Non-equilibrium dynamics of isolated quantum systems

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Based on joint works with:

John Cardy, Mario Collura, Fabian Essler, Maurizio Fagotti, Marton Kormos, Spyros Sotiriadis

Goal of the talk

Show that an isolated many-body quantum system prepared in a state $|\Psi_0\rangle$

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle$$

can reach for $t=\infty$, in some sense, a stationary state

The steady state can be thermal or not

von Neumann in 1929 studied this problem **1003.2133**

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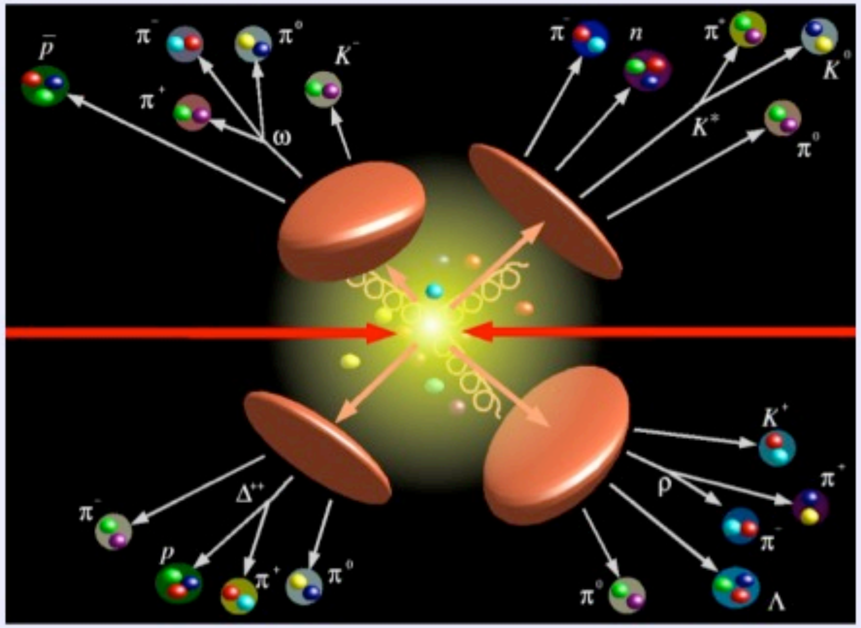
It stayed a purely academic question for many years, but it recently became a top question in many branches of physics:

- Cold atoms
- Condensed matter
- Cosmology
- Nuclear physics

The paradox of statistical mechanics in high energy collisions



$$\widehat{\rho} = |\psi\rangle\langle\psi| \longrightarrow U(t) \longrightarrow \widehat{\rho}' = |\psi'\rangle\langle\psi'|$$

