Overview

✧ Brief introductions to effective field theories (EFTs) and to lattice systematics.

✧ Uses of EFTs in lattice QCD.
  • ChPT, Symanzik effective theory, HQET.
  • Discretization errors.
  • Partial quenching.
  • Finite volume effects and twisted boundary conditions; heavy quarks; ....

✧ The payback: ChPT results from the lattice.
  • Mesons, mainly SU(2).
  • Nucleons (a little).
  • Preliminary results: SU(3) and 3-flavor chiral limit.
Effective Field Theories

✨ Powerful tool to describe physics in some limited range of scales.

• Useful when the fundamental theory is too difficult to handle (or unknown).

• Typically:
  1. “Integrate out” high energy modes of a theory (those above a cutoff $\Lambda$).
  2. Expand the resulting non-local theory in inverse powers of $\Lambda$ times local operators (an OPE).
  3. Left with a local effective field theory (EFT) at low energy.

    – In rare cases (e.g. heavy quark effective theory), steps can actually be carried out (perturbatively).
    – Usually just imagine performing steps 1-3; use symmetries to constrain EFT.
These days, often said that all field theories are effective theories.

- Unknown new physics must kick in at some higher scale.
- E.g., QCD could be supplemented by higher dimension terms, such as:
  \[
  \frac{1}{M} \bar{q} \sigma_{\mu\nu}G^{\mu\nu}q
  \]
  - \(M\) is mass scale of new physics.
- Distinction between renormalizable and unrenormalizable theories is less important than we used to think.
- Still, an important distinction:
  - If LO effective theory is nonrenormalizable (e.g. ChPT), it tells you the scale at which new physics must enter \((4\pi f_\pi)\), so sets natural scale for NLO terms.
  - If the LO effective theory is renormalizable (e.g., QCD), then scale of new physics undetermined.
    - must be found/bound by experiment,
    - or by knowing/guessing the more fundamental underlying theory.
Lattice QCD Systematic Errors

- Lattice computation of QCD path integral inherently includes systematic errors.
  - Continuum extrapolation error: need to take lattice spacing \( a \to 0 \).
  - (Residual) finite-volume errors: need to take space & time extent \( L, T \to \infty \).
  - Chiral extrapolation error: for practical reasons may choose \( m_u, m_d \) larger than physical; need to extrapolate to physical values.
    - Even if near-physical values chosen (now possible), need to interpolate to precise physical values (can only be found \textit{a posteriori}): chiral interpolation error.
Use of EFTs in Lattice QCD

✧ EFTs provide functional forms for relevant extrapolations/interpolations.
  • thereby reduce systematic errors.

✧ First use: ChPT, to guide quark mass extrapolations.
  • ChPT gives functional form of expansion in quark masses (and momenta).
    • all dependence explicit.
    • exactly as needed for extrapolations.
  • Soon realized that ChPT also gives leading finite volume corrections.
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\text{CL} = 0.30
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- Here, simple analytic terms, $\text{const.} \times a^2$, do the trick.


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\[ a = 0.08 \text{ fm} \]
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- Here, simple analytic terms, $\text{const.} \times a^2$, do the trick.
- In some other cases (very precise lattice data, many degrees of freedom, larger discretization errors...) this approach may not be adequate.
Use of EFTs in Lattice QCD

Key insight: ChPT can be modified to include lattice discretization errors. [Sharpe & Singleton, 1998]

- Relates $a$-dependence to mass dependence, so better controlled extrapolations.
- Non-analytic terms in $a$ arise from loops.
- Method uses another EFT: Symanzik Effective Theory (SET) [Symanzik, 1983].
  - For SET, the lattice QCD theory at fixed lattice spacing $a$ is taken as “fundamental.”
  - SET is the EFT that describes the lattice theory at energy scales small compared to the cutoff: $p \ll 1/a$.
  - Leading order Lagrangian is just the continuum QCD Lagrangian.
  - Since $ap \ll 1$, need to keep only low powers of $a$ as corrections:
    - add on local operators with dimension $> 4$, multiplied by appropriate powers of $a$.
  - Needed local operators $\rightarrow$ determined by the underlying lattice symmetries.
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    - new LECs \( c_1 \) and \( c_2 \) encode leading discretization effects in ChPT.
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  - New LECs $c_1$ and $c_2$ encode leading discretization effects in ChPT.
  - Sharpe & Singleton showed from this ChPT that a new lattice-artifact phase ("Aoki phase") was possible at fixed $a$ for very small $m$.  

C. Bernard, CD15
Lattice QCD: twisted-mass quarks

• Start with a doublet of Wilson quarks.
• Add a twisted mass [Frezzotti, Grassi, Sint & Weisz, 2001]:

\[ \bar{q} \left( \slashed{D} + m \right) q \rightarrow \bar{q} \left( \slashed{D} + m + i \mu \gamma_5 \tau_3 \right) q \]

• In continuum, \( \mu \) term can be rotated away by non-singlet SU(2) chiral rotation.
• But on lattice, since Wilson term (to remove doublers) is in “m direction”, twist is nontrivial:
  • Avoid “exceptional configurations” in which statistical fluctuations from Wilson term bring mass to zero.
  • If \( m \) tuned to 0, physical quantities have errors starting at \( O(a^2) \), not \( O(a) \) [Frezzotti and Rossi, 2004].
• Price is violation of isospin symmetry at nonzero \( a \).
  – twisted mass ChPT [Munster, Schmidt & Scholz, 2004; Sharpe & Wu, 2004]
    \[ \Rightarrow O(a^2) \text{ splitting of } \pi^0 \text{ from } \pi^+ \]
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  - where $\Sigma$ is now a $4n \times 4n$ matrix.
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- Expanding as usual \(\rightarrow\) 16 pions each (non-singlet) flavor combination.
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Pion taste splittings vs \((\alpha_s a)^2\) for two versions of staggered quarks: “asqtad” and newer, more highly improved version, “HISQ”.

MILC [A. Bazavov, et al.],
PRD 87, 054505 (2013)
[arXiv:1212.4768]
So lattice-spacing-dependent ChPT can explain/control lattice artifacts:
  • Aoki phase (Wilson)
  • pion isospin-violations (twisted mass)
  • pion taste-splittings (staggered)

Another key use to is guide continuum extrapolations:
  • Fit quark-mass dependence and lattice-spacing dependence together, using expressions from the appropriate chpt.
  • Can significantly reduce systematic errors.
  • Such fits often done in partially quenched context: choose valence quarks to have different masses than sea quarks.
    • Useful because valence quarks are cheap compared to sea quarks: extract as much as possible for a given configuration (generated with sea quark back-effects).
    • “Partially quenched” because valence quarks are quenched: forbidden from appearing in virtual loops, but sea quarks are not quenched.
    • Add corresponding ghost (bosonic!) quarks, with same mass matrix as the valence quarks, to cancel the virtual loops (determinant) of the valence quarks [Morel, 1987].
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C. Bernard, *CD15*
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- Then partially quenched ChPT (PQChPT) at LO is [CB & Golterman, 1993]:
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C. Bernard, CD15
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  - graded group has some Grassman generators, because some transformations take fermions into bosons, and vice-versa, as in supersymmetry. 

(C. Bernard, CD15)
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  • Is PQChPT really justified?
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Using Staggered ChPT
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- continuum and chiral extrapolation of partially quenched staggered lattice data from multiple lattice spacings:
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C. Bernard, CD15
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MILC [A. Bazavov et al., PoS(LAT2010), 074 (2010), arXiv:1012.0868.]

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  - Even though bulk of lattice data have ~10% discretization or mass corrections.

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Staggered Heavy Quarks
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- Get:
  
  \[
  f_{D^+} = 212.6(0.4)(^{+1.0}_{-1.2}) \text{ MeV} \\
  f_{D_s} = 249.0(0.3)(^{+1.4}_{-1.5}) \text{ MeV}
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ChPT for Twisted Boundary Conditions
With periodic boundary conditions, lattice momenta are limited:

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With periodic boundary conditions, lattice momenta are limited:

\[ p = \frac{2\pi n}{L}. \]

\([L = \text{spatial lattice dimension}, \, n = \text{integer}].\]

- Even with current large volumes, \(L = 5\) fm, momenta spaced by \(\approx 250\) MeV.
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Solution [Bedaque, 2004; de Divitiis, Petronzio & Tantalo, 2004]:

give (some) quarks *twisted boundary conditions*:

\[ q(x+L) = e^{i\theta} q(x). \]

- Then allowed momenta are: \(p = \frac{(2\pi n + \theta)}{L}.\)
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    \]

    - Infinite volume: odd integral
    - Finite volume: shift spoils \( k \rightarrow -k \) symmetry
Lattice allows first-principles computation of LECs of the effective theory from fundamental QCD.

- In practice, is easiest for LECs affecting pseudoscalar meson masses and leptonic decay constants.
  - Can be calculated from quark-mass dependence of 2-point Euclidean Green’s functions.
  - Nice complement to experiments, which give little constraint on quark-mass dependence since quark masses fixed in Nature.
- LECs affecting momentum dependence of scattering amplitudes are just the opposite:
  - Difficult on the lattice: n-point functions; must pull out (indirectly) Minkowski-space amplitudes from Euclidean space calculations. [Maiani & Testa, 1990; Lüscher, 1991].
SU(2) LECs

\[ \Sigma \equiv |\langle \bar{u}u \rangle| = (271(15) \text{ MeV})^3 \]

- in the two flavor chiral limit: \( m_u, m_d \rightarrow 0 \).
- values quoted are for \( N_f = 2+1 \) theory.
- For \( F_\pi/F \), looks like systematic errors of one or more calculations may be underestimated.
$N_f = 2$ Condensate from Banks-Casher
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RBC [T. Blum et al.], arXiv:1411.7017; physical quark masses; \( N_f=2+1 \):
SU(2) LECs

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Convergence of ChPT: SU(2)
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2+1; $a \sim 0.12$ to 0.10 fm
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- Convergence good for \( f_\pi \) and reasonable for \( M_\pi^2/\hat{m} \).
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  - MILC lattice data is partially quenched.
  - corrected for in lattice ChPT.


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SU(2) LECs

FLAG2013

\[ \ell_6 \]

- \text{RBC/UKQCD 08A}
- \text{our estimate for } N_f = 2
- \text{Brandt 13}
- \text{JLQCD/TWQCD 09}
- \text{ETM 08}

- \text{Bijnens 98}
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H. Fukaya, et al., PRD 90, 034506 (2014); $N_f = 2+1$
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\[ \tilde{\ell}_6 = 7.5(1.3)(1.5) \]
\[ F_\pi/F = 1.6(2)(3) \]

C. Bernard, CD15 26
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- Fit including \( \Delta \) is very similar.

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- Update of RBC points.*
- Seems to be a curious accident; doesn’t contradict expected chiral behavior.

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SU(3) LECs

\[ L_{10} \text{ from lattice+continuum: P. Boyle et al., PRD 89, 094510 (2014)}. \]
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![Graph 1](image1)

![Graph 2](image2)

- $1/a=1.37$ GeV, $m_\pi=171$ MeV
- OPAL+DV model, central
- OPAL+DV model, $1\sigma$ errors
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    - This makes a big difference in size of NNLO terms!
  - Reliable control of the SU(3) ChPT seems only possible for the simulated strange-quark mass, \( m_s' \), chosen less than its physical value, \( m_s \).
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![Graph 1](image1)

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function of strange sea mass, with $u,d$ mass at chiral limit (& extrapolated to continuum).

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- see:
  - C. Bernard, CD15
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“Paramagnetic effect:” Descotes, Girlanda & Stern, 1999

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  - Need data with smaller discretization errors and smaller $m_s'$ (in progress). [Higher order staggered ChPT would also help.]
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    – If want interesting, but not experimentally accessible, quantities like $f_0$, (decay constant in 3-flavor chiral limit), staggered ChPT will still be needed for foreseeable future.
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    – can reduce already small sea-quark effect on scale even further by fitting to their formula.
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  - Many paybacks to come!
QCD Simulations

- Generate an ensemble of gluon fields according to a probability distribution given by the QCD gluon action and back-effect of sea quarks (virtual quark loops).
  - Expensive! (mainly because of sea quarks, whose effect is encoded in a determinant).

- In each gluon-field background (a “configuration”), calculate propagation of valence quarks.
  - Relatively cheap (for each quark, need one column of a matrix inverse).
  - E.g., for \( \langle 0 | A_\mu | \pi(p) \rangle = i f_{\pi} p_\mu \), in a given background:

Then average over the configurations ties together background gluon fields to make:
Some SU(2) Fits

• Note: Borsanyi et al paper includes physical quark masses.

• Discretization errors small in both cases.


2+1+1


2+1

a~0.12 fm
a~0.10 fm
Nucleon Chiral Extrapolation

- Good agreement among groups.
- Note QCDSF point close to physical, with relatively small errors.
Nucleon isospin violation

C. Aubin, W. Detmold, E. Mereghetti, K. Orginos, S. Syritsyn, B. Tiburzi, A. Walker-Loud

NNLO χPT

\[ \delta M_{n-p}^{m_d-m_u} = \delta \left\{ \alpha \left[ 1 - \frac{m_{\pi}^2}{(4\pi f_{\pi})^2} (6g_A^2 + 1) \ln \left( \frac{m_{\pi}^2}{\mu^2} \right) \right] \right\} \]

\[ (g_A = 1.27, f_{\pi} = 130 \text{ MeV}) + \beta(\mu) \frac{2m_{\pi}^2}{(4\pi f_{\pi})^2} \]

\[ \chi^2/dof = 1.66/5 = 0.33 \]

this is striking evidence of a chiral logarithm
**SU(3) LECs**

\[ L_5 = 0.84(38) \times 10^{-3} \]

\[ L_4 = 0.04(14) \times 10^{-3} \]

- Results quoted are for \( N_f = 2+1 \).
- From decay constant.
- \( L_5 \) controls valence mass dependence; \( L_4 \) controls sea mass dependence.
\[2L_8 - L_5 = -0.12(22) \times 10^{-3}\]

\[2L_6 - L_4 = 0.10(12) \times 10^{-3}\]

- Results quoted are for \(N_f = 2+1\).
- From meson mass.
- \(2L_8-L_5\) controls valence mass dependence; \(2L_6-L_4\) controls sea mass dependence.
- Small because \(m_\pi^2\) is nearly linear in quark mass (small NLO corrections).
Meson (mass)$^2$

- $m^2_{\pi}$ and $m^2_K$ vs. $\hat{m}$
- Shows how linear the (mass)$^2$ is.
- Old: from MILC, 2004!
- Usually people divide by $\hat{m}$ to show non-linearity.