# Pion–nucleon scattering at low energies

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Chiral Dynamics 2015

Pisa

July 2nd, 2015

# Outline

**Why is pion–nucleon scattering important?**

**Chiral perturbation theory with nucleons**

- regularisation schemes, $\Delta$ or $\not\Delta$, ...
- phase shift analyses with chiral amplitudes

**A new dispersive analysis: Roy–Steiner equations**

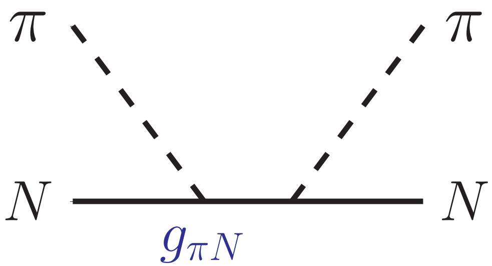
$\longrightarrow$ "leading talk" by J. Ruiz de Elvira, Tue. 14:30

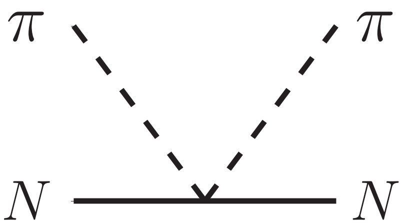
- phase shifts, $\sigma$-term, and low-energy constants

in collaboration with Martin Hoferichter, Jacobo Ruiz de Elvira, and Ulf-G. Meißner

# Pion–nucleon interaction

- simplest process for chiral pion interaction with nucleons




- leading-order $\mathcal{O}(p) = \mathcal{O}(M_\pi)$ predictions for $\pi N$:

  scattering lengths:

$$a^- = \frac{M_\pi m_N}{8\pi(m_N + M_\pi)F_\pi^2} + \mathcal{O}\big(M_\pi^3\big) \qquad a^+ = \mathcal{O}\big(M_\pi^2\big)$$

Weinberg 1966

  Goldberger–Treiman relation: $g_{\pi N} = \frac{g_A m_N}{F_\pi}$

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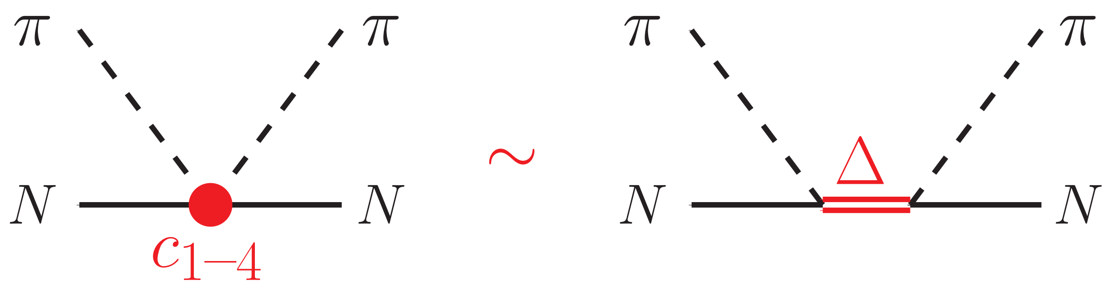
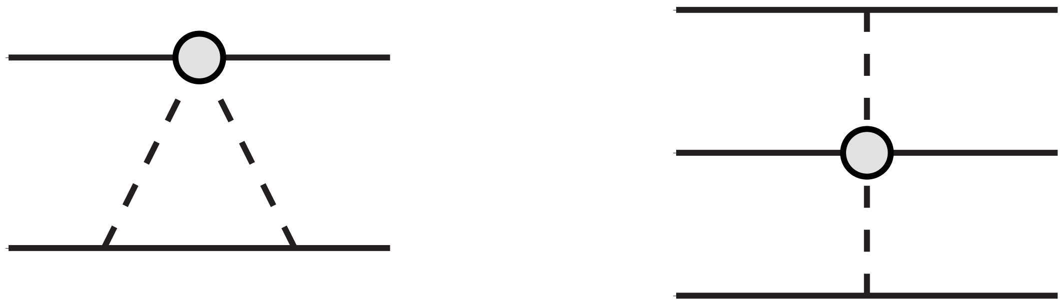



- next-to-leading order $\mathcal{O}(p^2)$: low-energy constants $c_{1-4}$ effectively incorporate effects of the $\Delta(1232)$ resonance: low mass $m_\Delta - m_N \approx 2M_\pi$ and strong couplings
- determination of $c_i$ very important for nuclear physics: $\pi N$ important for $NN$ / determines longest-range $3N$ forces



# The pion–nucleon $\sigma$-term

- scalar form factor of the nucleon:

$$\langle N(p')|\hat{m}(\bar{u}u + \bar{d}d)|N(p)\rangle = \sigma(t)\bar{u}(p')u(p) \qquad t = (p - p')^2$$

$$\sigma_{\pi N} \equiv \sigma(0) = \frac{\hat{m}}{2m_N}\langle N|\bar{u}u + \bar{d}d|N\rangle \qquad \hat{m} = \frac{m_u + m_d}{2}$$

- $\sigma_{\pi N}$ determines light quark contribution to nucleon mass:
  Feynman–Helmann theorem

$$\sigma_{\pi N} = \hat{m}\frac{\partial m_N}{\partial \hat{m}} = -4c_1 M_\pi^2 + \mathcal{O}(M_\pi^3)$$

$\longrightarrow$ at leading order, related to the chiral coupling $c_1$

- $\sigma_{\pi N}$ determines scalar couplings wanted for
  direct-detection dark matter searches

e.g. Ellis et al. 2008

$\longrightarrow$ S. Beane's talk Tue.

# Meson–baryon ChPT and loops

- loop integrals cover all energy scales
- Goldstone boson sector: all mass scales "small"
  naive power counting has to work
- with baryons: new mass scale $m_N \approx \Lambda_\chi \approx 1$ GeV
  loop integration picks up momenta $p \sim m_N$

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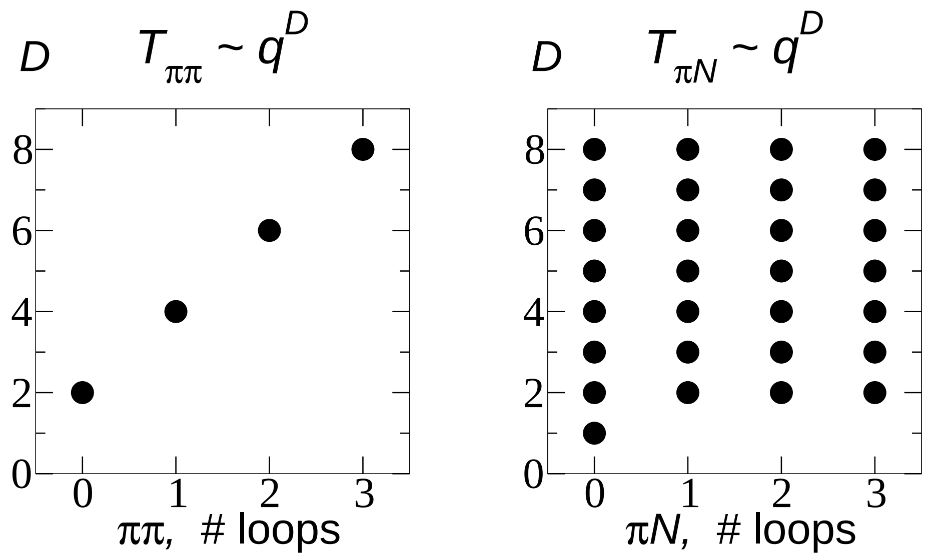
- schematically: Gasser, Sainio, Švarc 1988




$\longrightarrow$ higher-order loops renormalise lower-order couplings

# Remedies (1): Heavy-baryon ChPT

Jenkins, Manohar 1991; Bernard, Kaiser, Meißner 1995

- decompose baryon momentum according to

$$p_\mu = \underbrace{m_N v_\mu}_{\text{large}} + \underbrace{l_\mu}_{\text{residual}} \;, \quad v^2 = 1 \,, \quad v \cdot l \ll m_N$$

- nucleon propagator in the heavy-baryon limit:

$$\frac{1}{p^2 - m_N^2} \to \frac{1}{2m_N} \frac{1}{v \cdot l} + \mathcal{O}\big(1/m_N^2\big)$$

- eliminates mass scale $m_N$ from propagator
  re-enters as parametrical suppression factor
- two-fold expansion $\left(\frac{p}{\Lambda_\chi}\right)^m \times \left(\frac{p}{m_N}\right)^n$

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- counting schemes:

$$\pi N : \quad \frac{p}{m_N} \sim \frac{p}{\Lambda_\chi} \qquad NN : \quad \frac{p}{m_N} \sim \left(\frac{p}{\Lambda_\chi}\right)^2$$

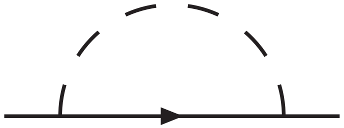
$\longrightarrow$ recoil corrections effectively suppressed in $NN$ counting

# Remedies (2): Infrared regularisation

Ellis, Tang 1998; Becher, Leutwyler 1999

consider (relativistic) nucleon self-energy graph:

$$H \; = $$



$$= \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}(d-3)} \frac{m_N^{d-3} + M_\pi^{d-3}}{m_N + M_\pi} = R + I$$

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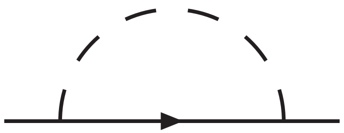
$H =$  $= \dfrac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}(d-3)} \dfrac{m_N^{d-3} + M_\pi^{d-3}}{m_N + M_\pi} = R + I$

"regular" part $R$

- fractional powers in $m_N$, regular in $M_\pi$, $p^2$
- violates naive power counting
- can be expanded as polynomial in $M_\pi$, $p$

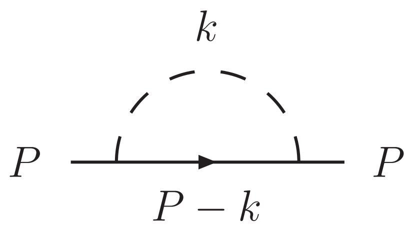
$\Rightarrow$ can be absorbed by re-definition of contact terms

"infrared" part $I$

- fractional powers in $M_\pi$, $p$
- obeys power counting rules
- non-analytic terms, imaginary parts . . .