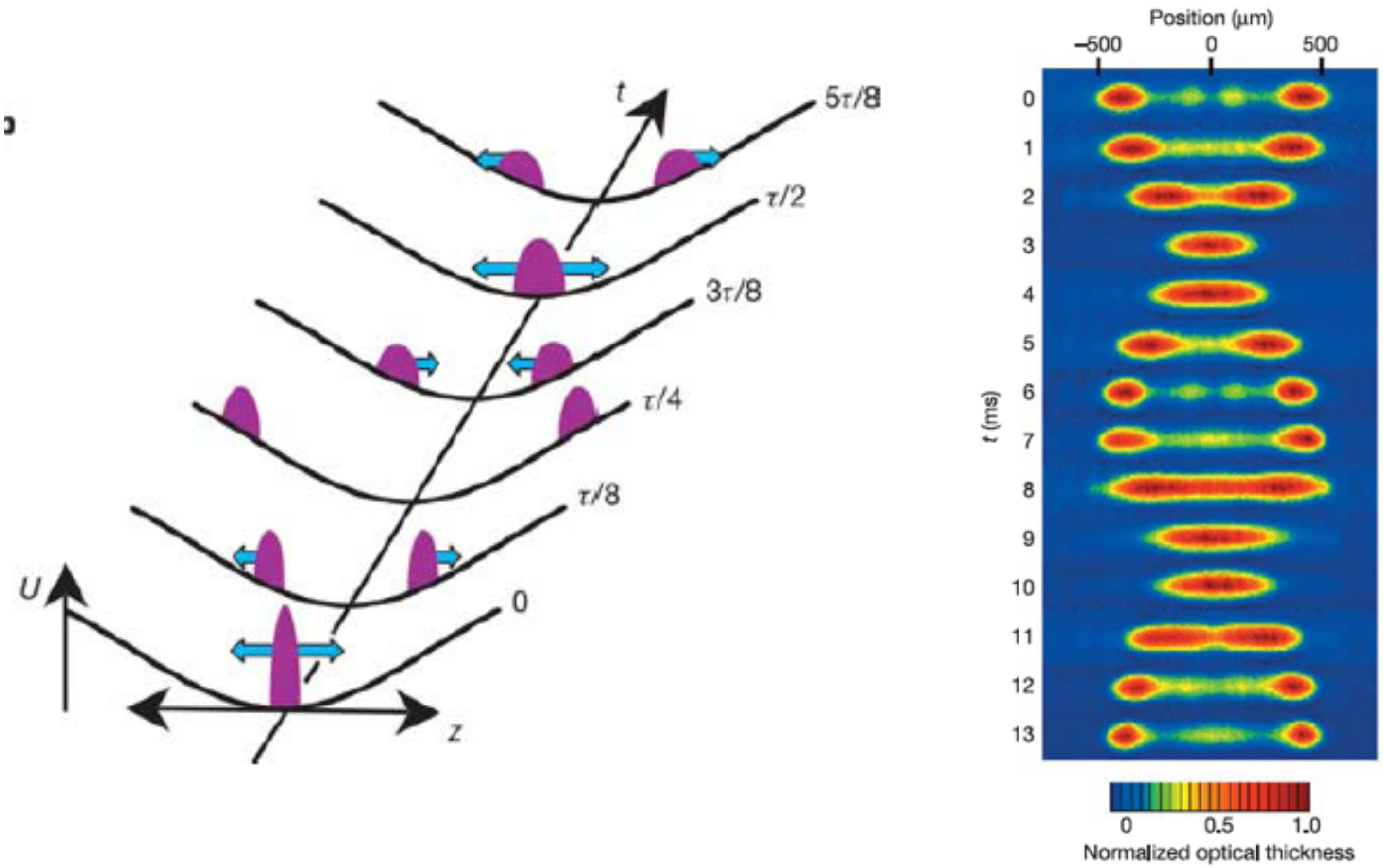
A unitary evolution cannot generate a mixed state from a pure state

$$\widehat{\rho} \propto \delta^4(\widehat{P} - P_0) \qquad\qquad \widehat{\rho} = \exp[-\widehat{\beta} \cdot \widehat{P}]$$

Quantum Newton cradle

T. Kinoshita, T. Wenger and D.S. Weiss, Nature 440, 900 (2006)

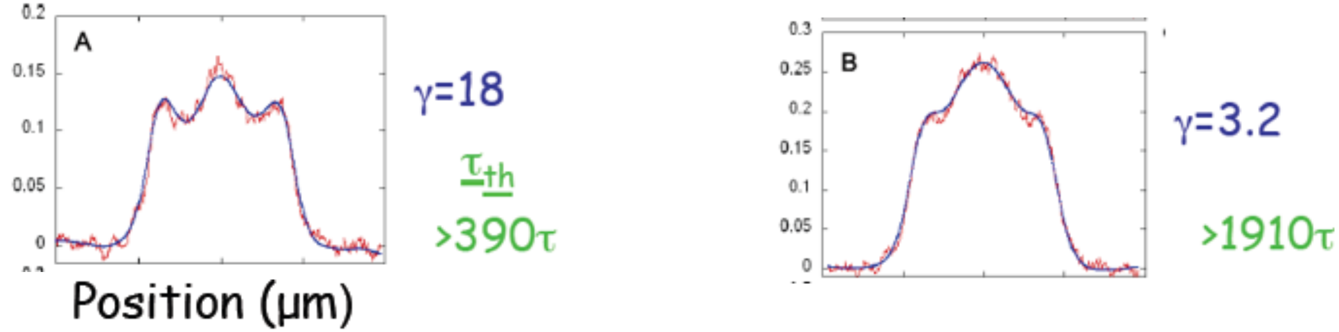
few hundreds ^{87}Rb atoms in a 1D trap



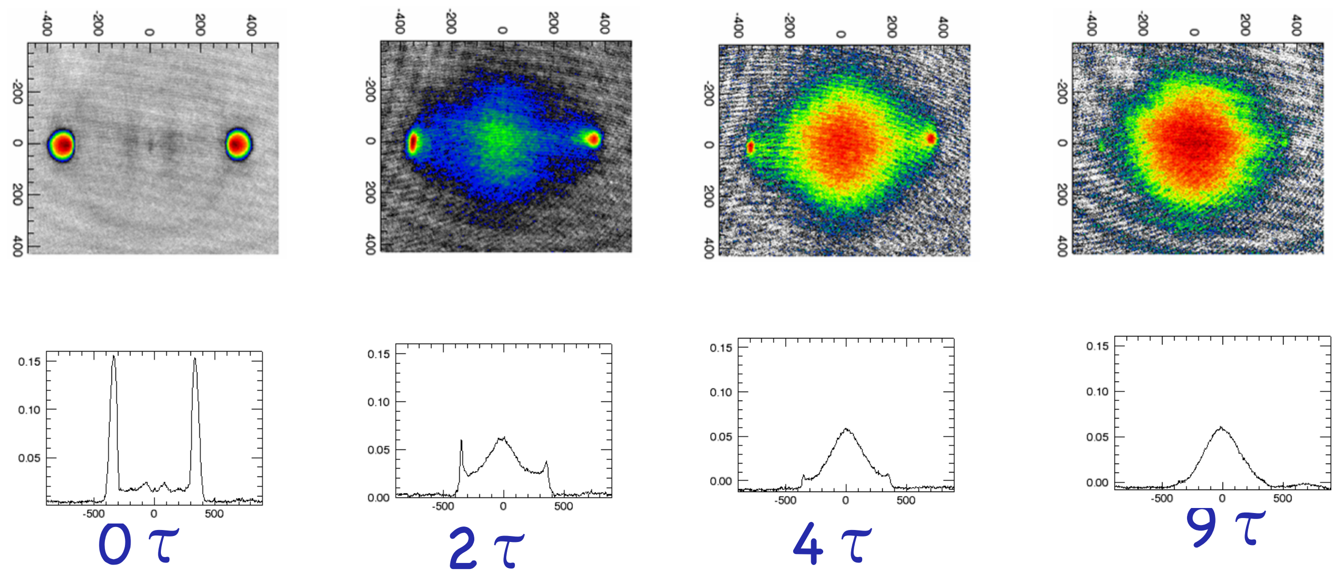
Essentially unitary time evolution

Can a steady state be attained? Surprisingly, YES

- 1D system relaxes slowly in time, to a non-thermal distribution



- 2D and 3D systems relax quickly and thermalize:



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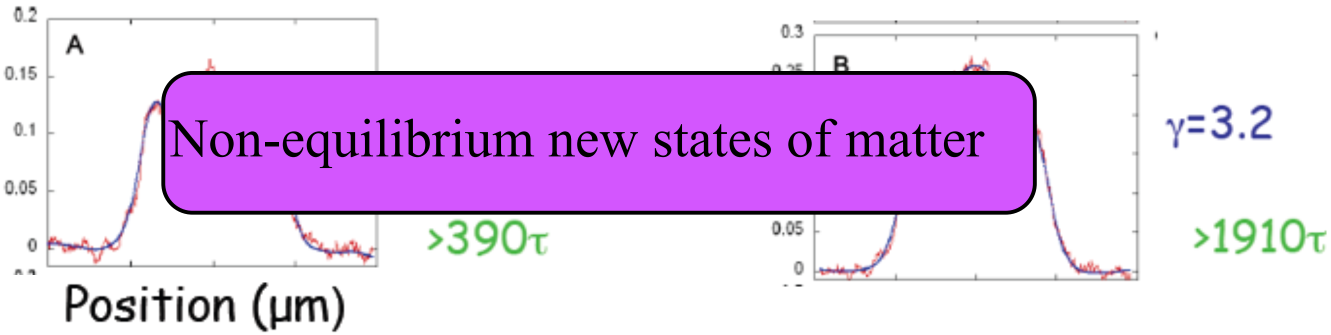

Non-equilibrium new states of matter

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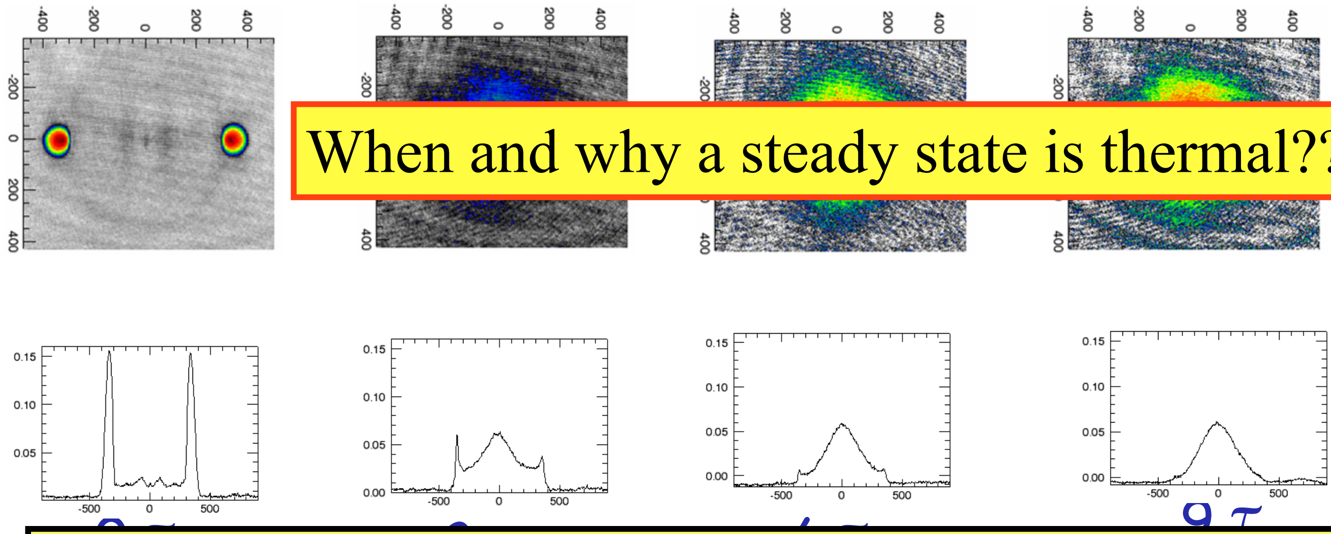
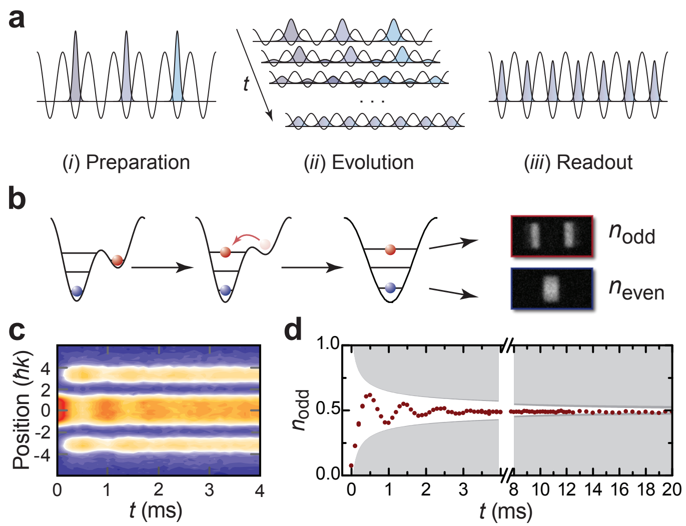


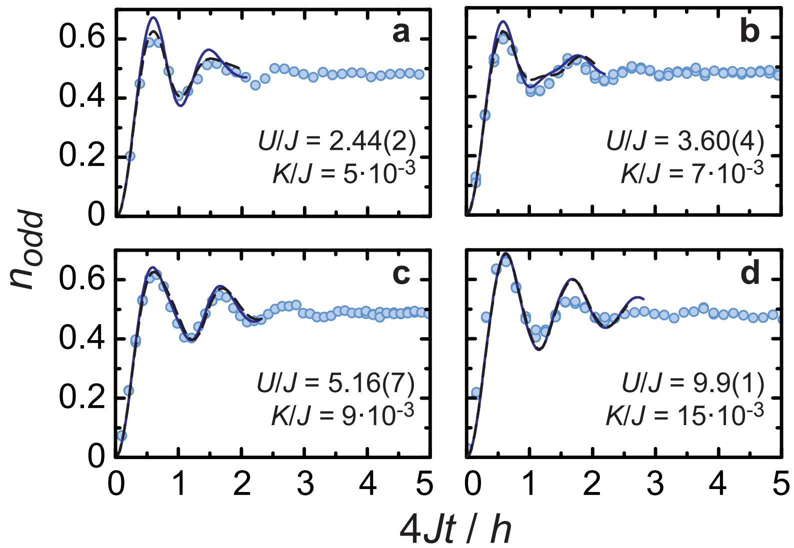
When and why a steady state is thermal??

The 1D case is special because the system is almost integrable

Probing relaxation

S Trotzky et al, Nature Phys. 8, 325 (2012)





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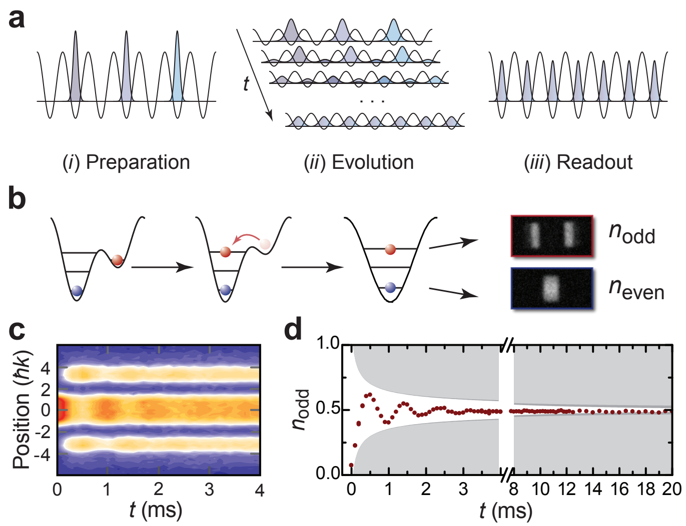

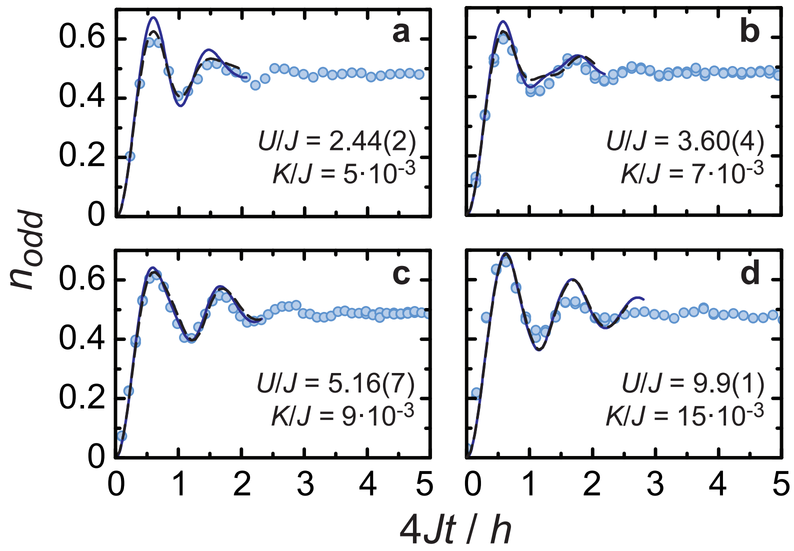

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COMMON BELIEF: - Generic systems “thermalizes”

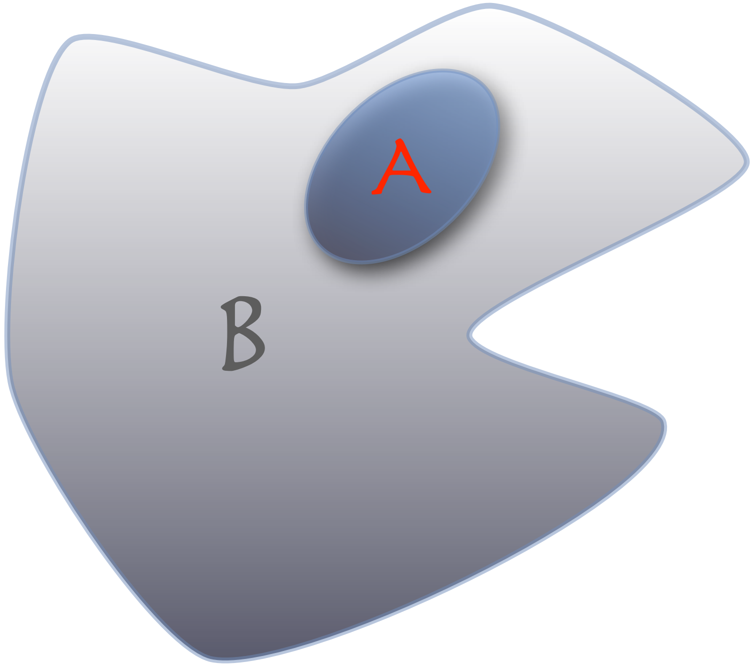
- Integrable systems are different

Deutsch ’91, Srednicki ’95

Rigol et al ’07

But the system is always in a pure state!

Reduced density matrix



$|\Psi(t)\rangle$ time dependent pure state

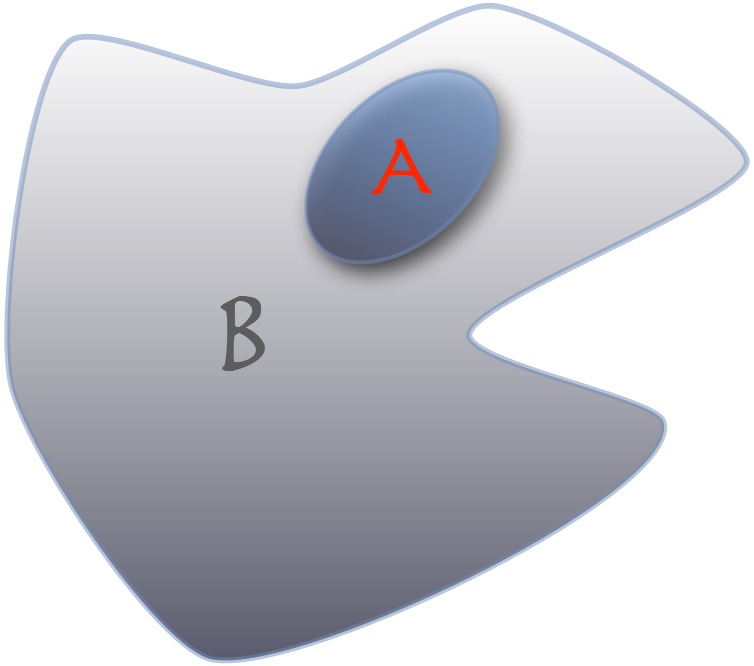
$\rho(\mathrm{t}) = |\Psi(t)\rangle\langle\Psi(t)|$ density matrix of A∪B

Reduced density matrix: $\rho_A(\mathrm{t}) = \mathrm{Tr}_B\, \rho(t)$

The expectation values of all local observables in A are

$$\langle\Psi(\mathrm{t})|O_A(\mathrm{x})\,|\Psi(\mathrm{t})\rangle = \mathrm{Tr}[\rho_A(\mathrm{t})\, O_A(\mathrm{x})]$$

Reduced density matrix



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The expectation values of all local observables in A are

$$\langle\Psi(t)|O_A(x)\,|\Psi(t)\rangle = \mathrm{Tr}[\rho_A(t)\,O_A(x)]$$

Stationary state: if for any finite subsystem A of an infinite system, it exists the limit

$$\lim_{t\to\infty} \rho_A(t) = \rho_A(\infty)$$

Note: in a finite system, the stationary state is the regime $|A| \ll vt \ll L$

Thermalization

Consider the Gibbs ensemble for the entire system A∪B

$$\rho_T = e^{-H/T_{eff}}/Z \qquad \text{with} \qquad \langle\Psi_0|\, H\, |\Psi_0\rangle = \mathrm{Tr}[\rho_T\, H]$$

T_{eff} is fixed by the energy in the initial state: no free parameter!!

Reduced density matrix for subsystem A: $\rho_{A,T} = \mathrm{Tr}_B\, \rho_T$

The system thermalizes if for any finite subsystem A

$$\rho_{A,T} = \rho_A(\infty)$$

The infinite part B of the system "acts as an heat bath for A"

Generalized Gibbs Ensemble

What about integrable systems?

Proposal by Rigol et al 2007: The GGE density matrix

$$\rho_{\mathrm{GGE}} = e^{-\sum \lambda_m I_m} / Z$$

with λ_m fixed by $\langle \Psi_0 | I_m | \Psi_0 \rangle = \mathrm{Tr}[\rho_{\mathrm{GGE}} I_m]$

Again no free parameter!!

I_m are the integrals of motion of H, *i.e.* $[I_m, H] = 0$

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Reduced density matrix for subsystem A: $\rho_{\mathrm{A,GGE}} = \mathrm{Tr}_{\mathrm{B}}\, \rho_{\mathrm{GGE}}$

The system is described by GGE if for any finite subsystem A of an infinite system

$$\rho_{\mathrm{A,GGE}} = \rho_{\mathrm{A}}(\infty)$$

[Barthel-Schollwock '08]
[Cramer, Eisert, et al '08] + ...
[PC, Essler, Fagotti '12]

Generalized Gibbs Ensemble

Which integral of motions must be included in the GGE?

Any quantum system has too many integrals of motion, regardless of integrability, e.g.

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New proposal: [PC, Essler, Fagotti '12]

$$\rho_{GGE} = e^{-\sum \lambda_m I_m} / Z$$

where I_m is a complete set of local (in space) integrals of motion

$$[I_m, I_n]=0 \quad [I_m, H]=0 \qquad I_m = \sum_x O_m(x)$$

In this case B is not a standard heat bath for A:
infinite information on the initial state is retained!

Quantum mechanics exercise

Quenching the frequency in one harmonic oscillator

$$H_0 = \frac{p^2}{2} + \frac{\omega_0^2}{2}x^2 \qquad H = \frac{p^2}{2} + \frac{\omega^2}{2}x^2 \qquad H_0 \to H$$

Solving Heisenberg equation of motion

$$\langle x^2(t)\rangle = \frac{\omega^2 + \omega_0^2}{4\omega_0\omega^2} + \frac{\omega^2 - \omega_0^2}{4\omega_0\omega^2}\cos 2\omega t$$

Not surprisingly, the harmonic oscillator oscillates

Many oscillators

$$H(m) = \frac{1}{2}\sum_{n=0}^{N-1}\left[\frac{1}{a}\pi_n^2 + am^2\varphi_n^2 + \frac{1}{a}(\varphi_{n+1}-\varphi_n)^2\right] \qquad m_0 \longrightarrow m$$

Each momentum mode is a free oscillator

$$\Omega_k^2 = m^2 + \frac{2}{a^2}\left(1-\cos\frac{2\pi k}{N}\right)$$

$$\langle\phi_r(t)\phi_0(t)\rangle - \langle\phi_r(0)\phi_0(0)\rangle = \int_{\mathrm{BZ}} e^{ikr}\frac{(\Omega_{0k}^2-\Omega_k^2)(1-\cos(2\Omega_k t))}{\Omega_k^2\Omega_{0k}}dk \quad \xrightarrow[t\to\infty]{} \int_{\mathrm{BZ}} e^{ikr}\frac{(\Omega_{0k}^2-\Omega_k^2)}{\Omega_k^2\Omega_{0k}}dk$$

This compatible with the GGE

$$\rho_{\mathrm{GGE}} = \frac{e^{-\sum_k \lambda_k n_k}}{Z} \qquad n_k = a_k^\dagger a_k \qquad \lambda_k = \ln\left(1+\frac{4\Omega_k\Omega_{0k}}{(\Omega_k-\Omega_{0k})^2}\right)$$

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Non local...
but linear combinations of local charges I_m

$$\Rightarrow \sum_k \lambda_k n_k = \sum_m \gamma_m I_m \Rightarrow$$

The GGE built with n_k and with I_m with are equivalent!

Quantum quenches in free theories

- Mass quenches in (lattice) field theories

  PC-Cardy '07, Barthel-Schollwock '08, Cramer, Eisert, et al '08, Sotiriadis et al '09.....

- Luttinger model quartic term quench

  Cazalilla '06, Cazalilla-Iucci '09, