Precision α_s determination from low-z parton-to-hadron fragmentation functions

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(*) Based on work with D. d'Enterria JHEP 1408 (2014) 068 and arXiv:1408.2865

Outline

- Theoretical description of parton-to-hadron fragmentation functions at small z:
  - The production of jets in high energy collisions.
  - DGLAP vs MLLA evolution equations for FFs, from the LLA to the MLLA.
  - Infrared and collinear singularities $\to$ "NMLLA+NLO*" scheme.
  - Distorted Gaussian (mean multiplicity, mean peak position, width, skewness and kurtosis).
  - Comparison with e^+e^--annihilation, $2-200$ GeV and e^-p-DIS $3-175$ GeV data-sets.
  - Determination of $\Lambda_{\rm QCD}$ and $\alpha_s(M^2_{\rm Z^0})$.

- Conclusions.

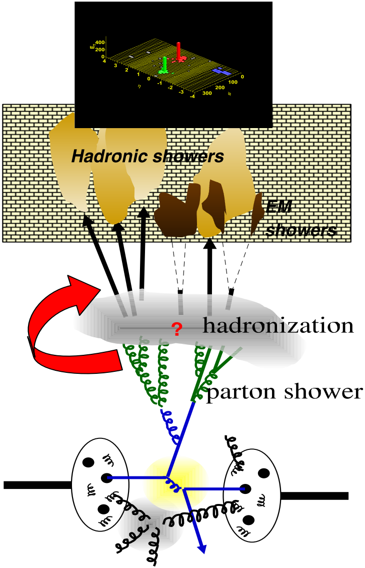
References:

JHEP 1408 (2014) 068 and arXiv:1408.2865 (with David d'Enterria)

Part I

Jet calculus in pQCD

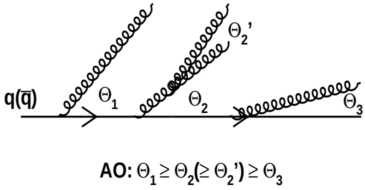
Production of jets

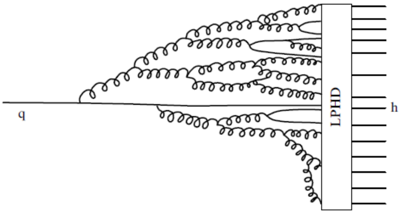


- Partonic cascade: treated in pQCD.
  - Planar gauge: tree amplitudes ⇒ parton shower picture (probabilistic interpretation).

- Hadronization (NMLLA+NLO*): advocates for Local Parton Hadron Duality Hypothesis (LPHD):
  - NMLLA+NLO* FFs ≃ hadron distributions: factor $\mathcal{K}^{ch}$.

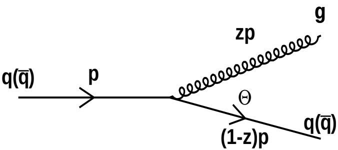
Parton shower evolution



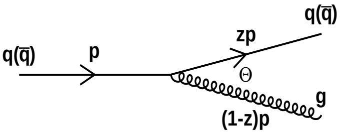


- $k_\perp$-ordering $\to$ DGLAP LLA evolution equations at large $x \sim 1$; ev. time variable "$t = \ln k_\perp$", $t_1 < t_2 < \ldots$.
- QCD coherence $\to$ Angular Ordering (AO) $\to$ MLLA evolution equations for FFs at small $x \ll 1$: ev. time variable $t = \ln \Theta$, $t_1 < t_2 < \ldots$.
- Include as many pQCD high-order log corrections in order to minimize the role of non-perturbative effects (hadronization): $\mathrm{Fragmentation}_{\mathrm{parton}}^{\mathrm{hadron}} \sim \delta(1 - \frac{x}{z})$ (LPHD)

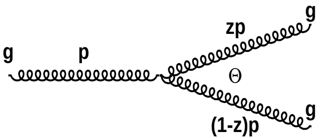
Both approaches involve DGLAP splitting functions $a[1] \to b[z]c[1-z]$:



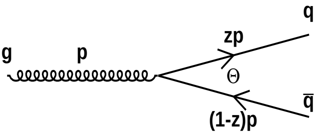
$$P_q^{gq}(z)=C_F\frac{1+(1-z)^2}{z}$$



$$P_q^{qg}(z)=C_F\frac{1+z^2}{1-z}$$



$$P_g^{gg}(z)=2C_A\left(\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right)$$



$$P_g^{q\bar{q}}(z)=T_R\left[z^2+(1-z)^2\right]$$

- Remark: $P_a^{bc}(z) = P_a^{cb}(1-z)$ (parton exchange), $k_\perp = z(1-z)E\Theta$.

DGLAP plus MLLA evolution equations:

- Renormalized QCD evolution equations for $a[1] \to b[z]c[1-z]$:

$$\frac{d}{d\ln Q} x D_a^b(x, \ln Q) = \sum_c \int_0^1 dz P_a^c(z) \frac{\alpha_s(\ln zQ)}{\pi} \left[\frac{x}{z} D_c^b(\frac{x}{z}, \ln zQ)\right]$$

- z: energy fraction of intermediate parton; x: energy fraction of the hadron.