$\Rightarrow$ all "interesting" loop contributions

# Infrared regularisation + variants

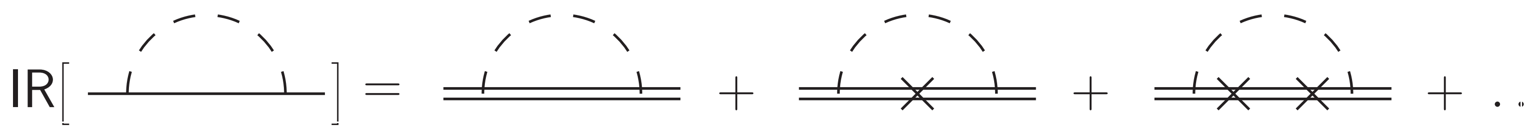



$$a = M_\pi^2 - k^2 - i\epsilon\,, \quad b = m_N^2 - (P-k)^2 - i\epsilon$$

$$H = \frac{1}{i}\int \frac{d^dk}{(2\pi)^d}\frac{1}{ab} = \int_0^1 dz \frac{1}{i}\int \frac{d^dk}{(2\pi)^d}\frac{1}{[(1-z)a + zb]^2}$$

$$= \int_0^{\boxed{\infty}} dz \ldots - \int_1^\infty dz \ldots = I + R$$

- infrared prescription: retain $I$, drop $H$

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$\longrightarrow$ unphysical cut for $P^2 \leq 0$

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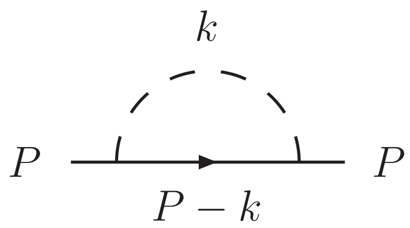


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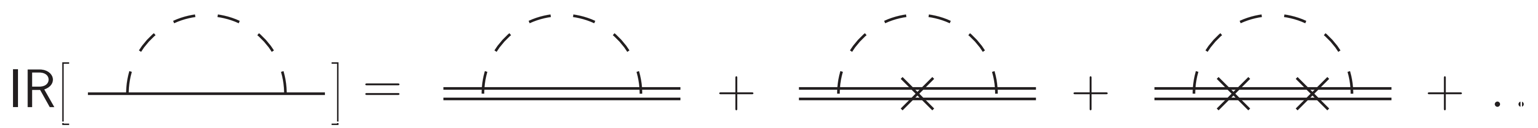


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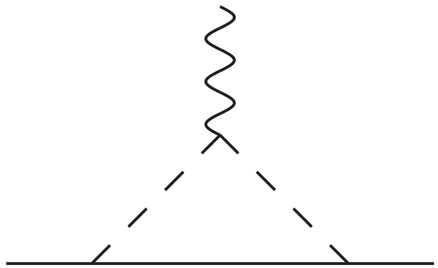
$\longrightarrow$ unphysical cut for $P^2 \leq 0$

- alternative: extended on-mass-shell renormalisation (EOMS): chirally expand $H$ (polynomial!), drop only explicitly power-counting-breaking terms Gegelia et al. 1999–

$\longrightarrow$ analytic structure preserved exactly

# Does it matter? sometimes ...

- consider the electromagnetic form factors of the nucleon: triangle-graph contribution to the spectral function $\mathrm{Im}\, F_1^v(t)$



- "normal" threshold at $t = 4M_\pi^2$
- anomalous threshold at

$$t = 4M_\pi^2 - \frac{M_\pi^4}{m_N^2} \overset{\text{HB}}{=} 4M_\pi^2 + \mathcal{O}(M_\pi^4)$$

$\longrightarrow$ analytic structure distorted

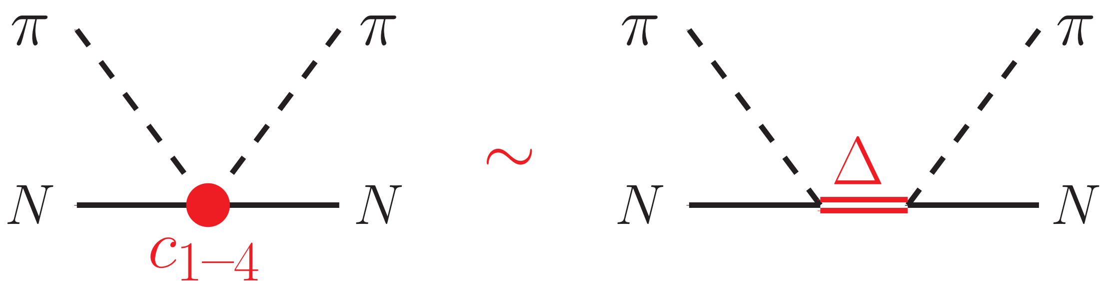
Bernard, Kaiser, Meißner 1996

$$\mathrm{Im}\, F_1^v(t) \overset{\text{IR}}{=} \frac{g_A^2}{192\pi F_\pi^2}(4m_N^2 - M_\pi^2)\left(1 - \frac{4M_\pi^2}{t}\right)^{3/2} + \ldots \quad \text{p-wave}$$

$$\mathrm{Im}\, F_1^v(t) \overset{\text{HB}}{=} \frac{g_A^2}{96\pi F_\pi^2}(5t - 8M_\pi^2)\left(1 - \frac{4M_\pi^2}{t}\right)^{1/2} + \ldots \quad \text{wrong!}$$

$\longrightarrow$ HBChPT fails to converge in parts of the low-energy region

# Including the $\Delta(1232)$ resonance explicitly




- large $\Delta$ effects slow down convergence of chiral series:

$$c_2^\Delta \approx 3.8 \qquad c_3^\Delta \approx -3.8 \qquad c_4^\Delta \approx 1.9$$

Bernard, Kaiser, Meißner 1997

- $N$ and $\Delta$ become degenerate in the large-$N_c$ limit
  $\longrightarrow$ include $\Delta$ as explicit degrees of freedom   Jenkins, Manohar 1991
- consistent EFT counting scheme: $\epsilon$-expansion   Hemmert et al. 1998

$$p = \mathcal{O}(\epsilon) \qquad M_\pi = \mathcal{O}(\epsilon) \qquad m_\Delta - m_N = \mathcal{O}(\epsilon)$$

- alternative: $\delta$ counting   Pascalutsa, Phillips 2003

$$p = \mathcal{O}(\delta) \qquad M_\pi = \mathcal{O}(\delta) \qquad m_\Delta - m_N = \mathcal{O}(\delta^{1/2})$$

$\longrightarrow$ loops with $\Delta$ shifted to higher orders

# Extracting LECs from pion–nucleon scattering

**Strategy:**

- fit results of phase shift analyses:
  - ▷ Karlsruhe–Helsinki (KH) $\longrightarrow$ dispersion theory based
    Koch, Pietarinen 1980, Höhler 1983
  - ▷ GWU/SAID (GW) $\longrightarrow$ modern data input    Workman et al. 2012
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- use chiral low-energy theorems to extract $\sigma_{\pi N}$

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- ChPT obeys unitarity only perturbatively
  de-facto unitarisation to calculate phase shifts from real parts:

$$\delta = \arctan\left(\frac{|\mathbf{p}|}{8\pi\sqrt{s}}\mathrm{Re}\,T\right) \approx \frac{|\mathbf{p}|}{8\pi\sqrt{s}}\mathrm{Re}\,T$$

# Recent chiral phase-shift analyses

- $\mathcal{O}(p^3)$ IR + unitarisation $\not\Delta$ [KH, GW] Alarcón et al. 2011
- $\mathcal{O}(p^3)$ EOMS $\not\Delta$, $\mathcal{O}(\delta^3)$ $\Delta$ [KH, GW, EM] Alarcón et al. 2013

  $\longrightarrow$ $\sigma_{\pi N} = 59(7)$ MeV [GW, EM] ($43(5)$ MeV [KH]) Alarcón et al. 2012

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- $\mathcal{O}(p^4)$ EOMS ($\not\Delta$), $\mathcal{O}(p^4, \delta^3)$ ($\Delta$) [GW] Y.-H. Chen et al. 2013

  $\longrightarrow \sigma_{\pi N} = 52(7)$ MeV ($\not\Delta$), $45(6)$ MeV ($\Delta$) *)

