric FF to pcm accuracy

Structure Effects through TPE

- proton structure effects at subleading orders arise through multi-photon processes



“polarizability” contribution:
“inelastic” contribution,
“subtraction” term (Lamb shift)



“elastic” contribution:
finite-size recoil,
3rd Zemach moment (Lamb shift),
Zemach radius (HFS)

- “blob” corresponds to doubly-virtual Compton scattering (VVCS):

$$T^{\mu\nu}(q,p) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu,Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2}\,q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}\,q^\nu\right) T_2(\nu,Q^2),$$

$$+\frac{1}{M}\gamma^{\mu\nu\alpha}q_\alpha S_1(\nu,Q^2) + \frac{1}{M^2}\left(\gamma^{\mu\nu}q^2 + q^\mu\gamma^{\nu\alpha}q_\alpha - q^\nu\gamma^{\mu\alpha}q_\alpha\right)S_2(\nu,Q^2)$$

or: $T_A^{\mu\nu}(q,p) = \frac{i}{M}\,\epsilon^{\mu\nu\alpha\beta}\,q_\alpha s_\beta\,S_1(\nu,Q^2) + \frac{i}{M^3}\,\epsilon^{\mu\nu\alpha\beta}\,q_\alpha(p\cdot q\;s_\beta - s\cdot q\;p_\beta)\,S_2(\nu,Q^2)$

Eur. Phys. J. C (2014) 74:2852
DOI 10.1140/epjc/s10052-014-2852-0

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Chiral perturbation theory of muonic-hydrogen Lamb shift: polarizability contribution

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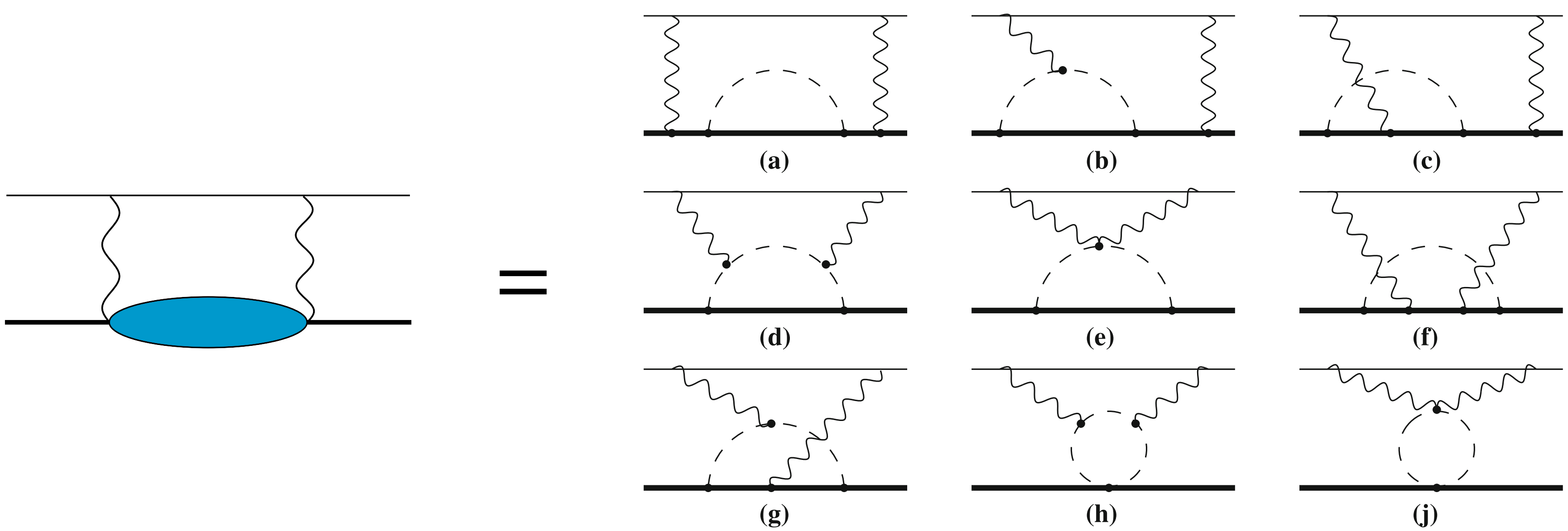
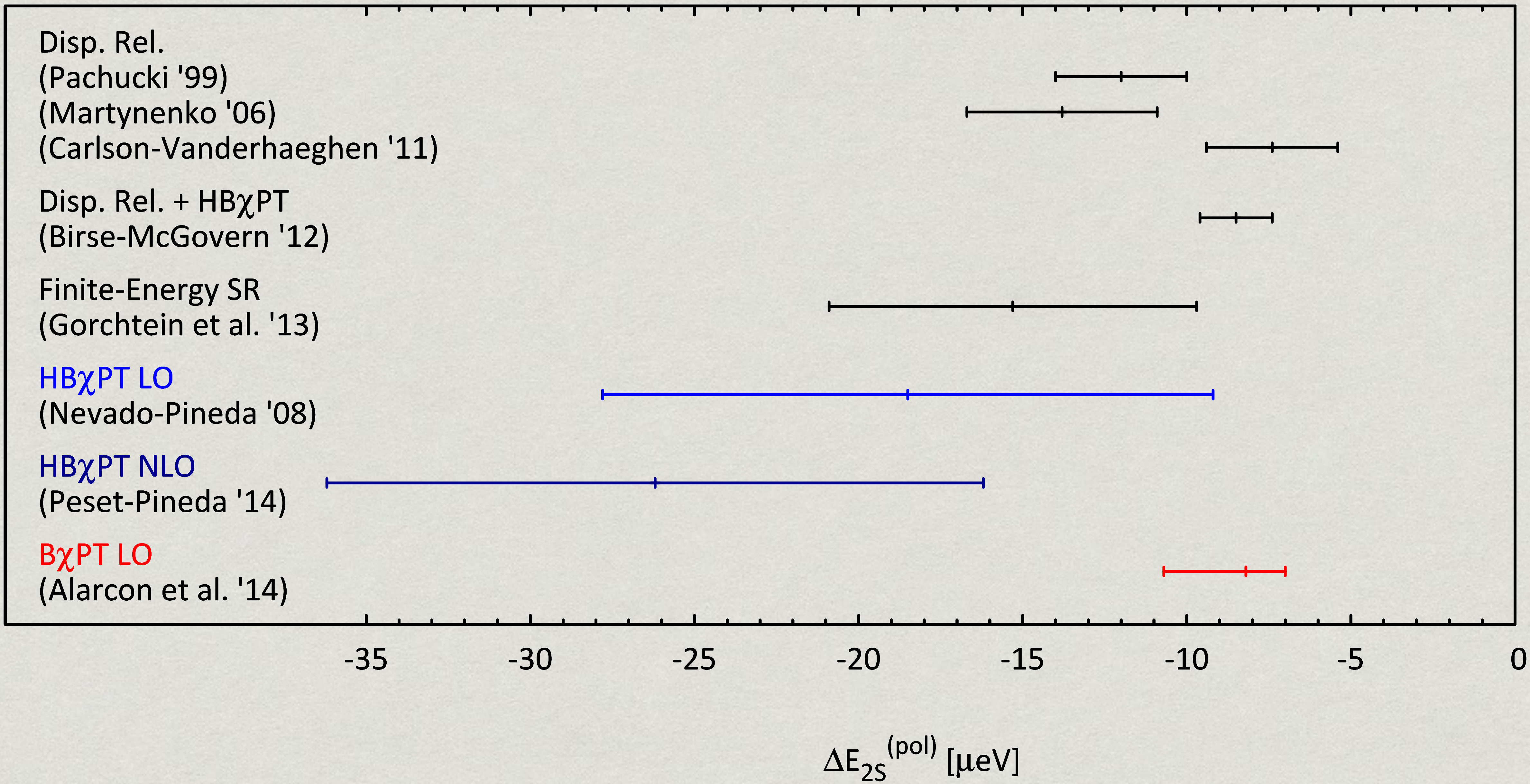
$$\Delta E_{nS} = \frac{\alpha m}{i\pi^3}\,\phi_n^2 \int_0^\infty \mathrm{d}\nu \int \mathrm{d}\boldsymbol{q}\, \frac{\left(Q^2 - 2\nu^2\right) T_1(\nu, Q^2) - (Q^2+\nu^2) T_2(\nu, Q^2)}{Q^4 (Q^4 - 4m^2\nu^2)}$$



with corrections due to "elastic" proton FFs subtracted, i.e., "polarizability" contribution alone

TPE in μH Lamb Shift: Polarizability Contribution



- BχPt result is in good agreement with calculations based on dispersive sum rules

χPt Calculations for TPE in μH Lamb Shift

[1] D. Nevado, A. Pineda, Phys. Rev. **C**77 (2008) 035202.
[2] A. Pineda, Physical Review **C**71 (2005) 065205.
[3] C. Peset, A. Pineda (2014).
[4] J. M. Alarcon, V. Lensky, V. Pascalutsa, Eur. Phys. J. **C**74 (2014) 2852.