- Identical but for one detail: for hard partons the shift in $\ln z$ in the argument of D and α_s is negligible:
  - for hard/collinear splittings $z \sim 1$: $\ln z \sim 0$, $\Theta \ll 1$: LLA.
  - for soft/collinear splittings $z \ll 1$: $|\ln z| \gg 1$, $\Theta \ll 1$: DLA.
  - for soft/collinear + hard/collinear corrections: Modified-LLA (MLLA).
- (DGLAP) computation of FFs and PDFs at large x: $k_\perp$-ordering.
- (MLLA) computation of FFs at small x: exact angular ordering.

Resummation schemes at small x and Θ

- DLA: $\alpha_s \log(1/x) \log\Theta$: resummation of soft and collinear gluons:
  - main ingredient to the estimation of inclusive observables in jets,
  - neglects the energy balance.
- Single Logs (SL): $\alpha_s \log\Theta$:
  - collinear splittings (i.e. LLA FFs, PDFs at large $x \sim 1$),
  - running of $\alpha_s(k_\perp \to Q_0)$ ($\propto \beta_0$).
- MLLA: $\underbrace{\alpha_s \log\log}_{\mathcal{O}(1)} + \underbrace{\alpha_s \log}_{\mathcal{O}(\sqrt{\alpha_s})}$: the SL corrections to DLA:
  - "restore" the energy balance,
  - take into account the running of $\alpha_s(k_\perp)$.
- Next-to-MLLA+NLO*: $\underbrace{\alpha_s \log\log}_{\mathcal{O}(1)} + \underbrace{\alpha_s \log}_{\mathcal{O}(\sqrt{\alpha_s})} + \underbrace{\alpha_s \log\log^{-1}}_{\mathcal{O}(\alpha_s)}$:
  - improve the restoration of the energy balance,
  - NLO running coupling effects ($\propto \beta_1$) $\to \Lambda_{\mathrm{QCD}}$ and evaluation of $\alpha_s(M^2_{Z^0})$.

Solving the evolution equations at NMLLA+NLO*

- Expressing the Mellin-transformed hadron distribution in terms of the anomalous dimension: $D \simeq C(\alpha_s(t)) \exp\left[\int^t \gamma(t')dt'\right]$, $t \simeq \ln Q$, one needs to solve the equation (iteratively) in the $\mathcal{D}^{\pm}$ basis:

$$(\omega+\gamma_\omega)\gamma_\omega - \frac{2N_c\alpha_s}{\pi} = -\beta(\alpha_s)\frac{d\gamma_\omega}{d\alpha_s} - a_1(\omega+\gamma_\omega)\frac{\alpha_s}{2\pi} - \frac{a_1}{2\pi}\beta(\alpha_s) + a_2(\omega^2+2\omega\gamma_\omega+\gamma_\omega^2)\frac{\alpha_s}{2\pi},$$

- with NLO β-function (necessary for competitive assessment of α_s):

$$\beta(\alpha_s) = -\beta_0\frac{\alpha_s^2}{2\pi} - \beta_1\frac{\alpha_s^3}{4\pi^2} + \mathcal{O}(\alpha_s^4)$$

$$\begin{aligned}\gamma_\omega^{\mathrm{NMLLA+NLO}*} &= \gamma_\omega^{\mathrm{MLLA}} + \frac{\gamma_0^4}{16N_c^2}\left\{a_1^2\frac{\gamma_0^2}{(\omega^2+4\gamma_0^2)^{3/2}} + \frac{a_1\beta_0}{2}\left(\frac{1}{\sqrt{\omega^2+4\gamma_0^2}} - \frac{\omega^3}{(\omega^2+4\gamma_0^2)^2}\right)\right.\\ &\left. + \beta_0^2\left(\frac{2\gamma_0^2}{(\omega^2+4\gamma_0^2)^{3/2}} - \frac{5\gamma_0^4}{(\omega^2+4\gamma_0^2)^{5/2}}\right) - \boxed{4N_c\frac{\beta_1}{\beta_0}\frac{\ln 2(Y+\lambda)}{\sqrt{\omega^2+4\gamma_0^2}}}\right\}\\ &+ \boxed{\frac{1}{4}a_2\gamma_0^2\left[\frac{\omega}{(\omega^2+4\gamma_0^2)^{1/4}} + (\omega^2+4\gamma_0^2)^{1/4}\right]^2} + \mathcal{O}(\gamma_0^4),\end{aligned}$$

- $\gamma_\omega = \gamma^{\mathrm{DLA}}\left[\mathcal{O}(\sqrt{\alpha_s})\right] + \delta\gamma^{\mathrm{MLLA}}\left[\mathcal{O}(\alpha_s)\right] + \delta\gamma^{\mathrm{NMLLA+NLO}*}\left[\mathcal{O}(\alpha_s^{3/2})\right]$.

- Obtained after combining the eigenvectors $\mathcal{D}^+$ and $\mathcal{D}^-$ $(= 0)$:

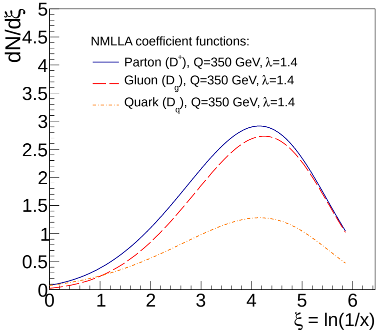
$$\text{Quark FF}: \quad \mathcal{D}_q(\omega, Y, \lambda) \approx C_q^g(\Omega)\mathcal{D}^+(\omega, Y, \lambda), \quad C_q^g(\Omega) = \frac{P_{qg}(\Omega)}{P_{++}(\Omega) - P_{--}(\Omega)},$$

$$\text{Gluon FF}: \quad \mathcal{D}_g(\omega, Y, \lambda) \approx C_g^g(\Omega)\mathcal{D}^+(\omega, Y, \lambda), \quad C_g^g(\Omega) = \frac{P_{++}(\Omega) - P_{qq}(\Omega)}{P_{++}(\Omega) - P_{--}(\Omega)}.$$

- Jet virtuality $Q = 350$ GeV evolves down to scale $\lambda = 1.4$ i.e to $Q_0 = 0.8$ GeV.
- NMLLA+NLO* multiplicity ratio:

$$\frac{\mathcal{N}_g}{\mathcal{N}_q} = \frac{N_c}{C_F}\left(1 - r_1\sqrt{\alpha_s} - r_2\alpha_s\right)$$

[Consistent with I. Dremin et al. Phys.Rep. 349(2001)]

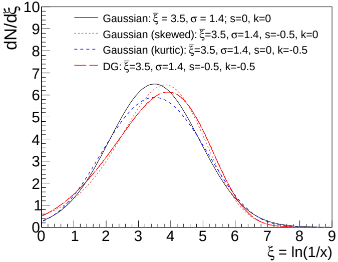


Part II

Distorted Gaussian

Single inclusive distribution: Distorted Gaussian

- $D^+(\xi, Y, \lambda) = \frac{\mathcal{N}}{\sigma\sqrt{2\pi}} \exp\left[\frac{1}{8}k - \frac{1}{2}s\delta - \frac{1}{4}(2+k)\delta^2 + \frac{1}{6}s\delta^3 + \frac{1}{24}k\delta^4\right]$



- $\delta = \frac{(\xi - \bar{\xi})}{\sigma}$
- Mean multiplicity: $\mathcal{N} = \mathcal{D}^+(\omega = 0, Y, \lambda)$
- Mean peak position: $\bar{\xi}$
- Dispersion (width): σ
- Skewness: s, kurtosis: k

- Moments of the Distorted Gaussian (from anomalous dimension):

$$\mathcal{N} = K_0, \quad \bar{\xi} = K_1, \quad \sigma = \sqrt{K_2}, \quad s = \frac{K_3}{\sigma^3}, \quad k = \frac{K_4}{\sigma^4}$$

$$K_{n\geq 0} = \int_0^Y dy \left(-\frac{\partial}{\partial\omega}\right)^n \gamma_\omega(\alpha_s(y+\lambda))\Big|_{\omega=0}, \; Y = \ln\frac{E\theta}{Q_0}$$

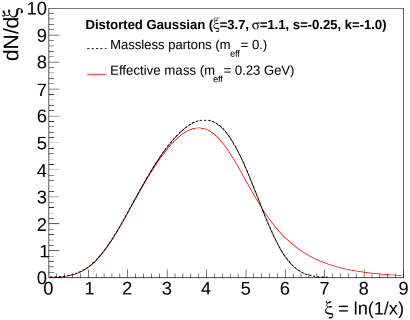
- Skewness and kurtosis (new ingredient) affect tails $\neq$ Gaussian shape!

Finite hadron-mass & heavy-flavour threshold corrections

- FF distribution measurement for massive hadrons ($\xi_{\rm p}$), but theory derived for massless partons/hadrons $\xi = \xi_{\rm E}$:

$$\frac{1}{\sigma_{\rm tot}}\frac{d\sigma^{\rm h}}{d\xi_p} \propto \frac{p_h}{E_h} D^+(\xi, Y)\,,\, \xi = \ln(1/x) = \ln\left(\frac{\sqrt{s}/2}{\sqrt{(s/4)e^{-2\xi_p} + m^2_{\rm eff}}}\right)$$

with $m_h \sim \mathcal{O}(\Lambda_{\rm QCD})$,
$E_h = \sqrt{p_h^2 + m^2_{\rm eff}}$,
$p_h = (\sqrt{s}/2) \cdot \exp -\xi_p$.



- Mass effects gauged by varying data fits with $m_{\rm eff} = 0 - 0.36$ MeV.
- Theoretical expressions depend weakly on the number of active flavours ($N_{\rm flavours} = 3, 4, 5$).

