Anisotropic hydrodynamics for the early stages of heavy-ion collisions

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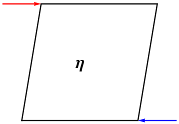
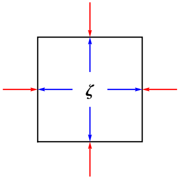
- relativistic hydrodynamics plays an important role in modeling of relativistic heavy-ion collisions and other physical processes
- general quantum mechanical arguments suggest that the shear viscosity η cannot be zero
- strongly-coupled $\mathcal{N} = 4$ SYM theory impose the lower bound of the ratio $\eta/\mathcal{S}$ (with $\mathcal{S}$ being the entropy density), dissipative corrections important
  (Kovtun, Son, and Starinets, Phys.Rev.Lett. 94, 111601 (2005))
- in the general case we deal with two types of viscosity

SHEAR VISCOSITY



BULK VISCOSITY



both the shear and bulk viscosities induce corrections to the equilibrium pressure, shear-stress tensor $\eta \to \pi^{\mu\nu}$, and bulk pressure $\zeta \to \Pi$

- canonical treatment based on an expansion of the phase-space distribution function around local equilibrium state (corrections give rise to dissipative currents)

$$f(x,p) = f_{\rm iso}\left(\frac{p^\mu u_\mu}{T(x)}\right) + \delta f(x,p)$$

$\Rightarrow$ early thermalization required

- large anisotropy at early times predicted by microscopic models (CGC, AdS/CFT, ...)
- studied systems are subject to rapid longitudinal expansion

$\Rightarrow$ large viscous corrections to the ideal energy-momentum tensor
$\Rightarrow$ canonical expansion breaks down
$\Rightarrow$ may cause unphysical results

- form of relaxation-type equations of motion for the shear-stress tensor $\pi^{\mu\nu}$ and bulk viscous pressure Π must be derived within a certain framework
- large uncertainties concerning transport coefficients that appear in the equations of motion, many of them not known yet
- at large T the coupling is weak, theory is nearly conformal, the bulk viscosity is expected to be small, however, near T_C it can be large enough to affect the time evolution of the produced matter

- methods to improve early-time dynamics:
  - complete second-order treatments (Denicol, Niemi, Molnar, Rischke)
  - third-order treatments (El, Xu, Greiner, Jaiswal)
  - anisotropic hydrodynamics (Florkowski, Martinez, Nopoush, Ryblewski, Strickland, Tinti, Bazow, Heinz)

W. Florkowski, R. Ryblewski, Phys.Rev. C83, 034907 (2011)
M. Martinez, M. Strickland, Nucl. Phys. A848, 183 (2010)

note/warning: still most commonly used Israel-Stewart formulation

W. Florkowski, R. Ryblewski, M. Strickland, Nucl.Phys. A916 (2013) 249-259
W. Florkowski, R. Ryblewski, M. Strickland Phys.Rev. C88 (2013) 024903

- anisotropic hydrodynamics → one expands around an anisotropic background, momentum-space anisotropies are built into the LO

$$f(x,p) = \underbrace{f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\Lambda(x)}\right)}_{\rm LO} + \underbrace{\delta\tilde{f}(x,p)}_{\rm NLO}$$

spheroidal ansatz for $\Xi_{\mu\nu}$ give (LRF) $p^\mu \Xi_{\mu\nu} p^\nu = p_x^2 + p_y^2 + (1+\xi)p_z^2$ (R-S form)

- dynamical equations derived from an underlying classical kinetic theory framework by taking moments of Boltzmann equation

- anisotropic hydrodynamics has various appealing features (no negative pressures, reproduced free-streaming limit, kinetic coefficients included implicitly ...)

one wants to check how various dissipative relativistic hydrodynamics approaches describe the non-equilibrium evolution of the system, one possibility for doing this is to compare predictions of hydrodynamic models with exact solutions of the underlying kinetic theory, in general situation this is not possible, however, there are some cases in which this can be done

1. Exact solution of the kinetic equation for a massive gas
2. Bulk viscous pressure evolution within viscous hydrodynamics
3. Bulk viscous pressure evolution in anisotropic hydrodynamics
4. Results