*) including lattice information; $\Delta$ amplitude may violate positivity constraints inside the Mandelstam triangle Sanz-Cillero et al. 2014

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  $\longrightarrow \sigma_{\pi N}$ large [GW] or small [KH]
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- $\mathcal{O}(p^3)$ N/D unitarisation, CDD-poles for $\Delta$ and $N(1440)$ [KH, GW]

  $\longrightarrow \sigma_{\pi N} \approx 77$ MeV ("puzzle") — Gasparyan, Lutz 2010

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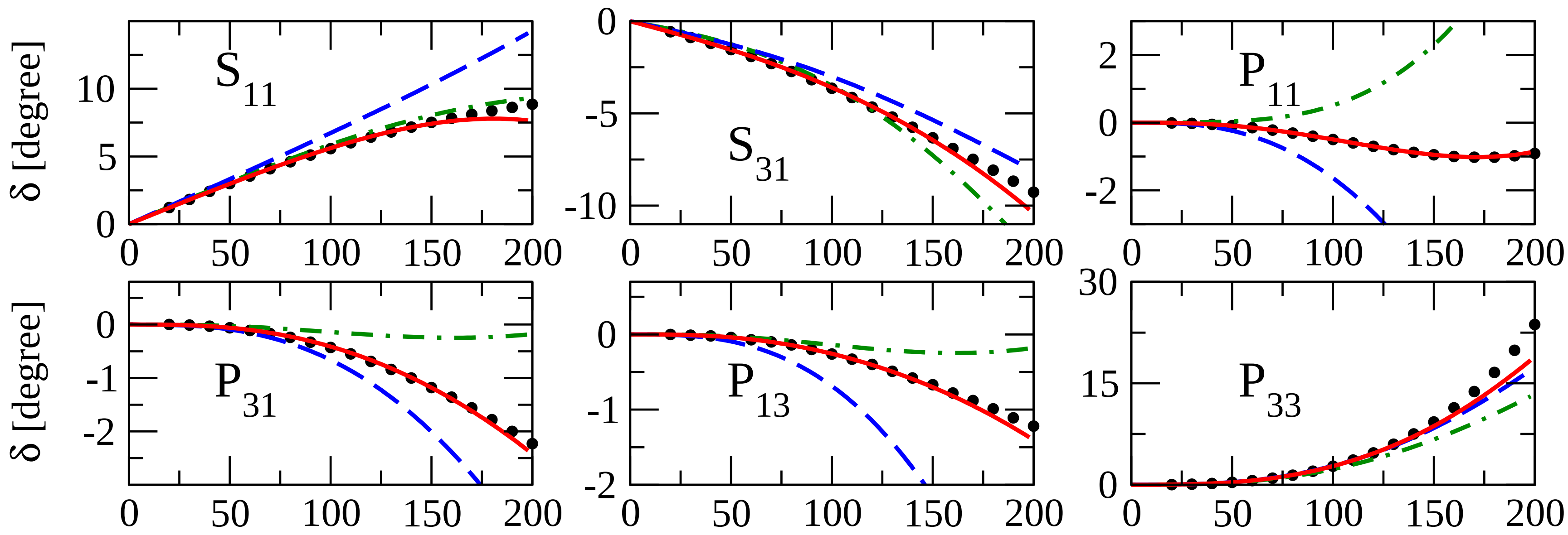
# Convergence of the chiral expansion

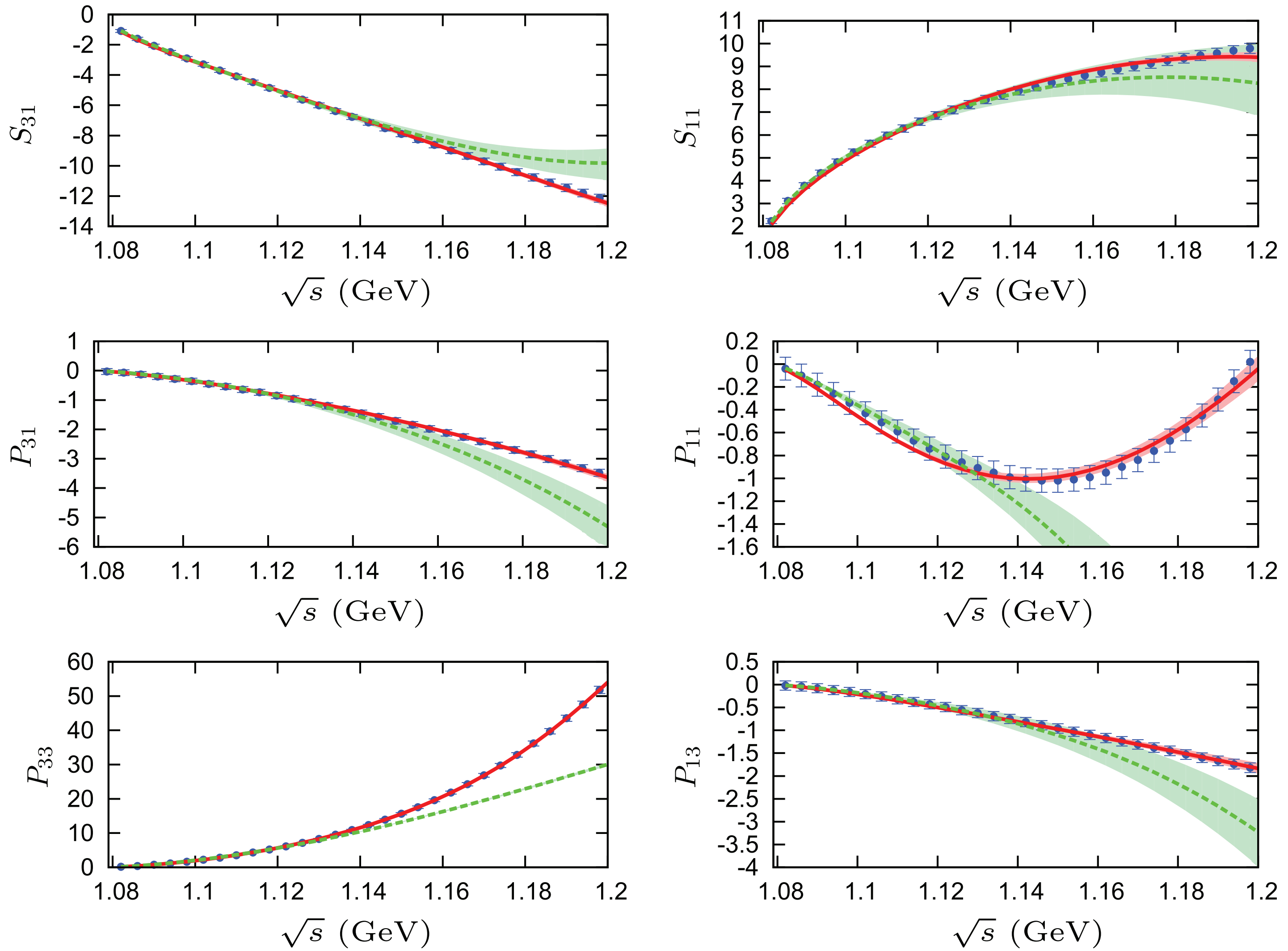



$\mathcal{O}(p^2)$ vs. $\mathcal{O}(p^3)$ vs. $\mathcal{O}(p^4)$

Krebs, Gasparyan, Epelbaum 2012

- fitted up to $p_{\text{Lab}} = 150$ MeV $\hat{=} \sqrt{s} \approx 1.13$ GeV, maximum energy shown $p_{\text{Lab}} = 200$ MeV $\hat{=} \sqrt{s} \approx 1.17$ GeV
- convergence assessed using LECs from highest-order fit
- D-waves also fitted

# ChPT with and without Δ




$\mathcal{O}(p^3)$ / $\mathcal{O}(\delta^3)$ Alarcón, Martin Camalich, Oller 2013

fit range: $\sqrt{s_{\text{max}}} = 1.13$ GeV / $\sqrt{s_{\text{max}}} = 1.20$ GeV

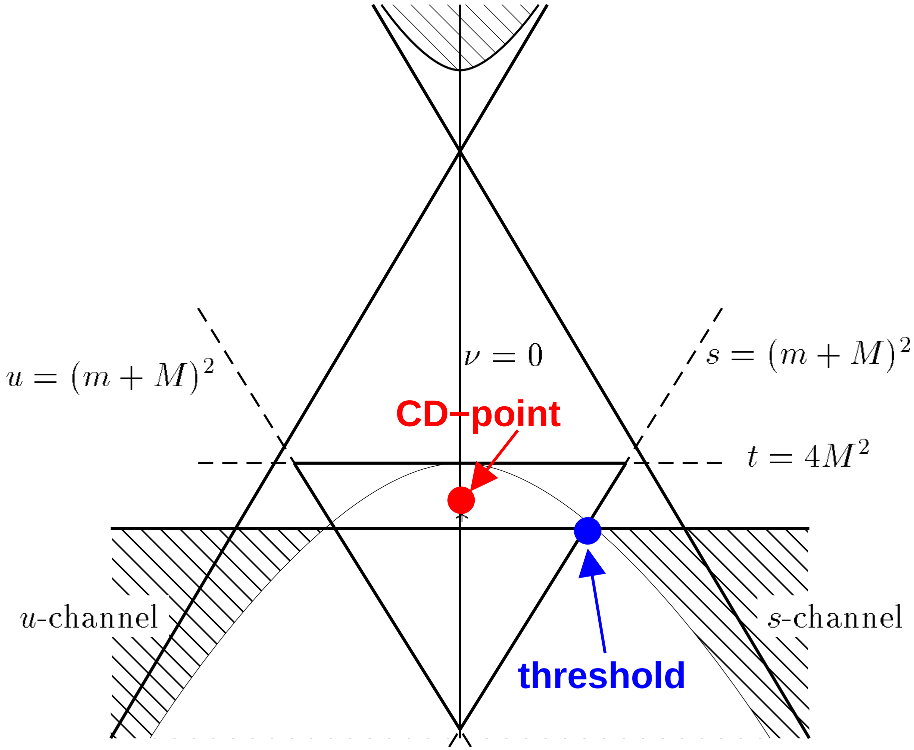
# On the chiral extractions of $\sigma_{\pi N}$

## The Cheng–Dashen theorem

- isoscalar amplitude at CD point related to scalar form factor

$$\underbrace{F_\pi^2 \bar{D}^+(s=u,t=2M_\pi^2)}_{F_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D} = \underbrace{\sigma(2M_\pi^2)}_{\sigma_{\pi N} + \Delta_\sigma} + \Delta_R$$

$|\Delta_R| \lesssim 2$ MeV   Bernard et al. 1996



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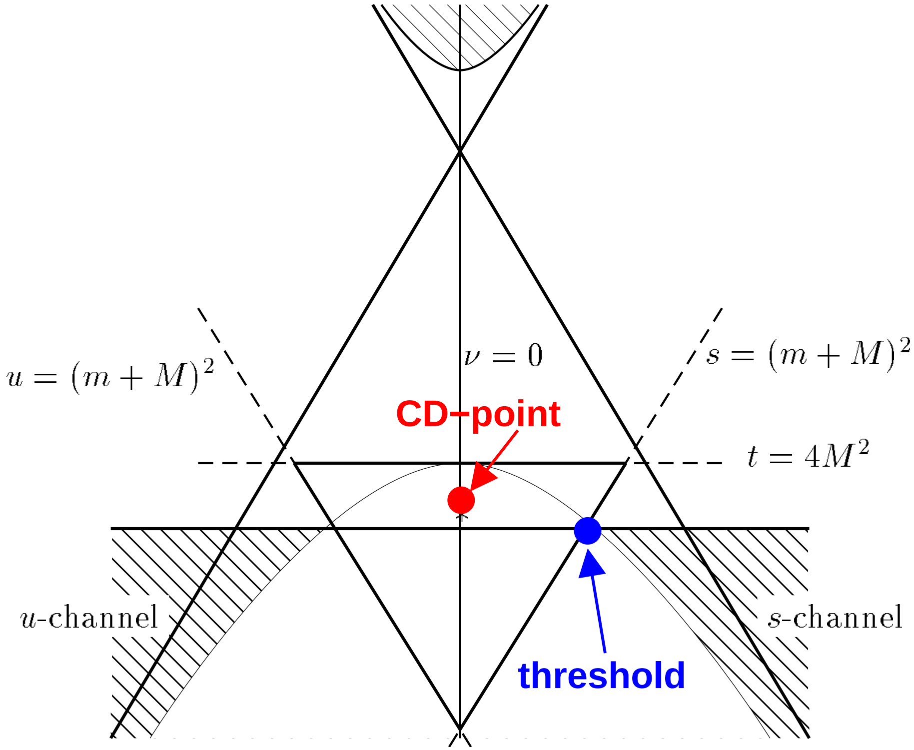


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  is known to fail at one loop for $\Delta_D$, $\Delta_\sigma$: Gasser, Leutwyler, Sainio 1991
  curvature $d_{02}^+$ not reproduced at one loop    Alarcón et al. 2013
- we're lucky: $\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)$ MeV cancels to large extent

$\longrightarrow$ one-loop ChPT does not describe pion–nucleon scattering accurately in the whole low-energy region

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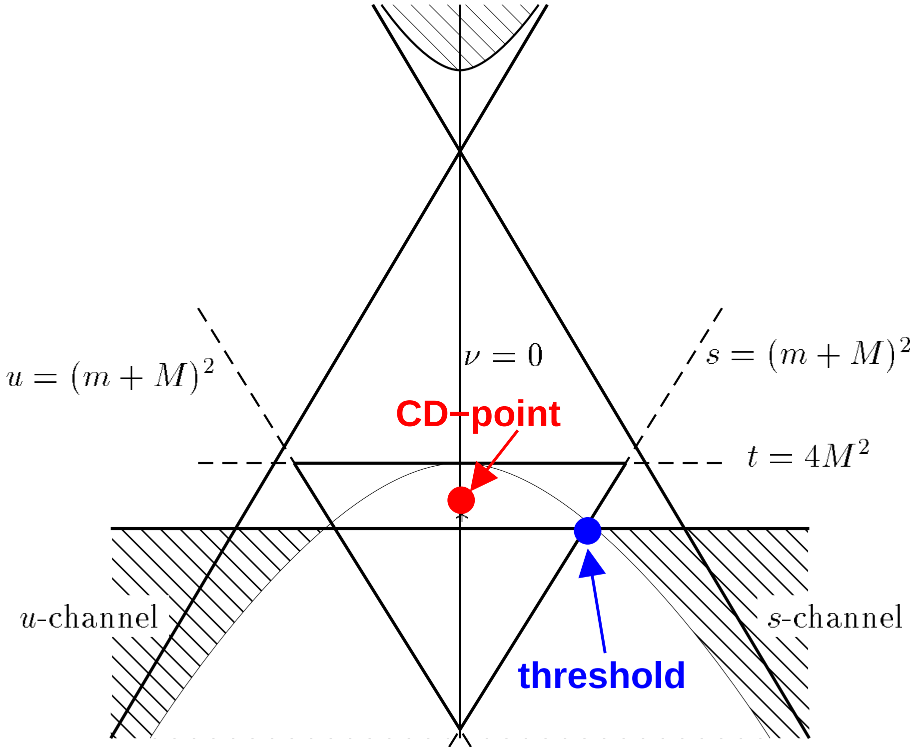


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$\longrightarrow$ update dispersive analysis, Roy–Steiner equations

Hoferichter, Ruiz de Elvira, BK, Meißner

# The well-known paradigm: $\pi\pi$ Roy equations

**Roy equations** = coupled system of partial-wave dispersion relations
+ crossing symmetry + unitarity

- twice-subtracted fixed-$t$ dispersion relation:

$$T(s,t) = c(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \left\{ \underbrace{\frac{s^2}{s'^2(s'-s)}}_{s\text{-channel cut}} + \underbrace{\frac{u^2}{s'^2(s'-u)}}_{u\text{-channel cut}} \right\} \mathrm{Im}T(s',t)$$

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- subtraction function $c(t)$ determined from crossing symmetry
- project onto partial waves $t_J^I(s)$ (angular momentum $J$, isospin $I$)
  expand $\mathrm{Im}T(s',t)$ in partial waves

$$t_J^I(s) = \text{polynomial}(a_0^0, a_0^2) + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int_{4M_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s,s')\mathrm{Im}t_{J'}^{I'}(s')$$

kernel functions $K_{JJ'}^{II'}(s,s')$ known analytically — Roy 1971

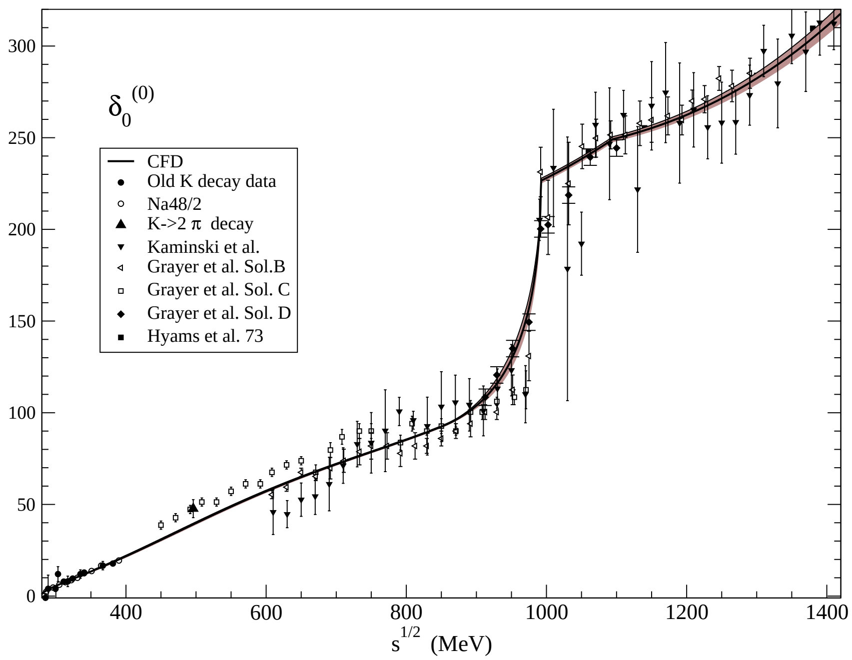
# $\pi\pi$ Roy equations

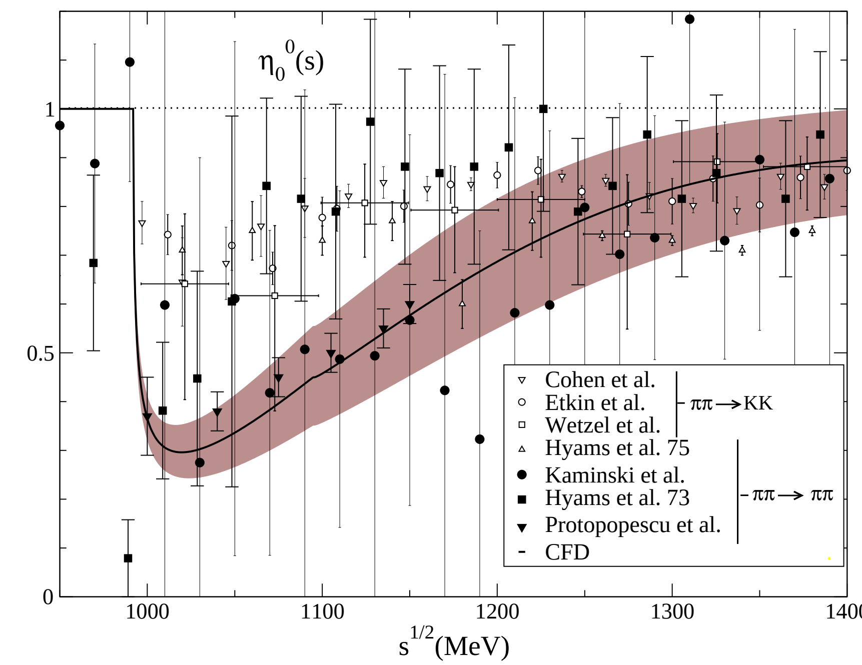
- elastic unitarity:

$$t_J^I(s) = \frac{e^{2i\delta_J^I(s)} - 1}{2i\sigma} \qquad \sigma = \sqrt{1 - \frac{4M_\pi^2}{s}}$$

$\longrightarrow$ coupled integral equations for phase shifts

- example: $\pi\pi$ $I = 0$ S-wave phase shift & inelasticity

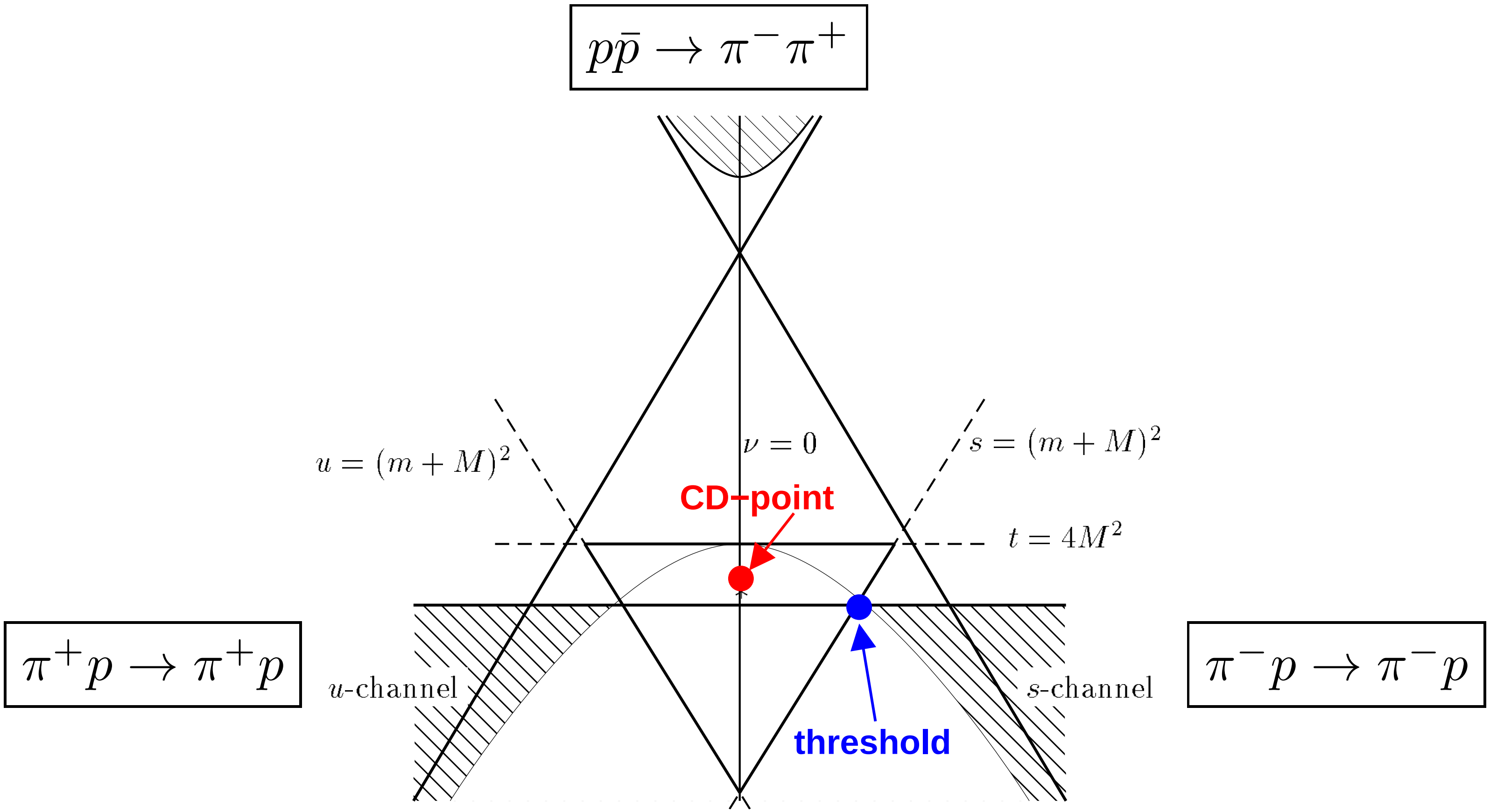



García-Martín et al. 2011

$\longrightarrow$ strong constraints on data from analyticity and unitarity!

# Pion–nucleon scattering, crossing symmetry



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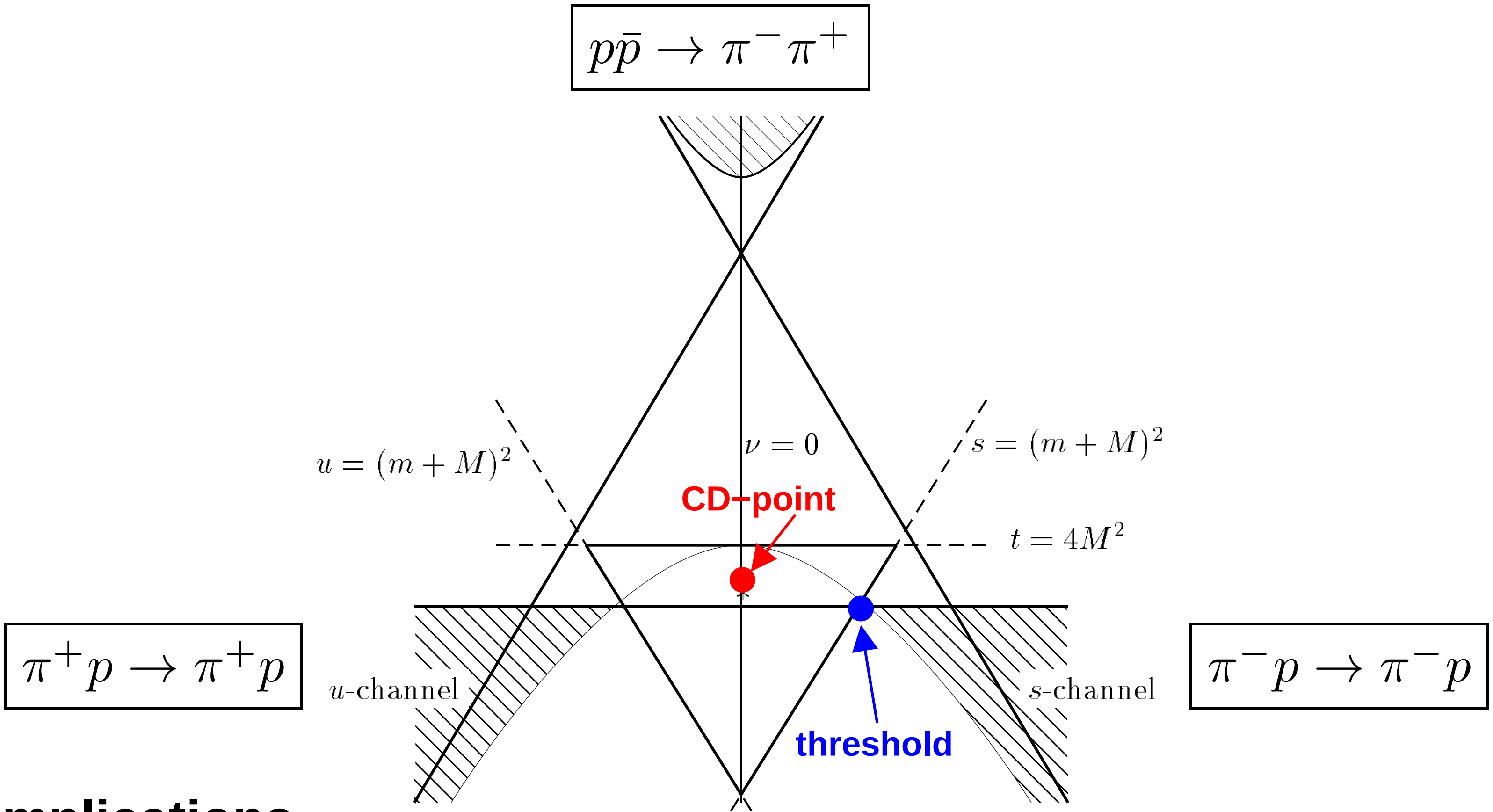


## Complications

- crossing links two different processes, $\pi N \to \pi N$ and $\pi\pi \to \bar{N}N$ $\longrightarrow$ use hyperbolic (instead of fixed-$t$) DR (Roy–Steiner)
- large pseudophysical region in the $t$-channel: $t = 4M_\pi^2 \longrightarrow 4m_N^2$, $\bar{K}K$ intermediate states ($f_0(980)$)

# Roy–Steiner equations for pion–nucleon scattering

**Limited range of validity:**

$$\sqrt{s} \leq \sqrt{s_m} = 1.38\,\text{GeV} \qquad \sqrt{t} \leq \sqrt{t_m} = 2.00\,\text{GeV}$$

**Input / constraints:**

- S-, P-waves above matching point $s > s_m$ ($t > t_m$)
- inelasticities
- higher waves (D-, F-. . . )
- scattering lengths from hadronic atoms Baru et al. 2011

**Output:**

- S- and P-waves at low energies $s < s_m$, $t < t_m$
- subthreshold parameters
  - ▷ pion–nucleon $\sigma$-term
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**Important analysis steps:**

- full analytic system Ditsche, Hoferichter, BK, Meißner 2012
- improved $t$-channel S-wave ($\pi\pi \leftrightarrow \bar{K}K \leftrightarrow \bar{N}N$) Hoferichter, Ditsche, BK, Meißner 2012
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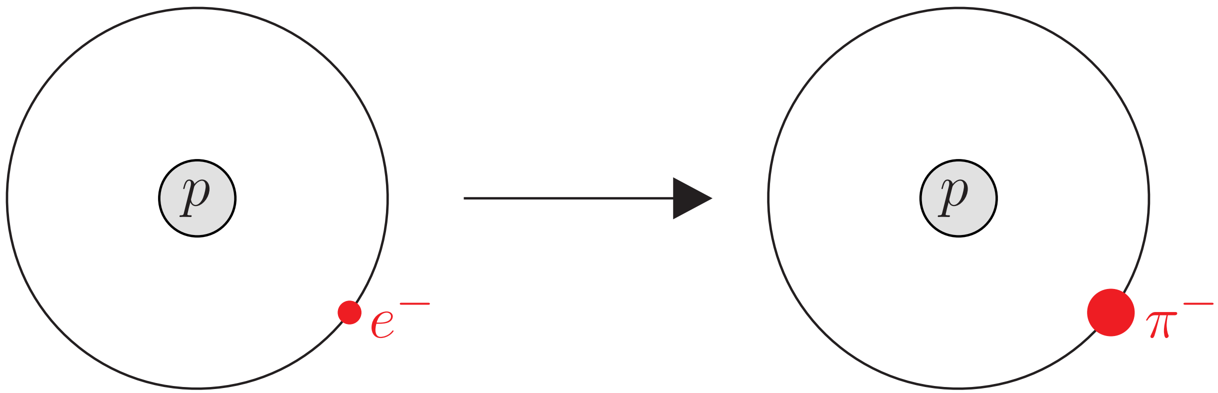
# Pionic atoms and pion–nucleon scattering lengths

cf. Gasser, Lyubovitskij, Rusetsky 2008

- pionic hydrogen $\pi H$, pionic deuterium $\pi D$: atoms with $e^- \to \pi^-$
  calculate energy levels as for hydrogen in quantum mechanics!



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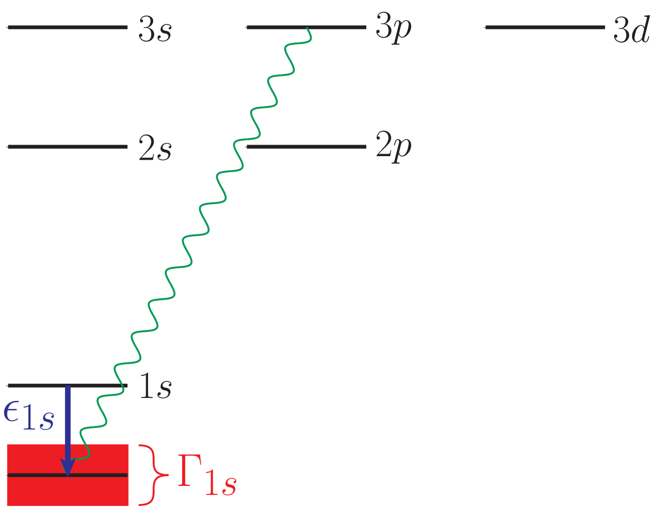
- energy levels perturbed by strong interactions:
  - ▷ ground state instable, decays:
    $\pi^- p \to \pi^0 n \quad \longrightarrow \quad$ width $\Gamma_{1s}$
  - ▷ ground state energy shift $\epsilon_{1s}$




- linked to $\pi N$ scattering at threshold:

$$\epsilon_{1s} \;\propto\; T(\pi^- p \to \pi^- p) \quad \propto\; a_0^+ + a_0^-$$
$$\Gamma_{1s} \;\propto\; |T(\pi^- p \to \pi^0 n)|^2 \propto\; |a_0^-|^2$$

Deser, Goldberger, Baumann, Thirring 1954

- $\pi D$: add. information from energy shift (diff. isospin combination)

# Pionic atoms and pion–nucleon scattering lengths

Measurements of $\pi H$ and $\pi D$ PSI 1995-2010

$\epsilon_{1s} = (7.120 \pm 0.012)$ eV $\Gamma_{1s} = (0.823 \pm 0.019)$ eV $\epsilon_{1s}^{D} = (2.356 \pm 0.031)$ eV

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Theory to match this accuracy requires

- isospin breaking in $\pi N$ Hoferichter, BK, Meißner 2009
- three-body corrections in $\pi D$ Weinberg 1992, . . .
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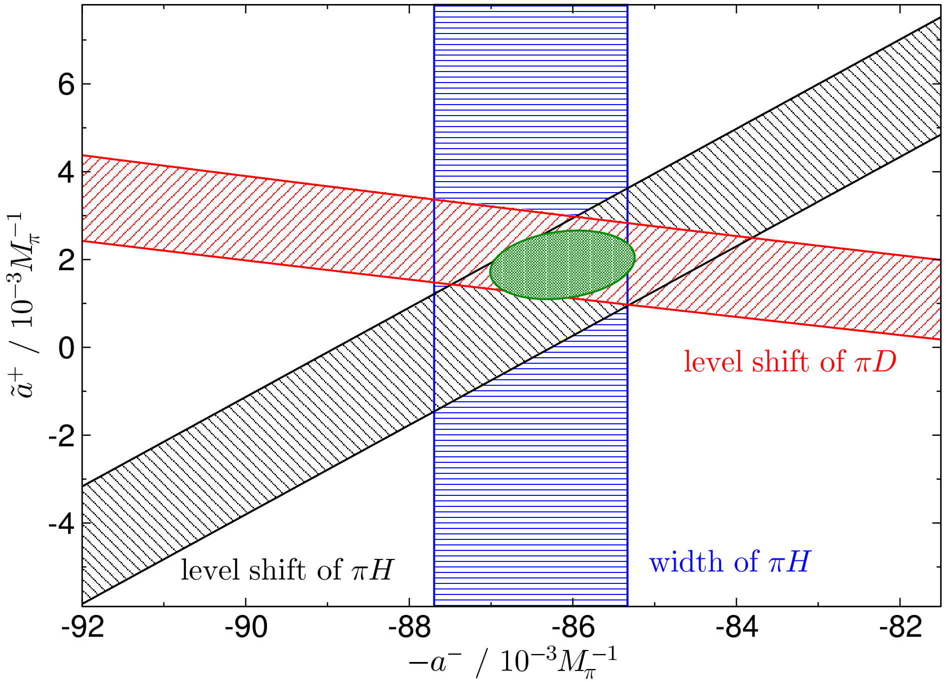


$$a_0^- = (86.1 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$$
$$a_0^+ = (7.6 \pm 3.1) \cdot 10^{-3} M_\pi^{-1}$$

but: $\frac{1}{2}(a_{\pi^- p} + a_{\pi^+ p}) = (-1.1 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$

$\longrightarrow$ large isospin-breaking effects in isoscalar sector

Baru et al. 2011

# Solving the coupled system: paradigms, uncertainties

**An update on Karlsruhe–Helsinki (KH) with modern input**

- $\pi N$ scattering lengths extracted from hadronic atoms
- Goldberger–Miyazawa–Oehme sum rule from those:

$$g_{\pi N}^2/4\pi = 13.7 \pm 0.2$$ Baru et al. 2011

compare: $$g_{\pi N}^2/4\pi = 14.28$$ Höhler 1983

$\longrightarrow$ check: always reproduce KH results with KH input

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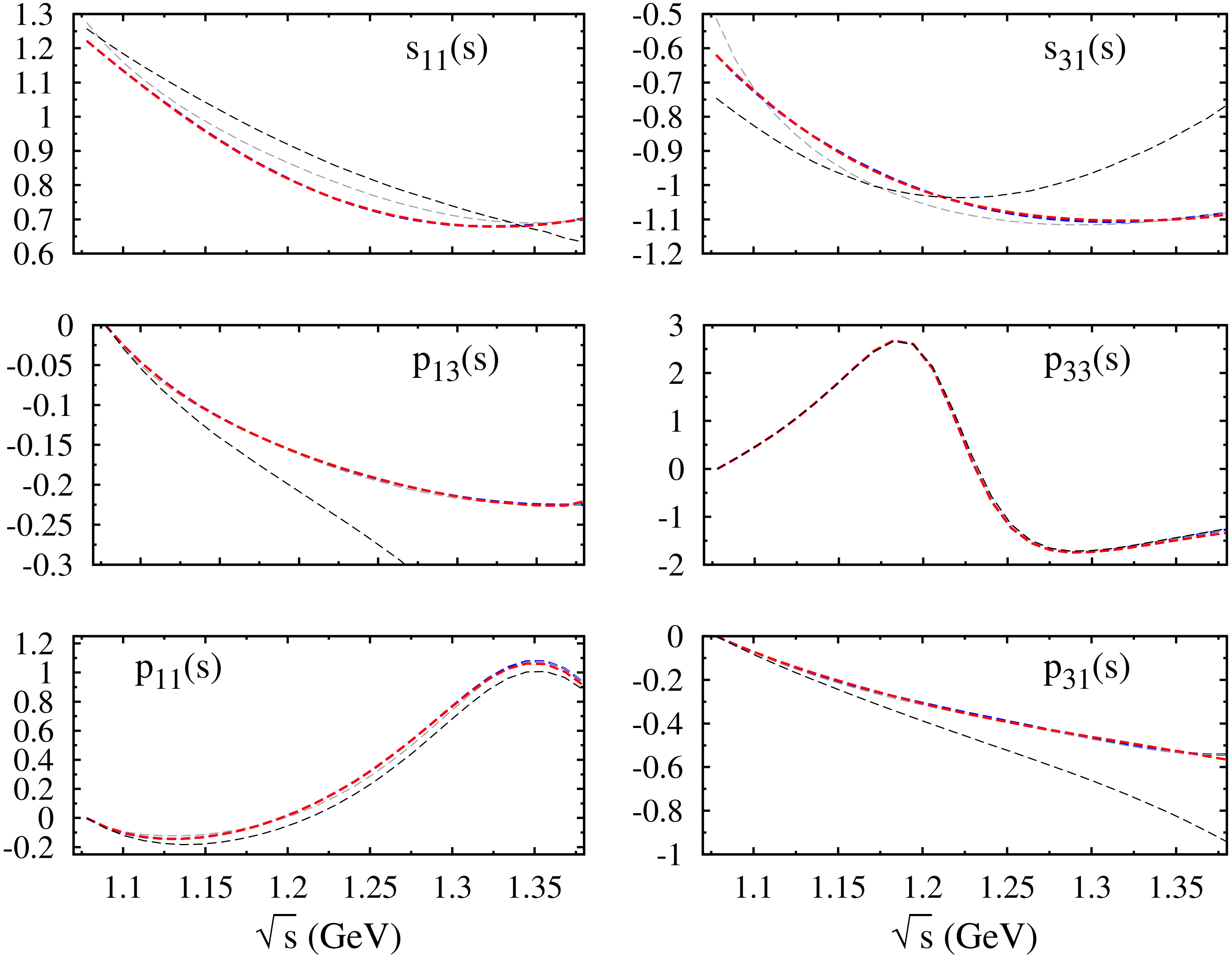
**Dominant uncertainties**

- near threshold: S-wave scattering lengths
- intermediate energies: significant correlations between 10 subtraction constants = subthreshold parameters ("flat minima")
- "large" energies: matching point uncertainties
- rather well under control:
  high-energy input, higher partial waves (in $s$- and $t$-channel)

# Results: *s*-channel solution

LHS+RHS of Roy–Steiner eqs. *before* / LHS+RHS *after* fit/iteration



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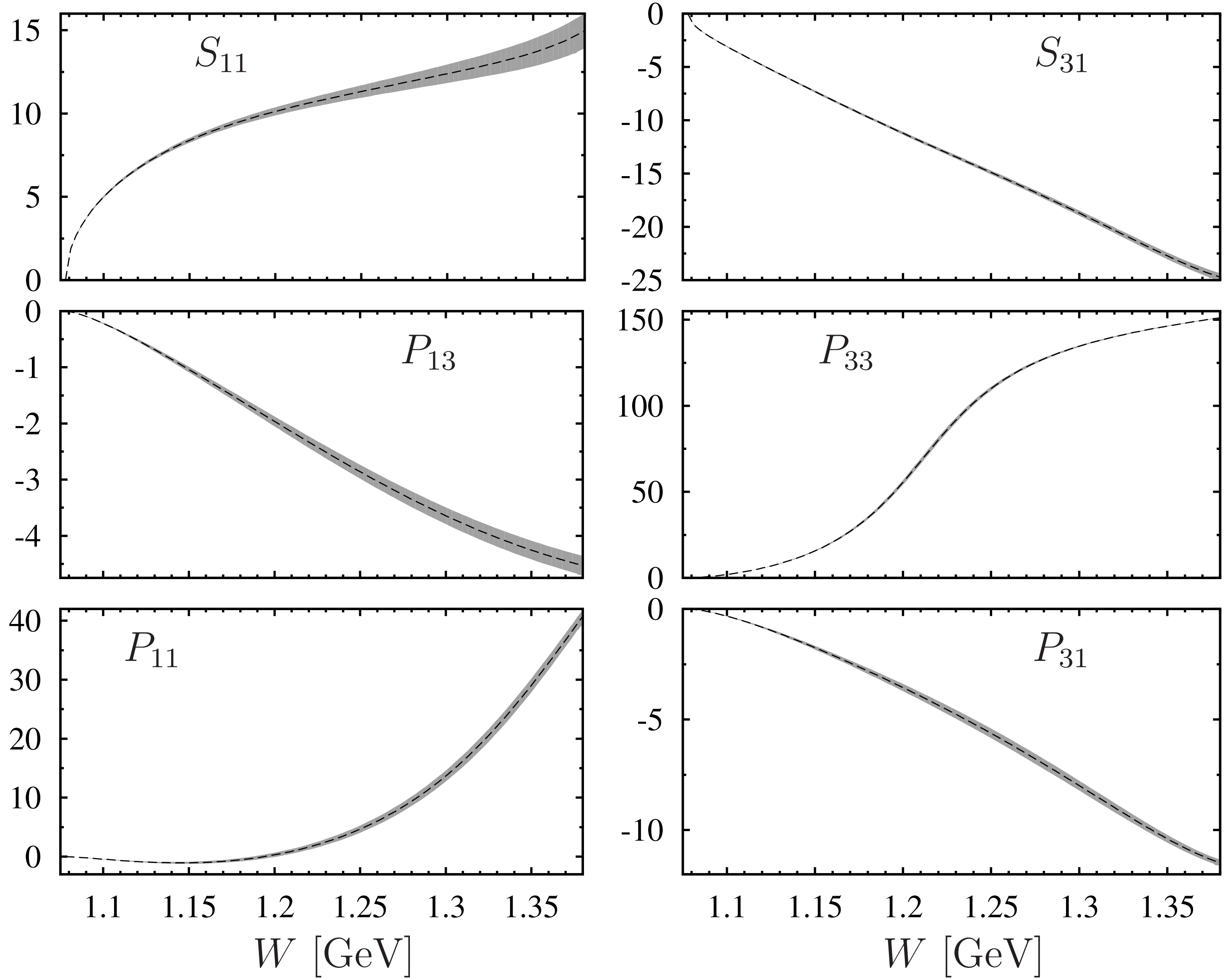


Hoferichter, Ruiz de Elvira, BK, Meißner 2015

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Hoferichter, Ruiz de Elvira, BK, Meißner 2015

- KH input $\longrightarrow$ $\sigma_{\pi N} \approx 46$ MeV — Gasser, Leutwyler, Sainio 1991
- compare also $\sigma_{\pi N} \approx (64 \pm 8)$ MeV — Pavan et al. 2002

# Nucleon strangeness

$\longrightarrow$ H. Leutwyler's talk Mon.

- relate $\sigma_{\pi N}$ to strangeness content of the nucleon:

$$\sigma_{\pi N} = \frac{\hat{m}}{2m_N} \frac{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle}{1-y} \; , \qquad y = \frac{2\langle N|\bar{s}s|N\rangle}{\langle N|\bar{u}u + \bar{d}d|N\rangle}$$

$(m_s - \hat{m})(\bar{u}u + \bar{d}d - 2\bar{s}s) \subset \mathcal{L}_{\text{QCD}}$ produces SU(3) mass splittings:

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- conclusion:
  - ▷ $\sigma_{\pi N} = (59.1 \pm 3.5)$ MeV not incompatible with small $y$
  - ▷ chiral convergence of $\sigma_0$ (hence $\langle N|\bar{s}s|N\rangle$) very doubtful

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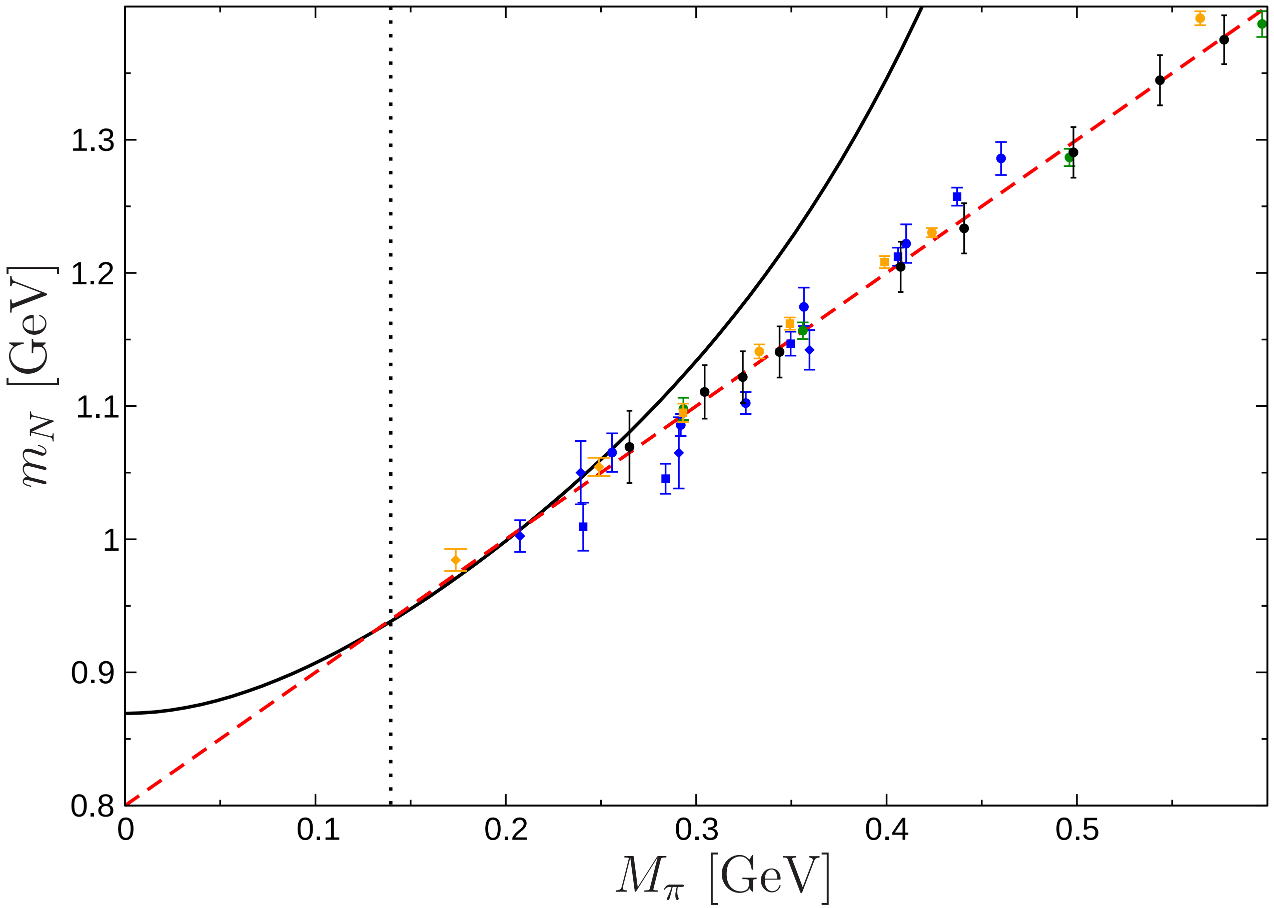
| | LO | NLO | NNLO |
|---|---|---|---|
| $c_1$ [GeV$^{-1}$] | $-0.74 \pm 0.02$ | $-1.07 \pm 0.02$ | $-1.11 \pm 0.03$ |
| $c_2$ [GeV$^{-1}$] | $1.81 \pm 0.03$ | $3.20 \pm 0.03$ | $3.13 \pm 0.03$ |
| $c_3$ [GeV$^{-1}$] | $-3.61 \pm 0.05$ | $-5.32 \pm 0.05$ | $-5.61 \pm 0.06$ |
| $c_4$ [GeV$^{-1}$] | $2.17 \pm 0.03$ | $3.56 \pm 0.03$ | $4.26 \pm 0.04$ |

$\longrightarrow$ subthreshold errors tiny, chiral expansion dominates uncertainty

# The “ruler plot” vs. ChPT

- pion mass dependence of $m_N$, using $\longrightarrow$ S. Beane's talk Tue.
  - ▷ $c_1$ from subthreshold matching to Roy–Steiner solution
  - ▷ combination of $e_i$ from $\sigma_{\pi N}$




thanks to A. Walker-Loud for providing the lattice data

# Summary

## Pion–nucleon Roy–Steiner equations

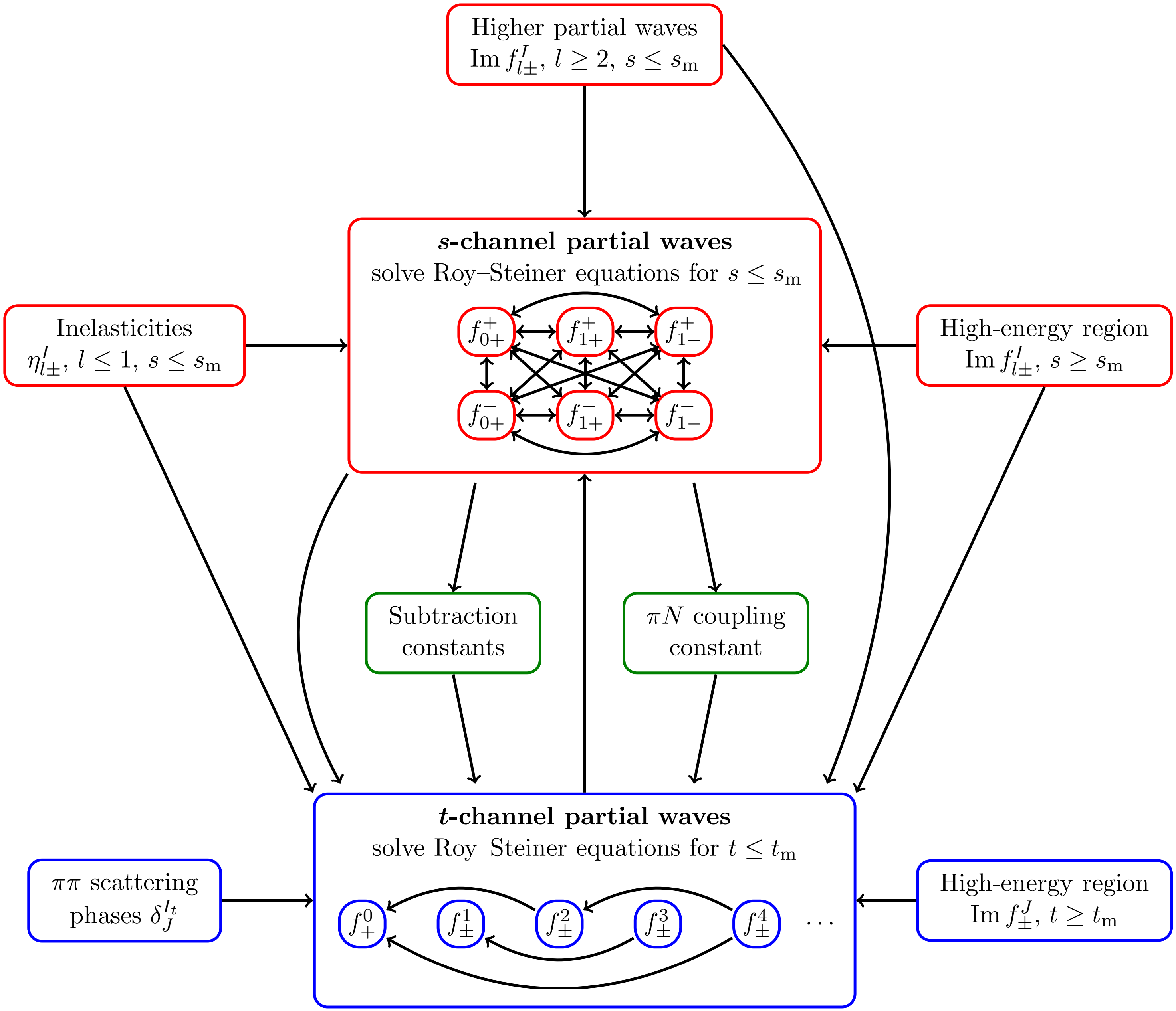
- allow to determine low-energy $\pi N$ scattering with precision
  - ▷ obeying analyticity, unitarity, crossing symmetry
  - ▷ new input on scattering lengths from hadronic atoms
- provide $\pi N$ phase shifts with systematic uncertainties
- similarly: $t$-channel $\pi\pi \to N\bar{N}$ spectral functions
- phenomenological determination of sigma term:

$$\sigma_{\pi N} = 59.1 \pm 3.5\,\text{MeV}$$

- consistency check: Karlsruhe–Helsinki input leads to Karlsruhe–Helsinki results
- chiral low energy constants obtained algebraically from subtreshold coefficients

# Spares

# Roy–Steiner equations: information flowchart



# Results: *t*-channel S-, P-, D-waves (compared to KH)