NMLLA+NLO* moments of the (DG) FF moments

Expressions as a functions of $Y = \overbrace{\ln(E\Theta/Q_0)}^{\sim\text{energy}}$ and $\overbrace{\lambda = \ln(Q_0/\Lambda_{\rm QCD})}^{\sim\text{collinear cut-off parameter}}$:

$$*\text{Multiplicity}: \quad \mathcal{N}(Y) \propto \exp\left[2.50217\left(\sqrt{Y+\lambda}-\sqrt{\lambda}\right) - 0.491546\ln\frac{Y+\lambda}{\lambda} - (0.06889 - 0.41151\ln(Y+\lambda))\frac{1}{\sqrt{Y+\lambda}} + (0.06889 - 0.41151\ln\lambda)\frac{1}{\sqrt{\lambda}}\right].$$

$$*\text{Peak position}: \quad \xi_{\rm max} = \frac{Y}{2} + \frac{a_1}{\sqrt{16N_c\beta_0}}\left(\sqrt{Y+\lambda}-\sqrt{\lambda}\right) - \frac{1}{2}\sigma s - 2N_c\frac{a_2}{\beta_0}(\ln(Y+\lambda)-\ln\lambda),$$

$$*\text{Width}: \quad \sigma(Y) = 0.36499\sqrt{(Y+\lambda)^{3/2}-\lambda^{3/2}}\left\{1 - 0.299739f_1(Y,\lambda)\frac{1}{\sqrt{Y+\lambda}} - \left[1.12479f_2(Y,\lambda) + 0.0449219f_1^2(Y,\lambda) + (0.32239 - 0.246692\ln(Y+\lambda))f_3(Y,\lambda)\right]\frac{1}{Y+\lambda}\right\}$$

$$*\text{Skewness}: \quad s(Y) = -\frac{1.94704}{\sqrt{(Y+\lambda)^{3/2}-\lambda^{3/2}}}\left[1 - 0.299739f_1(Y,\lambda)\frac{1}{\sqrt{Y+\lambda}}\right]$$

$$*\text{Kurtosis}: \quad k(Y) = -\frac{2.15812}{\sqrt{Y+\lambda}}\frac{1-\left(\frac{\lambda}{Y+\lambda}\right)^{5/2}}{\left[1-\left(\frac{\lambda}{Y+\lambda}\right)^{3/2}\right]^2}\left\{1 + [1.19896f_1(Y,\lambda) - 1.99826f_4(Y,\lambda)]\frac{1}{\sqrt{Y+\lambda}} + \left[1.07813f_1^2(Y,\lambda) + 4.49915f_2(Y,\lambda) + 1.28956f_3(Y,\lambda) - 2.39583f_1(Y,\lambda)f_4(Y,\lambda) - 3.76231f_5(Y,\lambda) + 0.0217751f_6(Y,\lambda) - (0.986767f_3(Y,\lambda) - 0.822306f_6(Y,\lambda))\ln(Y+\lambda)\right]\frac{1}{Y+\lambda}\right\}. \quad (1)$$

Evolution of the moments of the (DG) FFs: limiting spectrum

Expressions evolved down to $\Lambda_{\rm QCD}$ $Q_0 \sim \Lambda_{\rm QCD}$:

$$*\text{Multiplicity}: \quad \mathcal{N}(Y) = \mathcal{K}^{\rm ch} \exp\Big[2.50217\sqrt{Y} - 0.491546 \ln Y - (0.06889 - 0.41151 \ln Y)\frac{1}{\sqrt{Y}} + (0.00068 - 0.161658 \ln Y)\frac{1}{Y}\Big]$$

$$*\text{Peak position}: \quad \xi_{\max}(Y) = 0.5Y + 0.592722\sqrt{Y} - 0.351319 + 0.002$$

$$*\text{Width}: \quad \sigma(Y) = 0.36499 Y^{3/4}\left[1 - 0.299739\frac{1}{\sqrt{Y}} - (1.4921 - 0.246692 \ln Y)\frac{1}{Y} + \frac{1.98667}{Y^{3/2}}\right]$$

$$*\text{Skewness}: \quad s(Y) = -\frac{1.89445}{Y^{3/4}}\left[1 - 0.312499\frac{1}{\sqrt{Y}} - \frac{1.64009}{Y}\right]$$

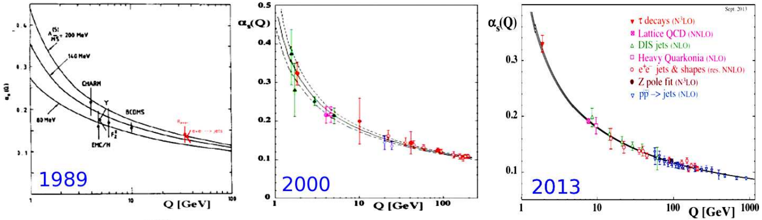
$$*\text{Kurtosis}: \quad k(Y) = -\frac{2.15812}{\sqrt{Y}}\left[1 - 0.799305\frac{1}{\sqrt{Y}} + (0.730466 - 0.164461 \ln Y)\frac{1}{Y} - \frac{8.05771}{Y^{3/2}}\right] \qquad (2)$$

$*$ Evolution of all moments depend on 1 single free parameter $\Lambda_{\rm QCD}$, which can be extracted from fits of exp. e^+e^- and $e^-p \to$ jets(hadrons) data.

Part III

Extraction of $\alpha_s(M^2_{Z_0})$ from fits

What for?

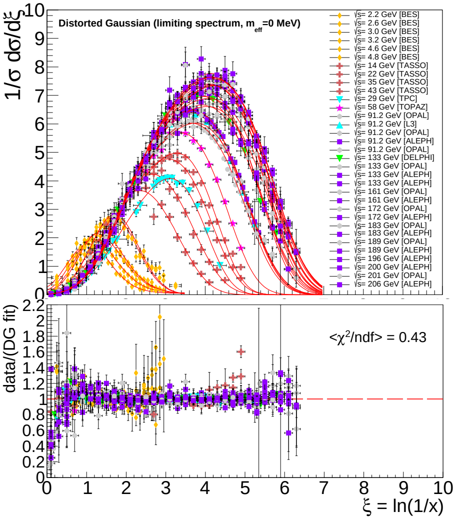


$\alpha_s(M_Z) = 0.110^{+0.006}_{-0.008}$ (NLO) — G. Altarelli, Ann. Rev. Nucl. Part. Sci. 39, 1989

$\alpha_s(M_Z) = 0.1184 \pm 0.0031$ (NNLO) — S. B., J. Phys. G 26, 2000

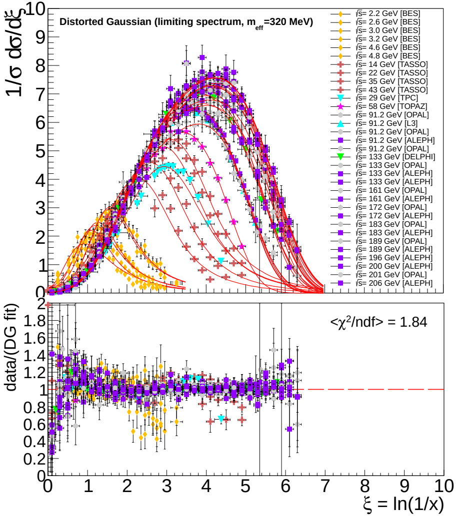
$\alpha_s(M_Z) = 0.1185 \pm 0.0006$ (NNLO)

- Single free parameter in QCD (in the $m_q \to 0$ limit). Determined at a given ref. scale (e.g. m_Z). Decreases as $\sim \ln(Q^2/\Lambda^2_{\rm QCD})$, with $\Lambda_{\rm QCD} \sim 0.25$ GeV
- Least precisely known of all couplings:
  - Impacts all LHC x-sections
  - Key for SM precision fits (e.g. uncertainties H_b, H_c Yukawa)
  - Key for BSM physics (e.g. couplings at GUT scale)



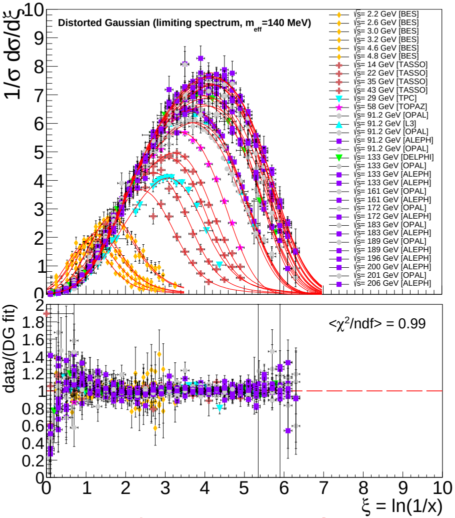
- 34 e^+e^- data-sets at $\sqrt{s} = 2.2 - 206$ GeV ~ 1200 data points
- Peak shifts to right (skewness), width increases as $E \nearrow$ and (tails) DG→Gaussian
- Excellent fit to DG at all energies, with 5 free parameters: $\mathcal{N}$, $\xi_{\rm max}$, σ, s and k

FFs ($n_f = 5$) for e^+e^- $\sqrt{s} = 2 - 200$ GeV, $m_{\rm eff} = 320$ GeV



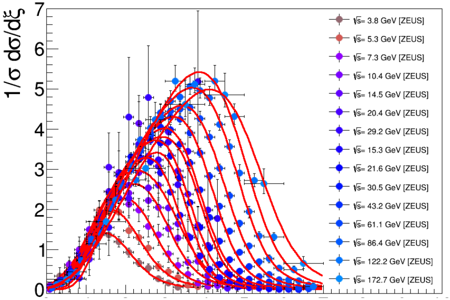
- For $m_h = 0.32$ GeV, goodness-of-fit per degree of freedom is not as good: $\chi^2/ndf = 1.84$ than $m_h = 0$

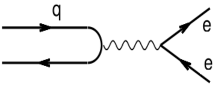
FFs ($n_f = 5$) for e^+e^- $\sqrt{s} = 2 - 200$ GeV, $m_{\rm eff} = 140$ GeV



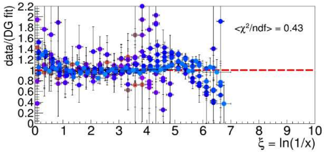
- Best agreement reached for $m_h = 0.14$ GeV: consistent with a dominant pion composition of the inclusive charged hadron spectra.
- Fits with $\sqrt{s} \geq 50$ GeV: insensitive to the choice of m_h!

DG fits to DIS FFs ($m_{\rm eff} = 0.14$ GeV)





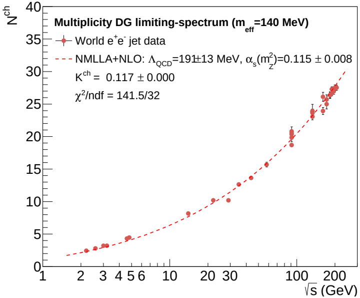
- Breit frame in DIS: "Brick wall frame:" Incoming quark scatters off photon & returns along the same axis

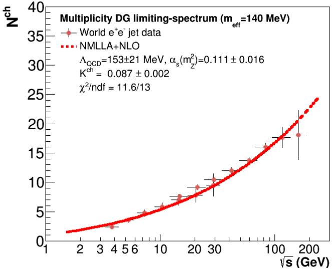


- 15 ZEUS data-sets at $\sqrt{s} = 3.8 - 173$ GeV $\sim$ 250 data points (H1 data to be added)

- Goods fits to DG but larger uncertainties than e^+e^- measurements

$$\mathcal{N}(Y) = \mathcal{K}^{\rm ch} \exp\left[2.50217\sqrt{Y} - 0.491546 \ln Y - (0.06889 - 0.41151 \ln Y)\frac{1}{\sqrt{Y}} + (0.00068 - 0.161658 \ln Y)\frac{1}{Y}\right]$$

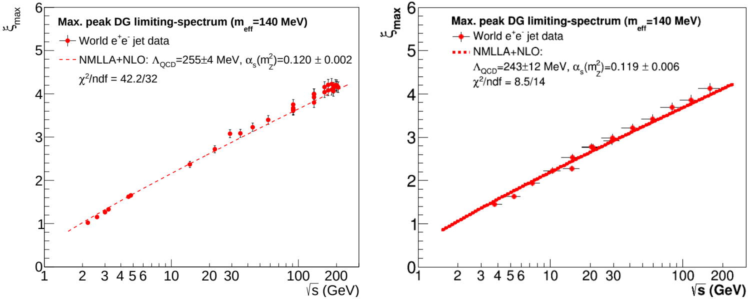




- Good data vs. (NMLLA+NLO*) agreement for multiplicity evolution ($\chi^2/\rm ndf \sim 1.3$) from $\mathcal{N}^{\rm ch} \sim 1 - 30$: $\mathcal{K}^{\rm ch} \sim 0.12$ (local hadron-parton duality norm.)
- DIS $\mathcal{N}^{\rm ch}(e^-p) \sim$ lower than $\mathcal{N}^{\rm ch}(e^+e^-)$, but with larger uncertainties

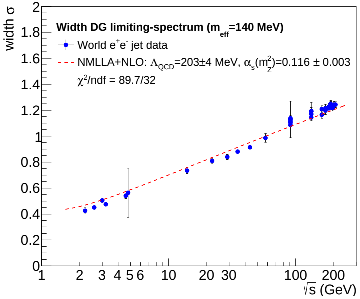
Maximum $\xi_{max}(\sqrt{s})$ for e^+e^- (left) and DIS (right)

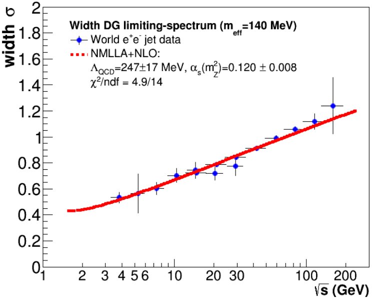
$$\xi_{\max}(Y) = 0.5Y + 0.592722\sqrt{Y} - 0.351319 + 0.002\ln Y,\; Y = ln\left(\frac{\sqrt{s}}{2\Lambda_{\rm QCD}}\right)$$



- Good data vs (NMLLA+NLO*) agreement for peak evolution ($\chi^2/\mathrm{ndf} \sim 1.3$)
- Consistent DIS e^-p and e^+e^- peak positions & extracted $\Lambda_{\rm QCD}$

$$\sigma(Y) = 0.36499\,Y^{3/4}\left[1 - 0.299739\frac{1}{\sqrt{Y}} - (1.4921 - 0.246692\ln Y)\frac{1}{Y} + \frac{1.98667}{Y^{3/2}}\right]$$

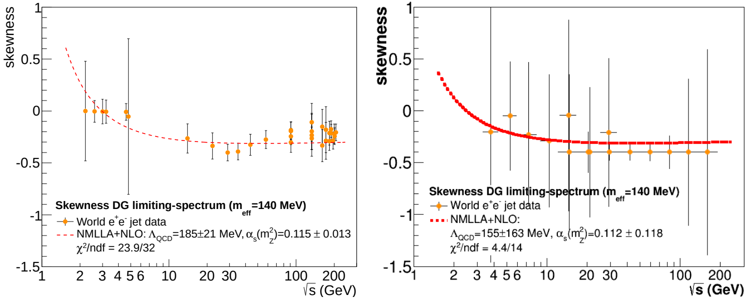




- Good data vs (NMLLA+NLO*) agreement for width evolution ($\chi^2/\mathrm{ndf} \sim$ 1.2)
- Consistent DIS e^-p and e^+e^- widths (but larger DIS uncertainties)

Skewness $s(\sqrt{s})$ for e^+e^- (left) and DIS (right)

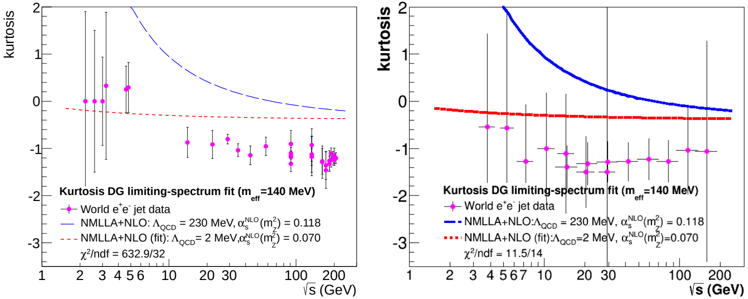
$$s(Y) = -\frac{1.94704}{Y^{3/4}}\left[1 - 0.299739\frac{1}{\sqrt{Y}} - \frac{1.64393}{Y}\right], \; Y = ln\left(\frac{\sqrt{s}}{2\Lambda_{\mathrm{QCD}}}\right)$$



- Very good data-theory agreement, though overestimated point-to-point uncertainties in skewness extraction ($\chi^2/\mathrm{ndf} \ll 1$)
- Consistent DIS e^-p & e^+e^- skewness (but larger DIS uncertainties)

Kurtosis $k(\sqrt{s})$ for e^+e^- (left) and DIS (right)

$$k(Y) = -\frac{2.15812}{\sqrt{Y}}\left[1 - 0.799305\frac{1}{\sqrt{Y}} + (0.730466 - 0.164461\ln Y)\frac{1}{Y} - \frac{8.05771}{Y^{3/2}}\right]$$



- Data show smaller kurtosis than NMLLA+NLO* prediction. Missing higher-order corrections? Mass effects more pronounced?

Extraction of $\Lambda_{\rm QCD}$ and $\alpha_s(M^2_{Z^0})$ from the low z-evolution of FFs moments in e^+e^- and DIS

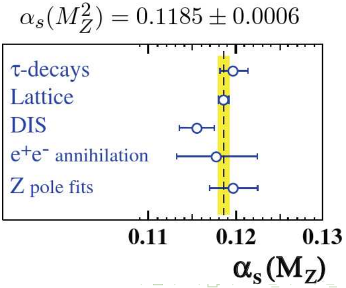
- Extracted values of $\Lambda_{\rm QCD}$ & associated $\alpha_s(\rm m^2_Z)$ at NMLLA+NLO* accuracy ($\overline{\rm MS}$ scheme, $n_f = 5$ quark flavors) combining e^+e^- and DIS data:

DG comp:	Peak position	Multiplicity	Width	Skewness	Combined
$\Lambda_{\rm QCD}$	255 ± 4	240 ± 75	212 ± 6	167 ± 48	241 ± 5
$\alpha_s(\rm m^2_Z)$	0.120 ± 0.002	0.119 ± 0.04	0.117 ± 0.004	0.113 ± 0.03	0.1193 ± 0.0018

- The last column provides the weighted-average of the individual measurements with its total propagated uncertainty.

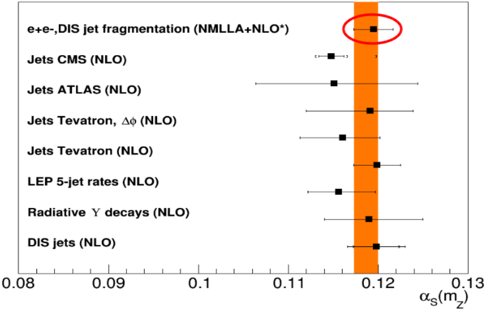
Excellent agreement with 2013 world-average: $\alpha_s(\rm m^2_Z) = 0.1185 \pm 0.0006$ (NNLO):
Peak ($\pm 1.5\%$) & width ($\pm 3.5\%$) are the most precise quantities.
Final α_s uncertainty ($\sim 1.5\%$) includes:

- Mass effects variations.
- Not (yet) scale uncertainties:
- To be determined by redoing fits varying e.g. $Q_0 = \Lambda_{\rm QCD} - 4\Lambda_{\rm QCD}$
- Non-pQCD corrections are small: method valid down to $\Lambda_{\rm QCD}$



World average: $\alpha_s(M_{Z_0}^2)$

This work provides the most precise measurement of as among those at NLO(*) accuracy (with totally different systematics):



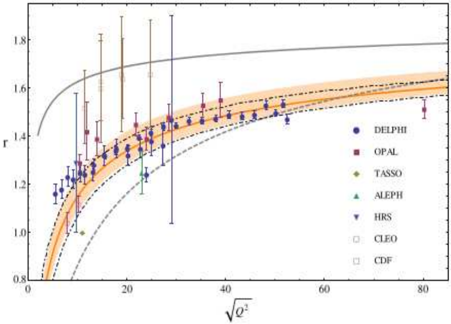
$*$ The LPHD and the "limiting spectrum" ($Q_0 = \Lambda_{\rm QCD}$) are successful on describing the parton vs hadron dynamics in the NMLLA+NLO* scheme; previously confirmed by the CDF data and also ALICE (to appear) for the inclusive k_t-distribution of intra-jet produced hadrons.

$**$ Outlook: higher-order corrections being worked out . . .

Part IV

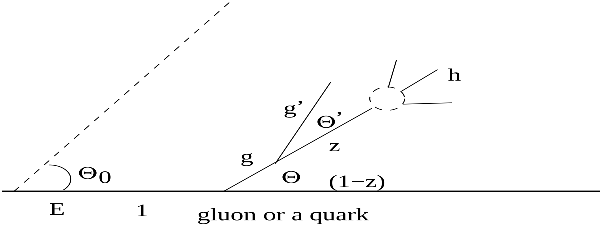
Backup slides

Gluon-to-Quark multiplicity ratio



- The average gluon-to-quark jet multiplicity ratio in the LO + NNLL (dashed/gray lines).
- The N3 LOapprox + NLO + NNLL (solid/orange lines) approximations compared with experimental data with $\alpha_s = 0.118$ for $n_f = 5$.
- The experimental and theoretical uncertainties in the N3 LOapprox + NLO + NNLL result are indicated by the shaded/orange bands.

Exact Angular Ordering



$$d^2\sigma^h_{q\to qg} = \frac{\alpha_s}{\pi} P_q^{qg}(z)dz\, V(\vec{n})\frac{d\Omega}{8\pi}, \quad V^{g'}_{g(q)}(\vec{n}) = \frac{a_{g'q} + a_{gq} - a_{g'g}}{a_{g'g}a_{g'q}}.$$

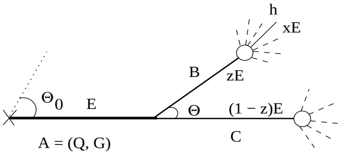
$$\left\langle V^{g'}_{g(q)} \right\rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} V^{g'}_{g(q)}(\vec{n}) = \frac{2}{a_{g'g}} \vartheta(a_{gq} - a_{g'g})$$

$$a_{gq} = 1 - \cos\Theta, \quad a_{g'g} = 1 - \cos\Theta'$$

- $\Theta' \leq \Theta$ (MLLA exact AO), $\Theta_0 \geq \Theta$ (kinematics)

MLLA Master Equation for the splitting process $A \to BC$

- From $Z \to$ MLLA Master Equation (exact Angular Ordering):



$$\frac{d}{d\ln\Theta} Z_A\left(p,\Theta;\{u\}\right) = \frac{1}{2}\sum_{B,C}\int_0^1 dz\,\Phi_A^{B[C]}(z)\,\frac{\alpha_s\left(k_\perp^2\right)}{\pi}$$
$$\Big(Z_B\big(zp,\Theta;\{u\}\big)\ Z_C\big((1-z)p,\Theta;\{u\}\big) - Z_A\big(p,\Theta;\{u\}\big)\Big)$$
$$Z \propto \exp\int^t \gamma(t')dt'; \quad \gamma \simeq 1+\sqrt{\alpha}_s+\alpha_s+\dots$$

- Exact solution of approached integro-differential equations: MLLA evolution equations at $x \ll 1$:
  - for the one-particle inclusive distributions: $D = \frac{\delta}{\delta u} Z(u)$,
  - for n-particle correlations inside jets: $D^{(n)} = \frac{\delta^n}{\delta u_1 \dots \delta u_n} Z(u)$.

Low-Barnett-Kroll wisdom

- $d\sigma^{(1)}(p_i,\omega) \propto \frac{\alpha}{\pi}\frac{d\omega}{\omega}\left[\left(1-\frac{\omega}{E}\right)\sigma^{(0)}(p_i)+\left(\frac{\omega}{E}\right)^2\tilde{\sigma}(p_i,\omega)\right]$

- 1st term is $\propto$ to the non-radiative cross-section $\sigma^{(0)}$, proves to be of classical origin.

- 2nd term involves ω-dependence cross-section $\tilde{\sigma}$, finite at $\omega = 0$ such that this contribution is suppressed for small photon energies as $\left(\frac{\omega}{E}\right)^2$.