- Boltzmann equation in the relaxation time approximation

$$p^\mu \partial_\mu f(x,p) = C[f(x,p)] \qquad C[f] = p^\mu u_\mu \, \frac{f^{\rm eq} - f}{\tau_{\rm eq}}$$

Bhatnagar, Gross, Krook, Phys. Rev. 94, 511 (1954)

background distribution (Boltzmann statistics)

$$f^{\rm eq} = \frac{g_s}{(2\pi)^3} \exp\left(-\frac{p^\mu u_\mu}{T}\right)$$

- boost-invariant variables (Bialas, Czyz)

$$w = tp_{\parallel} - zE \qquad v = tE - zp_{\parallel}$$

- for transversely homogeneous boost-invariant system

$$\frac{\partial f}{\partial \tau} = \frac{f^{\rm eq} - f}{\tau_{\rm eq}}$$

$$f^{\rm eq}(\tau, w, p_\perp) = \frac{g_s}{(2\pi)^3} \exp\left(-\frac{\sqrt{w^2 + (m^2 + p_\perp^2)\tau^2}}{T\tau}\right)$$

- particle density, energy density, transverse and longitudinal pressure

$$
\begin{aligned}
n(\tau) &= g_0 \int dP \, \frac{v}{\tau} \; f(\tau, w, p_\perp) \\
\mathcal{E}(\tau) &= g_0 \int dP \, \frac{v^2}{\tau^2} \, f(\tau, w, p_\perp) \\
\mathcal{P}_T(\tau) &= g_0 \int dP \, \frac{p_T^2}{2} \, f(\tau, w, p_\perp) \\
\mathcal{P}_L(\tau) &= g_0 \int dP \, \frac{w^2}{\tau^2} \, f(\tau, w, p_\perp)
\end{aligned}
$$

$$
\begin{aligned}
T^{\mu\nu} &= (\mathcal{E} + \mathcal{P}_T) u^\mu u^\nu - \mathcal{P}_T g^{\mu\nu} + (\mathcal{P}_L - \mathcal{P}_T) z^\mu z^\nu \\
T^{\mu\nu}_{\rm eq} &= (\mathcal{E}_{\rm eq} + \mathcal{P}_{\rm eq}) u^\mu u^\nu - \mathcal{P}_{\rm eq} g^{\mu\nu}
\end{aligned}
$$

$$
T^{\mu\nu}_{\rm LRF} = \mathrm{diag}(\mathcal{E}, \mathcal{P}_T, \mathcal{P}_T, \mathcal{P}_L)
$$

$$
T^{\mu\nu}_{\rm eq;LRF} = \mathrm{diag}(\mathcal{E}_{\rm eq}, \mathcal{P}_{\rm eq}, \mathcal{P}_{\rm eq}, \mathcal{P}_{\rm eq})
$$

- determination of effective temperature (Landau matching condition)

$$
\begin{aligned}
u_\mu T^{\mu\nu} &= u_\mu T^{\mu\nu}_{\rm eq} \\
\mathcal{E}(\tau) &= \mathcal{E}^{\rm eq}(\tau) \\
&= g_0 \int dP \, \frac{v^2}{\tau^2} \, f^{\rm eq}(\tau, w, p_\perp) \\
&= \frac{g_0 T m^2}{\pi^2} \left[3 T K_2\left(\frac{m}{T}\right) + m K_1\left(\frac{m}{T}\right) \right]
\end{aligned}
$$

- formal solution (generalization of Baym's result)

$$f(\tau, w, p_\perp) = D(\tau,\tau_0) f_0(w,p_\perp) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')} D(\tau,\tau')\, f^{\rm eq}(\tau', w, p_\perp)$$

$$D(\tau_2,\tau_1) = \exp\left[-\int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm eq}(\tau'')}\right]$$

- initial condition (Romatschke-Strickland form)

$$f_0(w,p_\perp) = \frac{g_s}{(2\pi)^3} \exp\left[-\frac{\sqrt{(1+\xi_0)w^2 + (m^2+p_\perp^2)\tau_0^2}}{\Lambda_0\, \tau_0}\right]$$

$\xi_0 = \xi(\tau_0)$ - initial value of the anisotropy parameter
$\Lambda_0 = \Lambda(\tau_0)$ - initial transverse-momentum scale

$$
Tm^2\left[3TK_2\left(\frac{m}{T}\right)+mK_1\left(\frac{m}{T}\right)\right]=\frac{g_s}{4}\left[D(\tau,\tau_0)\Lambda_0^4\widetilde{\mathcal{H}}_2\left(\frac{\tau_0}{\tau\sqrt{1+\xi_0}},\frac{m}{\Lambda_0}\right)\right.
$$
$$
\left.+\int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{\rm eq}(\tau')}D(\tau,\tau')T'^4\widetilde{\mathcal{H}}_2\left(\frac{\tau'}{\tau},\frac{m}{T'}\right)\right]
$$

- numerical (iterative) method
  1) use a trial function $T'=T(\tau')$ on the RHS of the dynamic equation
  2) the LHS of the dynamic equation determines the new $T=T(\tau)$
  3) use the new $T(\tau)$ as the trial one
  4) repeat steps 1-3 until the stable $T(\tau)$ is found

---

effective bulk pressure in the kinetic theory

$$
T^{\mu}_{\mu\,\rm LRF}=T^{\mu}_{\mu\,\rm visc;LRF}\qquad\rightarrow\qquad \Pi^k_\zeta=\frac{1}{3}\left[\mathcal{P}_\parallel(\tau)+2\mathcal{P}_\perp(\tau)-3\mathcal{P}_{\rm eq}(\tau)\right]
$$

Bulk viscous pressure evolution within viscous hydrodynamics

$$T^{\mu\nu}_{\rm visc} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P+\Pi) + \pi^{\mu\nu}$$

$$T^{\mu\nu}_{\rm visc;LRF} = {\rm diag}(\varepsilon, P+\Pi+\pi/2, P+\Pi+\pi/2, P+\Pi-\pi)$$

energy and momentum continuity equation (zero net charge, no charge diffusion)

$$\partial_\mu T^{\mu\nu}_{\rm visc} = 0 \qquad \rightarrow \qquad \partial_\tau \varepsilon = -\frac{\varepsilon + P + \Pi - \pi}{\tau}$$

- at first order (Fourier–Navier–Stokes, acausal)

$$\pi = \frac{4\eta}{3\tau} \qquad\qquad \Pi = -\frac{\zeta}{\tau}$$

correct bulk viscosity coefficient in the relaxation time approximation

$$\zeta(T) = \tau_{\rm eq} P_{\rm eq} \frac{\mu^2}{3}\left[-\frac{\mu K_2}{3(3K_3+\mu K_2)} + \frac{\mu}{3}\left(\frac{K_1}{K_2} - \frac{K_{i,1}}{K_2}\right)\right], \quad \mu = m/T$$

Redlich, Sasaki, Phys.Rev.C 79 055207 (2009); Bozek, Phys.Rev.C 81 034909 (2010)

$$\zeta(T) = \tau_{\rm eq} P_{\rm eq} \frac{\mu}{3}\left[\frac{3\left(G^2\mu - 5G - \mu\right)}{\mu^2 + 5G\mu - G^2\mu^2 - 1} + \frac{\mu^2}{3}\left(\frac{3G-\mu}{\mu^2} + \frac{K_1 - K_{i,1}}{K_2}\right)\right]$$

Anderson, Witting, Physica 74, 466 (1974)

Cercignani, Kremer, *The Relativistic Boltzmann Equation: Theory and Applications*

- at second order different methods employed:
  - → Grad's 14-moment approximation and second moment of Boltzmann equation
  - → Chapman-Enskog like expansion for distribution function close to equilibrium
- transport coefficients should be extracted by matching fluid dynamics to the underlying microscopic theory, most of them still not written in a convenient form to be implemented

- shear-stress evolution (no terms $\propto \lambda_{\pi\Pi}$ coupling)

$$\tau_\pi \dot{\pi} + \pi = \frac{4}{3\tau}\eta - \left(\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi}\right)\frac{\pi}{\tau}$$

$\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi} = 4/3\tau_\pi$
W. Israel, J. M. Stewart, Ann. Phys. (N.Y.) 118, 341 (1979)
$\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi} = 38/21\tau_\pi$
G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev. D 85, 114047 (2012)
A. Jaiswal, Phys. Rev. C87, 051901 (2013)

- bulk pressure evolution (no terms $\propto \lambda_{\Pi\pi}$ coupling)

$$\tau_\Pi \dot{\Pi} + \Pi = -\frac{\zeta}{\tau} - \frac{1}{2}\tau_\Pi \Pi \left[\frac{1}{\tau} - \left(\frac{\dot{\zeta}}{\zeta} + \frac{\dot{T}}{T}\right)\right] \qquad (A)$$

Muronga, Phys.Rev.C 69, 034903 (2004); Heinz, Song, Chaudhuri, Phys.Rev.C 73, 034904 (2006)

$$\tau_\Pi \dot{\Pi} + \Pi = -\frac{\zeta}{\tau} - \frac{4}{3}\tau_\Pi \Pi \frac{1}{\tau} \qquad (B)$$

Jaiswal, Bhalerao, Pal, Phys.Rev.C 87, 021901 (2013)

$$\tau_\Pi \dot{\Pi} + \Pi = -\frac{\zeta}{\tau} \qquad (C)$$

Heinz, Song, Chaudhuri, Phys.Rev.C 73, 034904 (2006)

M. Nopoush, R. Ryblewski, M.Strickland, Phys. Rev. C 90, 014908 (2014)

anisotropic one-particle distribution function

$$f(x,p) = f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\lambda}\right)$$

anisotropy tensor decomposition

$$\Xi^{\mu\nu} u^\mu u^\nu + \xi^{\mu\nu} - \Delta^{\mu\nu}\Phi$$

$$u_\mu \xi^{\mu\nu} = 0 \qquad u_\mu \Delta^{\mu\nu} = 0 \qquad \xi^\mu_\mu = 0 \qquad \Delta^\mu_\mu = 3$$

$$\xi^{\mu\nu} = \mathrm{diag}(0, \boldsymbol{\xi}) \qquad \boldsymbol{\xi} \equiv (\xi_x, \xi_y, \xi_z)$$

L. Tinti, W. Florkowski, Phys. Rev. C89 034907 (2014)
M. Nopoush, R. Ryblewski, M.Strickland, Phys. Rev. C 90, 014908 (2014)

equations of motion for ξ_z, Φ, λ, T for (0+1)d case are obtained by taking moments of the Boltzmann equation in the relaxation time approximation

$$p^\mu \partial_\mu f = p^\mu \frac{u_\mu}{\tau_{\rm eq}}(f^{\rm eq} - f) \qquad \rightarrow \qquad \partial_{\mu_1} \int dP\, p^{\mu_1} ... p^{\mu_{n+1}} f = u_{\mu_1} \int dP\, p^{\mu_1} ... p^{\mu_n} \frac{1}{\tau_{\rm eq}}(f^{\rm eq} - f)$$

0th moment (1 eq.)

$$\partial_\mu N^\mu = \frac{u_\mu}{\tau_{\rm eq}}(N^\mu_{\rm eq} - N^\mu)$$

1st moment (2 eq.)

$$u_\nu \partial_\mu T^{\mu\nu} = u_\nu \frac{u_\mu}{\tau_{\rm eq}}(T^{\mu\nu}_{\rm eq} - T^{\mu\nu})$$

$$u_\mu T^{\mu\nu}_{\rm eq} = u_\mu T^{\mu\nu}$$

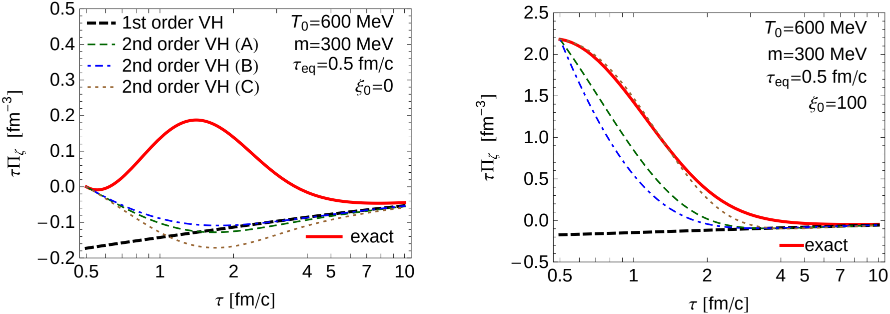
2nd moment (1 eq.)

$$X^i_\mu X^i_\nu \partial_\lambda \Theta^{\lambda\mu\nu} = X^i_\mu X^i_\nu \frac{u_\lambda}{\tau_{\rm eq}} \left(\Theta^{\lambda\mu\nu}_{\rm eq} - \Theta^{\lambda\mu\nu}\right)$$

$$i = 0, 1, 2, 3$$

...