	Nevado & Pineda HBχPT	Alarcón *et al.* BχPT	Alarcón *et al.* HBχPT	Peset & Pineda HBχPT^a
$\Delta E_{2S}^{(\text{subt})}$		-3.0	1.3	
$\Delta E_{2S}^{(\text{inel})}$		-5.2	-19.1	
$\Delta E_{2S}^{(\text{pol})}$	$-18.5(9.3)$	$-8.2(^{+1.2}_{-2.5})$	-17.85	$-26.2(10.0)$
$\Delta E_{2S}^{(\text{el})}$	$-10.1(5.1)$			$-8.3(4.3)$
ΔE_{2S}	-28.6			$-34.4(12.5)$

^{a}prediction at LO and NLO (including pions and deltas)

Table 1: Summary of available χPT calculations for the TPE correction to the Lamb shift in μH. Energy shifts are given in μeV, β_{M1} is given as $\times 10^{-4}$ fm^3.

"polarizability" contribution:

$$\overline{T}_1(\nu, Q^2) = \boxed{\overline{T}_1(0, Q^2)} + \frac{2\nu^2}{\pi} \int_{\nu_0}^{\infty} \mathrm{d}\nu' \, \frac{\sigma_T(\nu', Q^2)}{\nu'^2 - \nu^2},$$

$$\overline{T}_2(\nu, Q^2) = \frac{2}{\pi} \int_{\nu_0}^{\infty} \mathrm{d}\nu' \, \frac{\nu'^2 Q^2}{\nu'^2 + Q^2} \frac{\sigma_T(\nu', Q^2) + \sigma_L(\nu', Q^2)}{\nu'^2 - \nu^2}$$

subtraction function must be modelled!

TPE in μH HFS

$$\Delta E_{\mathrm{HFS}} = [1 + \Delta_{\mathrm{QED}} + \Delta^p_{\mathrm{weak}} + \Delta_{\mathrm{FSE}}]\, E_F$$

Fermi - Energy:

$$E_F(nS) = \frac{8}{3}\frac{Z\alpha}{a^3}\frac{1+\kappa}{mM}\frac{1}{n^3}$$

with $\Delta_{\mathrm{FSE}} = \Delta_Z + \Delta^p_{\mathrm{recoil}} + \Delta_{\mathrm{pol}}$

- TPE in HFS: $$\frac{E^{\mathrm{TPE}}}{E^p_F} = \frac{m}{2(1+\varkappa)\pi^4}\frac{1}{i}\int_0^\infty \mathrm{d}\nu \int \mathrm{d}\boldsymbol{q}\, \frac{1}{(Q^4 - 4m^2\nu^2)}\left\{\frac{(2Q^2-\nu^2)}{Q^2}S_1(\nu,Q^2) + 3\frac{\nu}{M}S_2(\nu,Q^2)\right\}$$

- TPE effect on the HFS is completely constrained by empirical information
- a BχPt calculation of the HFS in μH will put the reliability of both χPt and dispersive calculations to the test

$$S_1(\nu,Q^2) = 8\pi\alpha\left\{\frac{MQ^2F_1(Q^2)G_M(Q^2)}{Q^4-4M^2\nu^2} + \int_{\nu_0}^\infty \frac{\mathrm{d}\nu'}{\nu'^2-\nu^2}\, g_1(\nu',Q^2)\right\}$$

$$S_2(\nu,Q^2) = 8\pi\alpha\, M\nu\left\{-\frac{MF_2(Q^2)G_M(Q^2)}{Q^4-4M^2\nu^2} + \int_{\nu_0}^\infty \frac{\mathrm{d}\nu'}{\nu'^2-\nu^2}\frac{g_2(\nu',Q^2)}{\nu'^2}\right\}$$

TPE in μH HFS

$$\frac{E^{\mathrm{TPE}}}{E_F^p} = \frac{m}{2(1+\varkappa)\pi^4}\frac{1}{i}\int_0^\infty \mathrm{d}\nu \int \mathrm{d}\boldsymbol{q}\, \frac{1}{Q^4-4m^2\nu^2}\left\{\frac{(2Q^2-\nu^2)}{Q^2}S_1(\nu,Q^2)+3\frac{\nu}{M}S_2(\nu,Q^2)\right\}$$

$$= \frac{m}{4(1+\varkappa)\pi^4}\int_0^\infty \mathrm{d}Q\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi \mathrm{d}\theta\int_0^\pi \mathrm{d}\chi\;\sin^2\chi\sin\theta\times$$

$$\times\frac{Q}{Q^2+4m^2\cos^2\chi}\left\{(2+\cos^2\chi)S_1(\nu,Q^2)+\frac{3iQ\cos\chi}{M}S_2(\nu,Q^2)\right\}$$

- “polarizability” / “inelastic” contribution:

$$S_1(\nu,Q^2) = 8\pi\alpha\left\{\frac{MQ^2F_1(Q^2)G_M(Q^2)}{Q^4-4M^2\nu^2}+⨍_{\nu_0}^\infty\frac{\mathrm{d}\nu'}{\nu'^2-\nu^2}\,g_1(\nu',Q^2)\right\}$$

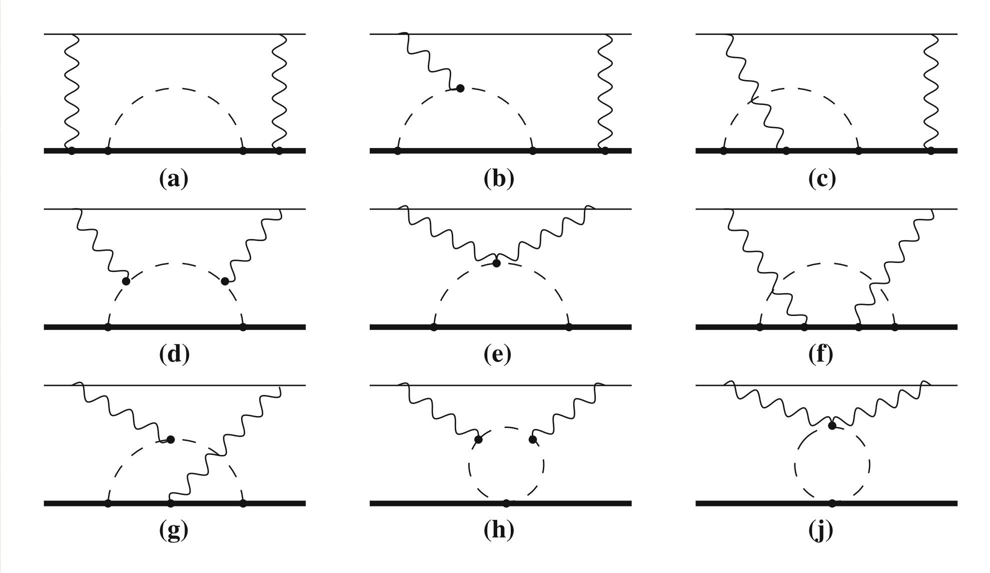
$$= S_1^{\mathrm{Born}}(\nu,Q^2) + \frac{2\pi\alpha}{M}F_2^2(Q^2)+8\pi\alpha⨍_{\nu_0}^\infty\frac{\mathrm{d}\nu'}{\nu'^2-\nu^2}\,g_1(\nu',Q^2) \quad \Delta_1$$

$$S_2(\nu,Q^2) = 8\pi\alpha\,M\nu\left\{-\frac{MF_2(Q^2)G_M(Q^2)}{Q^4-4M^2\nu^2}+⨍_{\nu_0}^\infty\frac{\mathrm{d}\nu'}{\nu'^2-\nu^2}\,\frac{g_2(\nu',Q^2)}{\nu'^2}\right\}$$

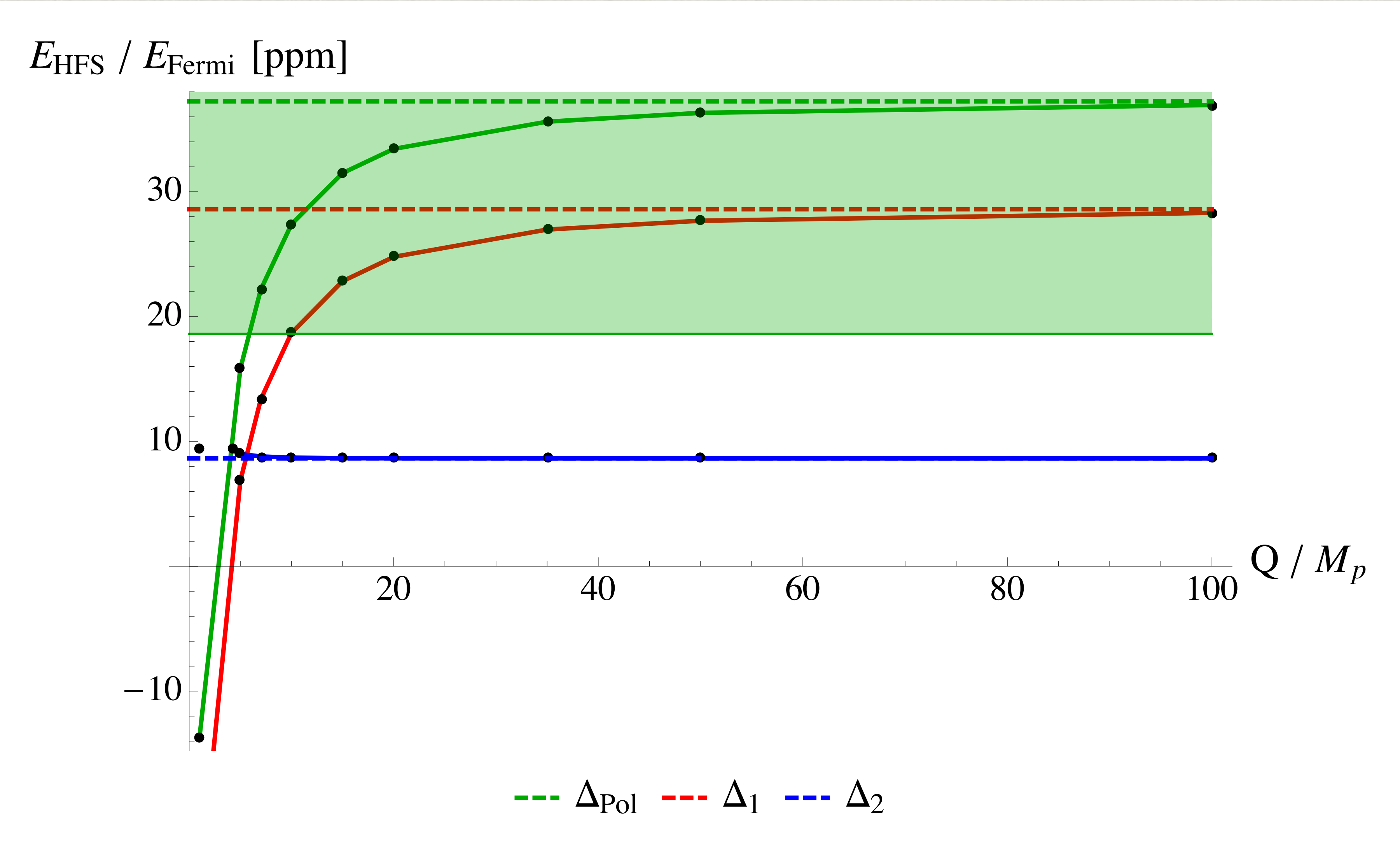
$$= S_2^{\mathrm{Born}}(\nu,Q^2) + 8\pi\alpha M\nu⨍_{\nu_0}^\infty\frac{\mathrm{d}\nu'}{\nu'^2-\nu^2}\,g_2(\nu',Q^2) \quad \Delta_2$$

Δ_1, Δ_2 → Δ_{Pol}

Pion-Nucleon-Loop Contribution



- cutoff dependence:



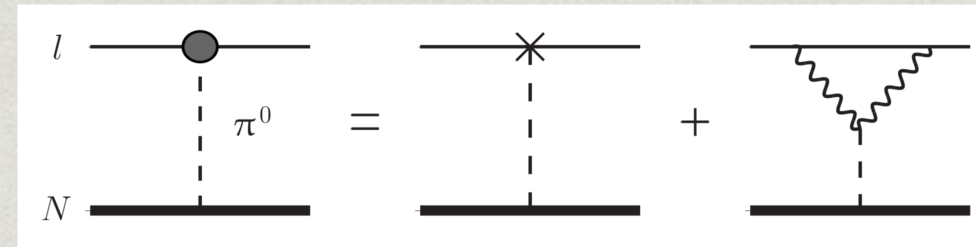
Fermi - Energy:

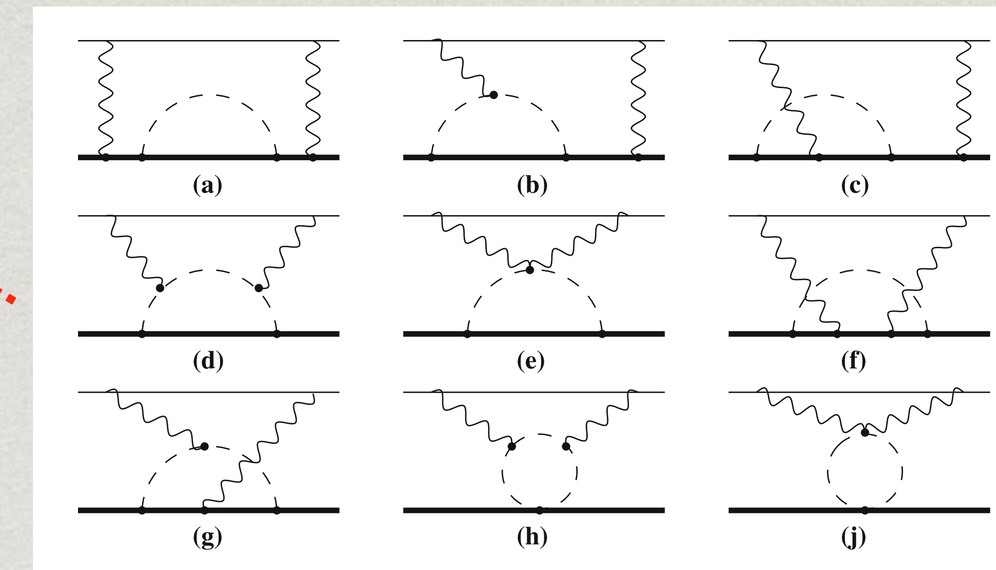
$$E_F(2S) = \frac{1}{3}\frac{Z\alpha}{a^3}\frac{1+\kappa}{mM} = 22.8054\ \mathrm{meV}$$

χPt Results for TPE in µH HFS

Fermi - Energy:

$$E_F(2S) = \frac{1}{3}\frac{Z\alpha}{a^3}\frac{1+\kappa}{mM} = 22.8054\ \text{meV}$$

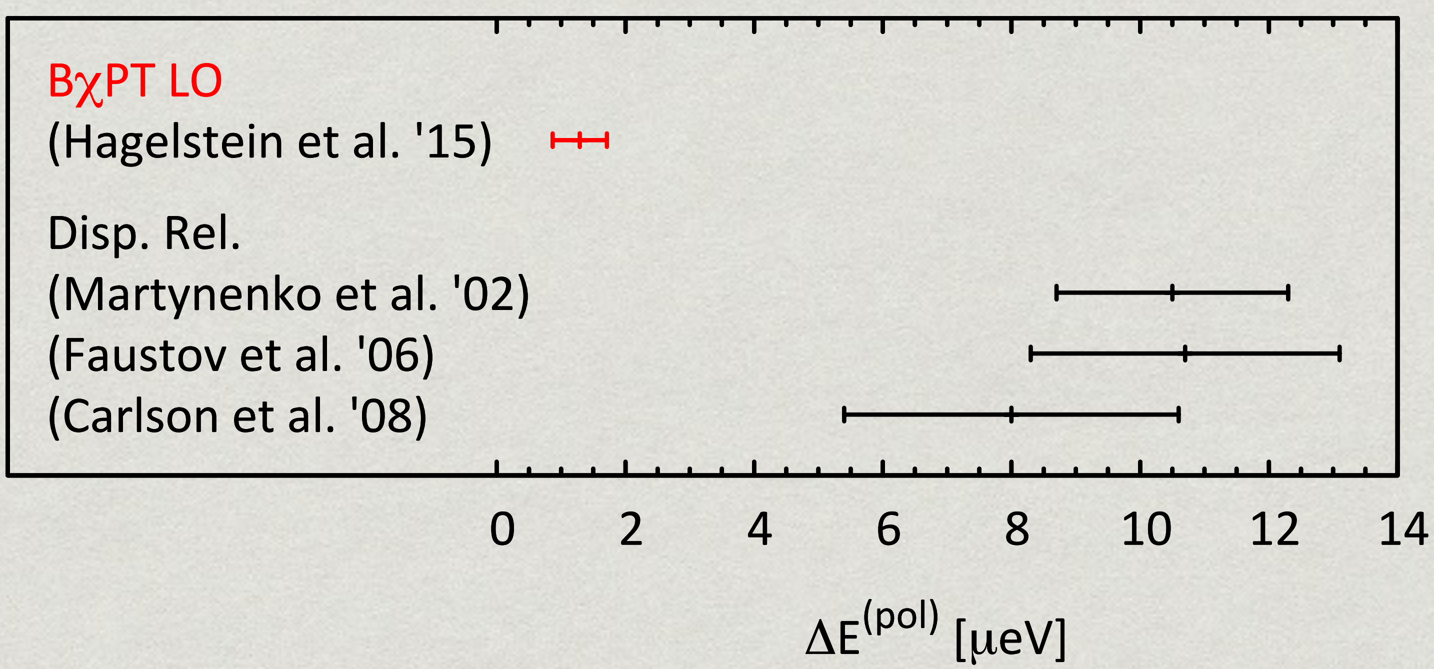




$$E^{\pi}_{\text{HFS}} = 0.44 \pm 0.04\,\mu eV$$

$$E^{\pi N-\text{loops}}_{\text{HFS}} = 0.85 \pm 0.42\,\mu eV$$

$$E^{\pi\,\&\,\pi N-\text{loops}}_{\text{HFS}} = 1.29 \pm 0.42\,\mu eV$$



[1] C. E. Carlson, et al., Phys. Rev. **A**83 (2011) 042509.

[2] C. E. Carlson, et al., Phys. Rev. **A**78 (2008) 022517.

[3] R. Faustov, et al., Proc. SPIE Int. Soc. Opt. Eng. 6165 (2006) 0M.

[4] A. Martynenko et al., Nucl. Phys. **A**703 (2002) 365–377.

[5] A. Martynenko, Phys. Rev. **A**71 (2005) 022506.

Dispersive Calculations for TPE in µH HFS

[1] C. E. Carlson, et al., Phys. Rev. **A**83 (2011) 042509.
[2] C. E. Carlson, et al., Phys. Rev. **A**78 (2008) 022517.
[3] R. Faustov, et al., Proc. SPIE Int. Soc. Opt. Eng. 6165 (2006) 0M.
[4] A. Martynenko et al., Nucl. Phys. **A**703 (2002) 365–377.
[5] A. Martynenko, Phys. Rev. **A**71 (2005) 022506.

Reference	FF	R_Z [fm]	Δ_Z [ppm]	Δ^p_{recoil} [ppm]	Δ_{pol} [ppm]	δ_1 [ppm]	δ_2 [ppm]	Δ_{FSE} [ppm]	$E_{2S\,\text{HFS}}$ [meV]
Carlson et al.[a]	AMT	1.080	−7703	931	351(114)	370(112)	−19(19)	−6421(140)	22.8123
	AS	1.091	−7782	931	353			−6498	22.8105
	Kelly	1.069	−7622	931	353			−6338	22.8141
	MAMI	1.045							22.8187
	combined[b]								22.8146(49)
Faustov et al.[c]					470(104)	518	−48		
Martynenko et al.[d]	Dipole	1.022	−7180		460(80)	514	−58		22.8138(78)[e]
Experiment		1.082(37)							22.8089(51)

[a] QED, higher-order and other small corrections included in $E_{2S\,\text{HFS}}$ are taken from Martynenko. The Zemach term includes radiative corrections: $\Delta_Z = -2\alpha m_r R_Z(1+\delta_Z^{\text{rad}})$.

[b] slightly moved average of the selected form factors

[c] The calculation is based on experimental data for the nucleon polarized structure functions obtained at SLAC, DESY and CERN.

[d] The calculation is based on experimental data for the nucleon polarized structure functions obtained at SLAC, DESY and CERN.

[e] Adjusted value; the original value, 22.8148(78) meV, is corrected by adding -1μeV, because the conventions of "elastic" and "inelastic" contributions, applied in Martynenko, are inconsistent.

Table 1: Summary of available dispersive calculations for the TPE correction to the HFS in μH.

Fermi - Energy:

$$E_F(2S) = \frac{1}{3}\frac{Z\alpha}{a^3}\frac{1+\kappa}{mM} = 22.8054\text{ meV}$$

Δ-Exchange Contribution

- TPE effect on the Lamb shift is dominated by the electric dipole polarizability α_{E1}, while the contribution from the magnetic dipole polarizability β_{M1} is suppressed

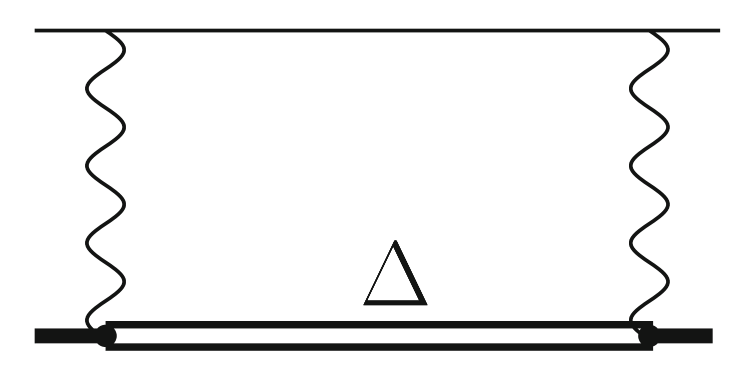
$$\Delta E^{(\mathrm{pol})}_{nS} = \frac{\alpha}{\pi}\,\phi_n^2 \int_0^\infty \frac{\mathrm{d}Q}{Q^2}\, w(\tau)\left[\overline{T}_1(0,Q^2) - \overline{T}_2(0,Q^2)\right] \qquad \text{with} \quad \phi_n^2 = 1/(\pi n^3 a^3)$$

low-energy theorem:

$$\lim_{\nu,Q^2\to 0} \overline{T}_1(\nu,Q^2) = 4\pi\,\nu^2\,(\alpha_{E1}+\beta_{M1}) + 4\pi\,Q^2\beta_{M1} + O(q^4)$$

$$\lim_{\nu,Q^2\to 0} \overline{T}_2(\nu,Q^2) = 4\pi\,Q^2\,(\alpha_{E1}+\beta_{M1}) + O(q^4)$$

- Does the Δ-excitation contribute significantly to the TPE effect in HFS?



$$\mathcal{L} = \frac{3e}{2M(M+M_\Delta)}\bar{N}T_3\left\{ig_M(\partial_\mu\Delta_\nu)\tilde{F}^{\mu\nu} - g_E\gamma_5(\partial_\mu\Delta_\nu)F^{\mu\nu} - i\frac{g_C}{M_\Delta}\gamma_5\gamma^\alpha(\partial_\alpha\Delta_\nu - \partial_\nu\Delta_\alpha)\partial_\mu F^{\mu\nu}\right\} + \mathrm{H.c.},$$

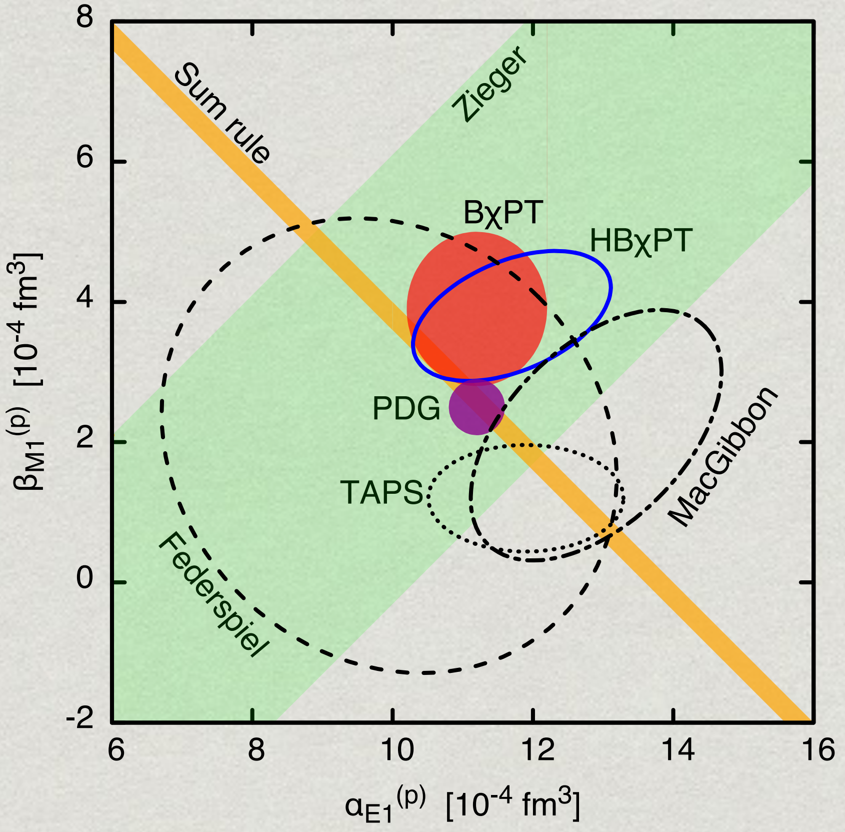
$$g_M = 2.88$$
$$g_E = -1.04$$
$$g_C = -2.6$$

Polarizabilties: Δ-Exchange Contribution

- scalar dipole polarizabilties:

$$\alpha_{E1} = \frac{e^2 g_E^2}{2\pi M_+^3} \approx 0.1 \times 10^{-4}\,\mathrm{fm}^3$$

$$\beta_{M1} = -\frac{e^2 g_M^2}{2\pi \Delta M_+^2} \approx -6.7 \times 10^{-4}\,\mathrm{fm}^3$$

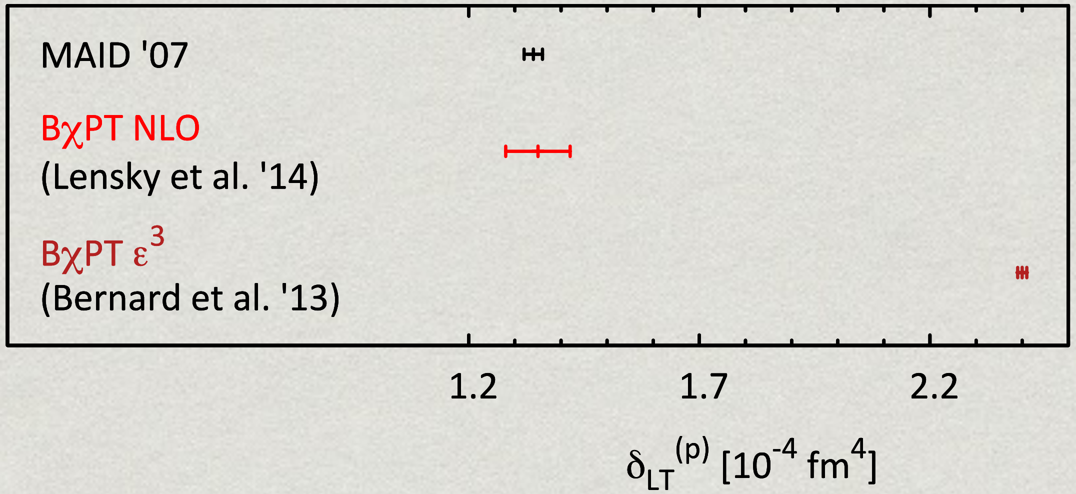


- spin polarizabilties:

$$\gamma_0 = -\frac{e^2}{4\pi M_+^2}\left(\frac{g_E^2}{M_+^2} + \frac{g_M^2}{\Delta^2} - \frac{4 g_M g_E}{M_+ \Delta}\right) \approx -2.7 \times 10^{-4}\,\mathrm{fm}^4$$

$$\delta_{\mathrm{LT}} = \frac{e^2}{4\pi M_+^3}\left(g_E^2 \frac{M_\Delta}{M_+ M} + g_E g_M \frac{M_\Delta}{M \Delta} + g_E g_C \frac{1}{M_\Delta}\right)$$

$$\approx (-0.138 + 0.023) \times 10^{-4}\,\mathrm{fm}^4 = -0.114 \times 10^{-4}\,\mathrm{fm}^4$$

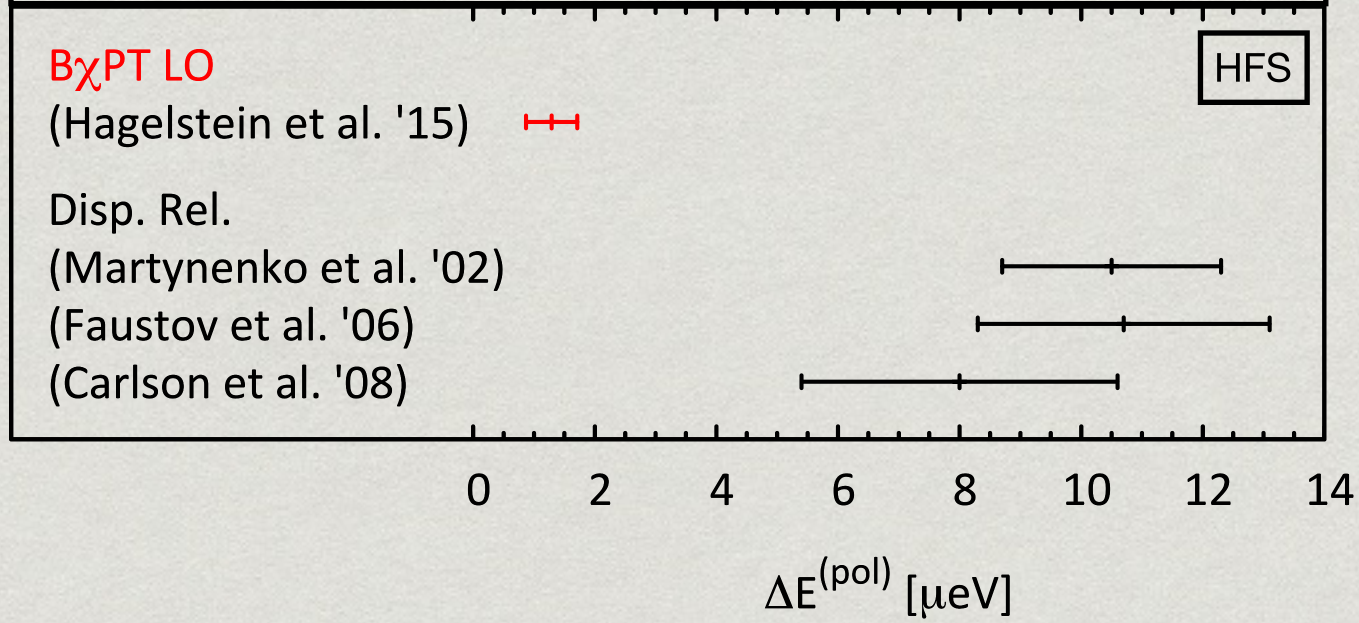


Conclusions

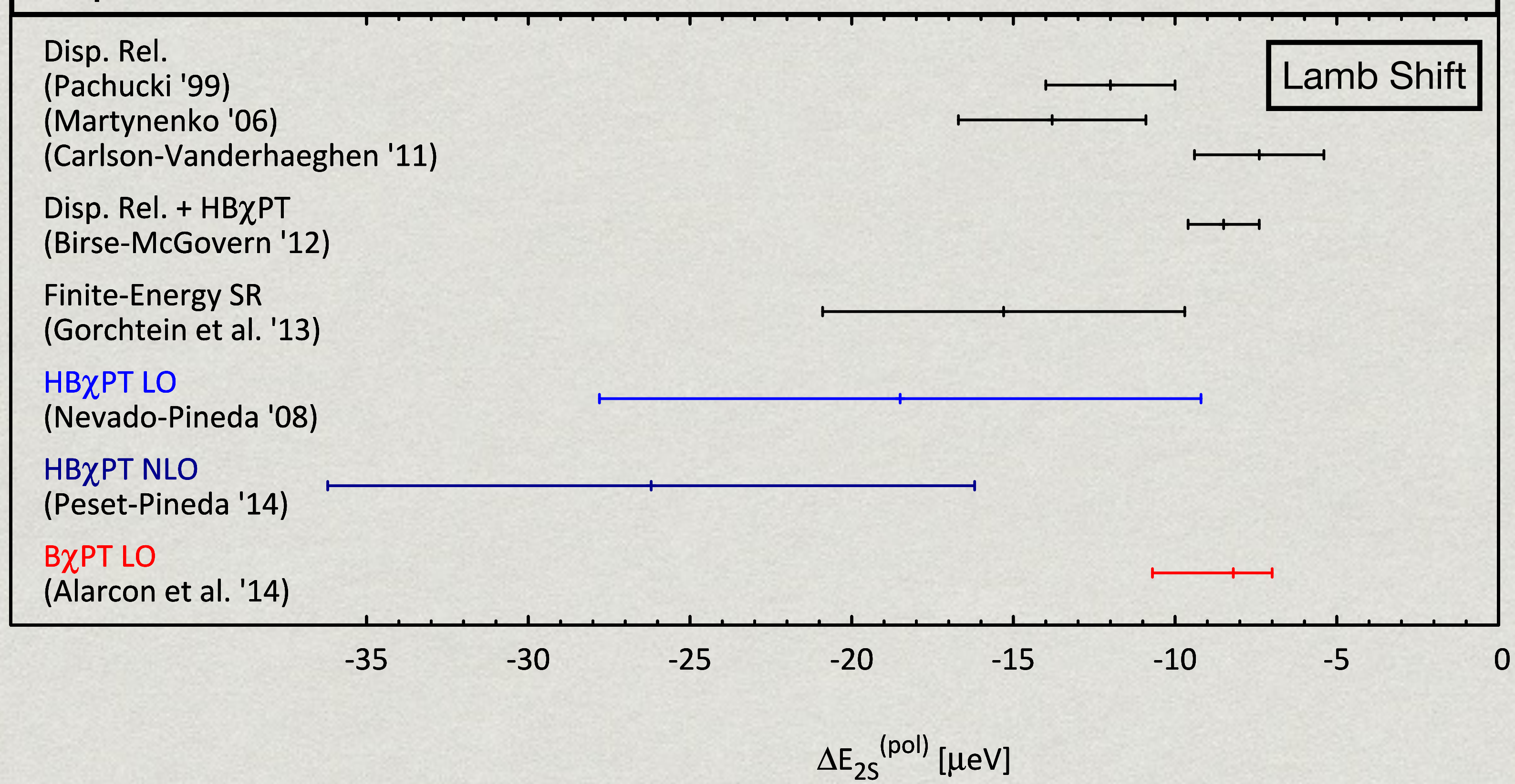
Why disagreement in HFS ???

- effect of the Δ-excitation might not be negligible
- empirical information on polarized (spin) structure functions is limited
  - little data on g_2
- problem in BχPT?

BχPt result is smaller than calculations based on dispersive sum rules



BχPt result is in good agreement with calculations based on dispersive sum rules



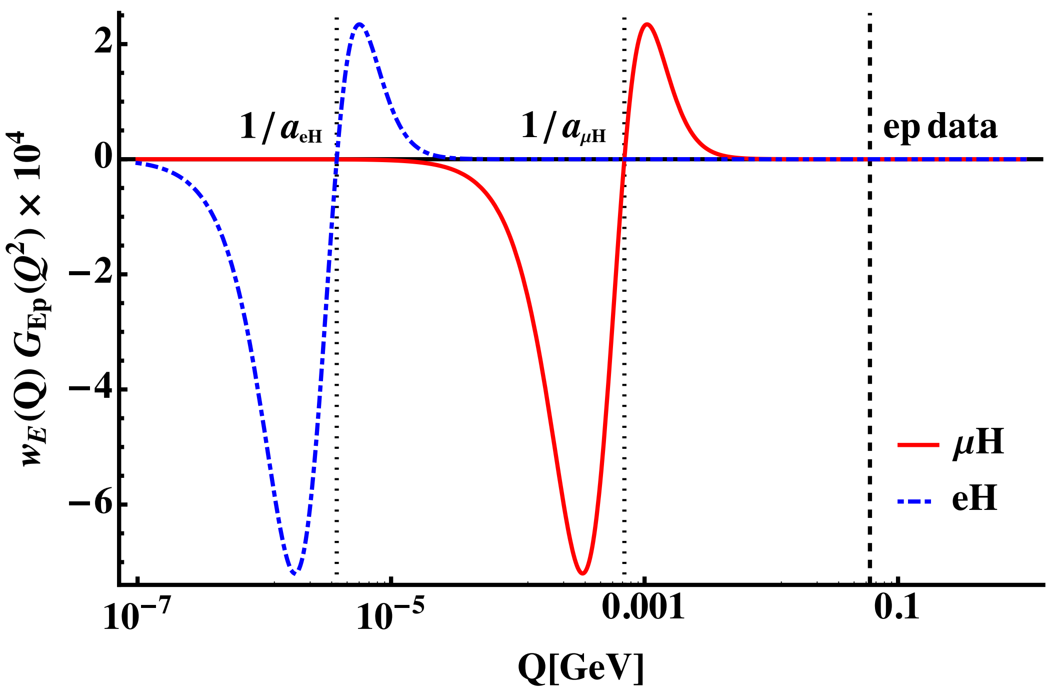
Backup Slides

PROTON FORM FACTOR IN HYDROGEN LAMB SHIFT

SFB 1044 — THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

Cluster of Excellence PRISMA — Precision Physics, Fundamental Interactions and Structure of Matter

$$E^{\mathrm{FF}(1)}_{2P-2S} = \int_0^\infty \mathrm{d}Q\, w(Q)\, G_E(Q^2)$$



with $w(Q) = -\frac{4}{\pi}(Z\alpha)^5 m_r^4\, Q^2\, \frac{(Z\alpha m_r)^2 - Q^2}{[(Z\alpha m_r)^2 + Q^2]^4}$

FH, V. Pascalutsa, Phys. Rev. **A91** (2015) 040502

"soft" contributions to the proton or lepton electric form factor could be able to explain the proton size puzzle ?!?

Yukawa-type potential:

$$V_Y(r) = \frac{Z\alpha}{r}\frac{1}{\pi}\int_{t_0}^\infty \frac{\mathrm{d}t}{t}\, e^{-r\sqrt{t}}\, \mathrm{Im}\, G_E(t)$$

$$\Delta E^{\mathrm{FF}(1)}_{2P-2S} = \langle 2P_{1/2}|V_Y|2P_{1/2}\rangle - \langle 2S_{1/2}|V_Y|2S_{1/2}\rangle$$
$$= -\frac{(Z\alpha)^4 m_r^3}{2\pi}\int_{t_0}^\infty \mathrm{d}t\, \frac{\mathrm{Im}\, G_E(t)}{(\sqrt{t}+Z\alpha m_r)^4}$$
$$= -\frac{(Z\alpha)^4 m_r^3}{12}\sum_{k=0}^\infty \frac{(-Z\alpha m_r)^k}{k!}\langle r^{k+2}\rangle_E$$
$$= -\frac{(Z\alpha)^4 m_r^3}{12}\left[\langle r^2\rangle_E - Z\alpha m_r \langle r^3\rangle_E\right] + O(\alpha^6)$$

convergence radius of the expansion is limited by t_0

- the finite-size effects are not always expandable in the moments of charge distribution
- a small variation in the form factor around the inverse Bohr radius scale may lead to significant effects !!!

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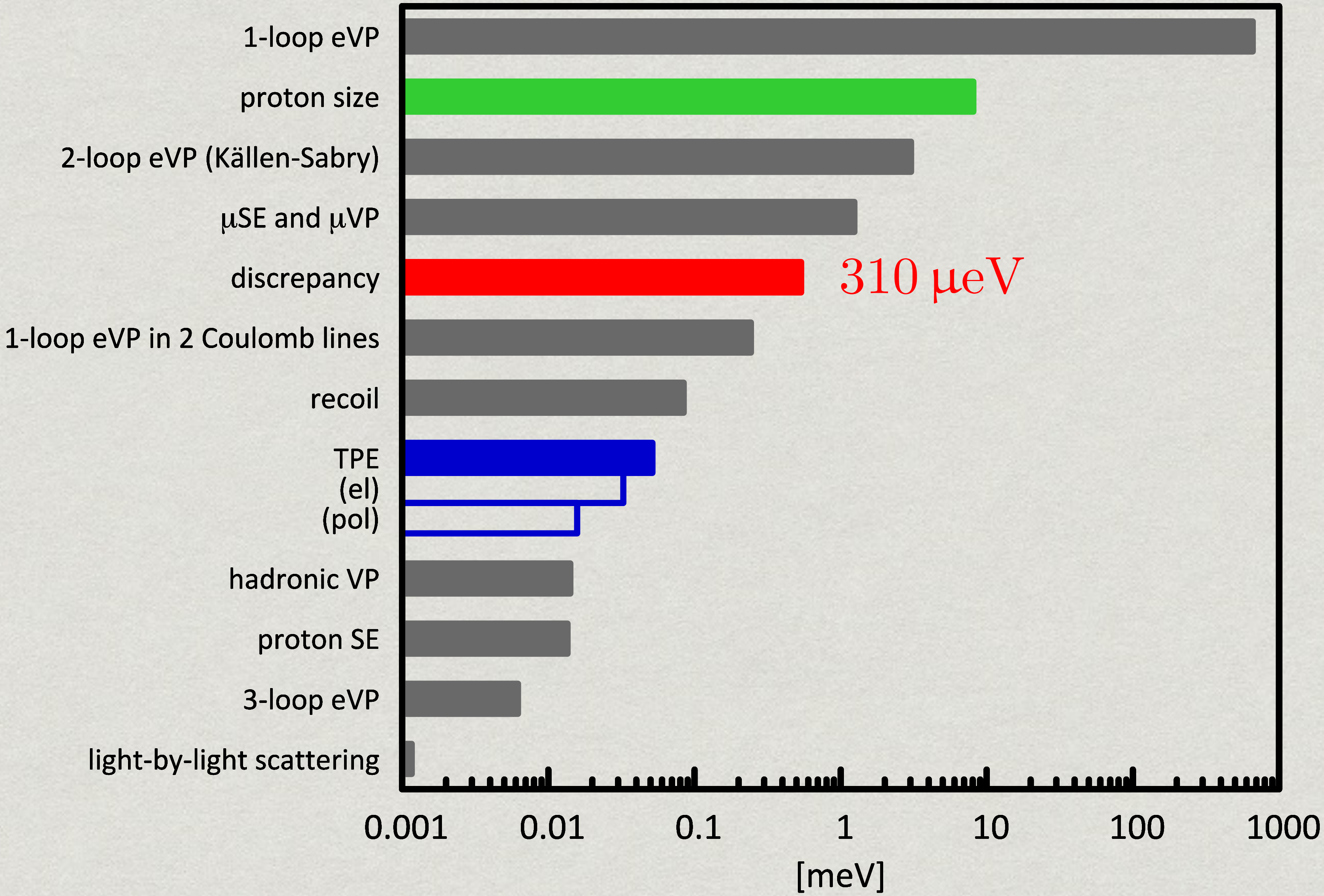
Theory of μH Lamb Shift

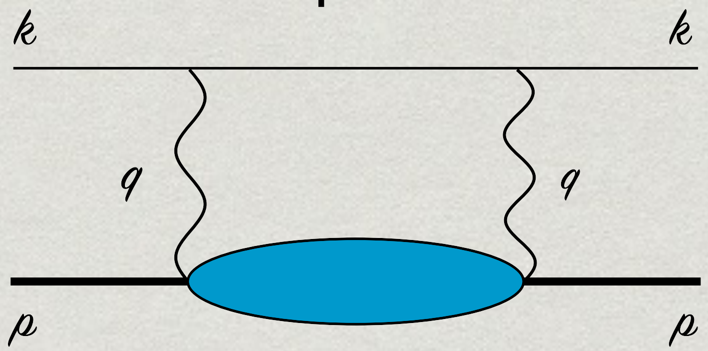
$$\Delta E^{\rm th}_{\rm LS} = 206.0668(25) - 5.2275(10)\,(R_E/{\rm fm})^2$$

numerical values reviewed in: A. Antognini *et al.*, Annals Phys. **331**, 127-145 (2013).

theory uncertainty:
$2.5\ \mu eV$



subleading effects of proton structure proposed to resolve the puzzle



$$\delta V^{(2\gamma)} = \delta V^{(2\gamma)}_{\rm elastic} + \delta V^{(2\gamma)}_{\rm polariz.}$$

A. De Rujula, Phys. Lett. B693 (2010)

G. A. Miller, Phys. Lett. B718 (2013)

TPE in μH Lamb Shift

- Lamb shift: $$\Delta E_{nS} = \frac{\alpha m}{i\pi^3}\,\phi_n^2 \int_0^\infty \mathrm{d}\nu \int \mathrm{d}\boldsymbol{q}\, \frac{(Q^2-2\nu^2)\,T_1(\nu,Q^2)-(Q^2+\nu^2)T_2(\nu,Q^2)}{Q^4(Q^4-4m^2\nu^2)}$$

with the hydrogen wavefunction at the origin $\phi_n^2 = 1/(\pi n^3 a^3)$

unitarity relations:

$$\mathrm{Im}\ T_1(\nu,Q^2) = \frac{4\pi^2\alpha}{M} f_1(\nu,Q^2) = \nu\,\sigma_T(\nu,Q^2)$$

$$\mathrm{Im}\ T_2(\nu,Q^2) = \frac{4\pi^2\alpha}{\nu} f_2(\nu,Q^2) = \frac{Q^2\nu}{\nu^2+Q^2}\left[\sigma_T+\sigma_L\right](\nu,Q^2)$$

$$f_1^{\mathrm{el}}(x,Q^2) = \frac{1}{2}G_M^2(Q^2)\delta(1-x)$$

$$f_2^{\mathrm{el}}(x,Q^2) = \frac{1}{1+\tau}\left[G_E^2(Q^2)+\tau G_M^2(Q^2)\right]\delta(1-x)$$

“polarizability” contribution:

$$\overline{T}_1(\nu,Q^2) = \boxed{\overline{T}_1(0,Q^2)} + \frac{2\nu^2}{\pi}\int_{\nu_0}^\infty \mathrm{d}\nu'\,\frac{\sigma_T(\nu',Q^2)}{\nu'^2-\nu^2},$$

$$\overline{T}_2(\nu,Q^2) = \frac{2}{\pi}\int_{\nu_0}^\infty \mathrm{d}\nu'\,\frac{\nu'^2 Q^2}{\nu'^2+Q^2}\,\frac{\sigma_T(\nu',Q^2)+\sigma_L(\nu',Q^2)}{\nu'^2-\nu^2}$$

subtraction function must be modelled!

“elastic” contribution:

$$T_1^{\mathrm{Born}}(\nu,Q^2) = \frac{4\pi\alpha}{M}\left\{\frac{Q^4\left[F_1(Q^2)+F_2(Q^2)\right]^2}{Q^4-4M^2\nu^2} - F_1^2(Q^2)\right\}$$

$$T_2^{\mathrm{Born}}(\nu,Q^2) = \frac{16\pi\alpha M Q^2}{Q^4-4M^2\nu^2}\left\{F_1^2(Q^2)+\frac{Q^2}{4M^2}F_2^2(Q^2)\right\}$$

“elastic” and “inelastic” contributions are well-constrained by empirical information

Dispersive Calculations for TPE in μH Lamb Shift

[1] K. Pachucki, Phys. Rev. **A**60 (1999) 3593–3598.
[2] A. Martynenko, Phys. Atom. Nucl. 69 (2006) 1309–1316.
[3] C. E. Carlson, M. Vanderhaeghen, hep-ph/1101.5965 (2011).
[4] M. C. Birse, J. A. McGovern, Eur. Phys. J. **A**48 (2012) 120.
[5] M. Gorchtein, et al., Phys. Rev. **A**87 (2013) 052501.

	Pachucki	Martynenko	Carlson & Vanderhaeghen	Birse & McGovern	Gorchtein *et al.*[a]
β_{M1}	1.56(57)	1.9(5)	3.4(1.2)	3.1(5)	
$\Delta E_{2S}^{(\text{subt})}$	1.9	2.3	5.3(1.9)	4.2(1.0)	−2.3(4.6)
$\Delta E_{2S}^{(\text{inel})}$	−13.9	−16.1	−12.7(5)	−12.7(5)[b]	−13.0(6)
$\Delta E_{2S}^{(\text{pol})}$	−12(2)	−13.8(2.9)	−7.4(2.0)	−8.5(1.1)	−15.3(4.6)
$\Delta E_{2S}^{(\text{el})}$	−23.2(1.0)		−27.8 **−29.5(1.3)** −30.8	−24.7(1.6)[c]	−24.5(1.2)
ΔE_{2S}	−35.2(2.2)		−36.9(2.4)	−33(2)	−39.8(4.8)

[a] Adjusted values; the original values, $\Delta E_{2S}^{(\text{subt})} = 3.3$ and $\Delta E_{2S}^{(\text{el})} = -30.1$, are based on a different decomposition into the "elastic" and polarizability contributions.

[b] taken from Carlson & Vanderhaeghen

[c] Result taken from Carlson & Vanderhaeghen with reinstated "non-pole" Born piece.

Table 1: Summary of available dispersive calculations for the TPE correction to the Lamb shift in μH. Energy shifts are given in μeV, β_{M1} is given as $\times 10^{-4}$ fm^3.

$$\lim_{Q^2\to 0} \frac{\overline{T}_1(0,Q^2)}{Q^2} = 4\pi\,\beta_{M1}$$

$$\beta_{M1}(Q^2) = \beta_{M1}\frac{\Lambda^8}{(\Lambda^2+Q^2)^4}$$

TPE in μH HFS

Fermi - Energy:
$$E_F(nS) = \frac{8}{3}\frac{Z\alpha}{a^3}\frac{1+\kappa}{mM}\frac{1}{n^3}$$

- HFS: $\Delta E_{\mathrm{HFS}} = [1 + \Delta_{\mathrm{QED}} + \Delta^p_{\mathrm{weak}} + \Delta_{\mathrm{FSE}}]\, E_F$

  with $\Delta_{\mathrm{FSE}} = \Delta_Z + \Delta^p_{\mathrm{recoil}} + \Delta_{\mathrm{pol}}$

- TPE in HFS: $$\frac{E^{\mathrm{TPE}}}{E^p_F} = \frac{m}{2(1+\varkappa)\pi^4}\frac{1}{i}\int_0^\infty \mathrm{d}\nu \int \mathrm{d}\boldsymbol{q}\, \frac{1}{(Q^4 - 4m^2\nu^2)}\left\{\frac{(2Q^2-\nu^2)}{Q^2}S_1(\nu,Q^2) + 3\frac{\nu}{M}S_2(\nu,Q^2)\right\}$$

$$S_1(\nu,Q^2) = 8\pi\alpha\left\{\frac{MQ^2F_1(Q^2)G_M(Q^2)}{Q^4-4M^2\nu^2} + \int_{\nu_0}^\infty \frac{\mathrm{d}\nu'}{\nu'^2-\nu^2}\, g_1(\nu',Q^2)\right\}$$

$$S_2(\nu,Q^2) = 8\pi\alpha\, M\nu\left\{-\frac{MF_2(Q^2)G_M(Q^2)}{Q^4-4M^2\nu^2} + \int_{\nu_0}^\infty \frac{\mathrm{d}\nu'}{\nu'^2-\nu^2}\frac{g_2(\nu',Q^2)}{\nu'^2}\right\}$$

unitarity relations:

$$\mathrm{Im}\ S_1(\nu,Q^2) = \frac{4\pi^2\alpha}{\nu}\, g_1(\nu,Q^2) = \frac{M\nu^2}{\nu^2+Q^2}\left[\frac{Q}{\nu}\sigma_{LT} + \sigma_{TT}\right](\nu,Q^2),$$

$$\mathrm{Im}\ S_2(\nu,Q^2) = \frac{4\pi^2\alpha M}{\nu^2}\, g_2(\nu,Q^2) = \frac{M^2\nu}{\nu^2+Q^2}\left[\frac{\nu}{Q}\sigma_{LT} - \sigma_{TT}\right](\nu,Q^2)$$

$$g_1^{\mathrm{el}}(x,Q^2) = 1/2\, F_1(Q^2)G_M(Q^2)\delta(1-x)$$
$$g_2^{\mathrm{el}}(x,Q^2) = -1/2\, \tau F_2(Q^2)G_M(Q^2)\delta(1-x)$$

- TPE effect on the HFS is completely constrained by empirical information

HFS Formalism

$$\Delta_Z = \frac{8\alpha m_r}{\pi}\int_0^\infty \frac{\mathrm{d}Q^2}{Q^2}\left[\frac{G_E(Q^2)G_M(Q^2)}{1+\varkappa}-1\right] \equiv -2\alpha m_r R_Z$$

$$\Delta^p_{\text{recoil}} = \frac{2\alpha m_r}{M^2}\frac{1}{\pi}\int_0^\infty \mathrm{d}Q\, F_2(Q^2)\frac{G_M(Q^2)}{1+\varkappa} - \frac{\alpha m}{2(1+\varkappa)M}\frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}Q^2}{Q^2}\beta_1(\tau_l)F_2^2(Q^2)$$
$$+\frac{\alpha m M}{2(1+\varkappa)(M^2-m^2)}\frac{1}{\pi}\left\{3\int_0^\infty \frac{\mathrm{d}Q^2}{Q^2}\left[\beta_2(\tau_p)-\beta_2(\tau_l)\right]F_2(Q^2)G_M(Q^2)\right.$$
$$\left.+\int_0^\infty \frac{\mathrm{d}Q^2}{Q^2}\left[\frac{\beta_1(\tau_p)-4\sqrt{\tau_p}}{\tau_p}-\frac{\beta_1(\tau_l)-4\sqrt{\tau_l}}{\tau_l}\right]F_1(Q^2)G_M(Q^2)\right\}$$

$$\Delta_{\text{FSE}} = \Delta_Z + \Delta^p_{\text{recoil}} + \Delta_{\text{pol}} = \frac{E^{\text{TPE}}}{E_F^p} - \frac{8\alpha m_r}{\pi}\int_0^\infty \frac{\mathrm{d}Q^2}{Q^2}$$

$$\Delta_{\text{pol}} = \frac{\alpha m}{2(1+\varkappa)\pi M}\left[\Delta_1+\Delta_2\right] = \delta_1+\delta_2$$

$$\Delta_1 = \int_0^\infty \frac{\mathrm{d}Q^2}{Q^2}\left\{\beta_1(\tau_l)F_2^2(Q^2) + 4M\int_{\nu_0}^\infty \frac{\mathrm{d}\nu}{\nu^2}\frac{Q^4\beta_1(\tau)-4m^2\nu^2\beta_1(\tau_l)}{Q^4-4m^2\nu^2}g_1(\nu,Q^2)\right\}$$

$$\Delta_2 = -12M^2\int_0^\infty \frac{\mathrm{d}Q^2}{Q^2}\int_{\nu_0}^\infty \frac{\mathrm{d}\nu}{\nu^2}\frac{Q^4\left[\beta_2(\tau)-\beta_2(\tau_l)\right]}{Q^4-4m^2\nu^2}g_2(\nu,Q^2)$$