W. Florkowski, E. Maksymiuk, R. Ryblewski, M.Strickland, Phys.Rev. **C 89** (2014) 054908

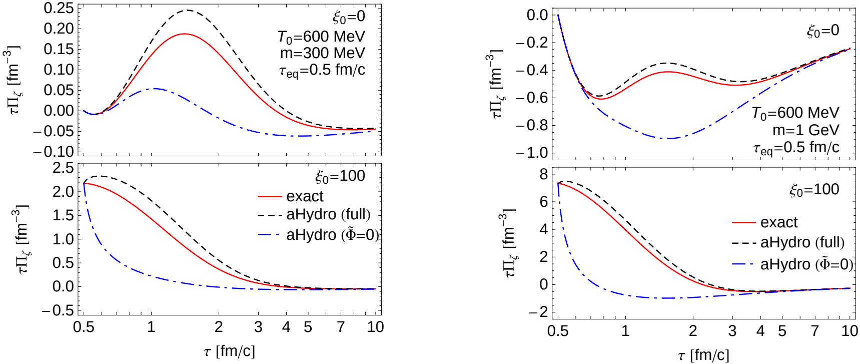


exact solution and all 2nd order viscous hydrodynamics variations tend toward the 1st order solution at late times

none of the 2nd order viscous hydrodynamics variations seems to qualitatively describe the early-time evolution of the bulk viscous pressure in all cases

there is something incomplete in the manner in which 2nd order viscous hydrodynamics treats the bulk pressure (neglected shear–bulk coupling)

M. Nopoush, R. Ryblewski, M.Strickland, Phys. Rev. C 90, 014908 (2014)

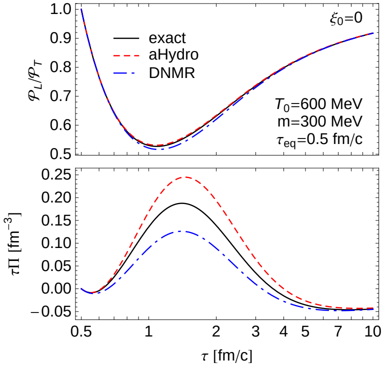


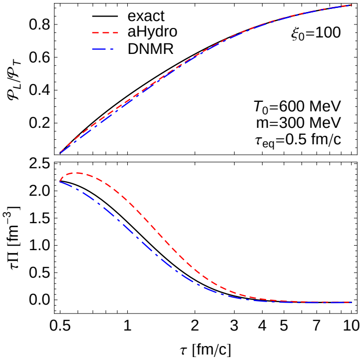
allowing for the bulk degree of freedom significantly improves agreement between anisotropic hydrodynamics and the exact solution

kinetic coefficients implicitly included

G. Denicol, S. Jeon, C. Gale, arXiv:1403.0962
G.Denicol, R. Ryblewski, W. Florkowski, M.Strickland, arXiv:1407.4767

SHEAR-BULK COUPLING





the shear-bulk couplings are extremly important for correct description of the bulk viscous correction

Conclusions

- exact solution of kinetic equation allows for testing various approximation schemes
- commonly used 2nd order viscous hydrodynamics equations do not describe early-time evolution of bulk viscous pressure correctly (shear–bulk coupling missing!)
- shear–bulk coupling is crucial for understanding the bulk viscous pressure evolution in 2nd order viscous hydrodynamics
- forms and values of the kinetic coefficients entering hydrodynamic equations are extreemly important
- caution: some references provide incorrect formulas for bulk viscosity
- explicit inclusion of parameter accounting for bulk viscous correction within anisotropic hydrodynamics helps to improve agreement with exact solution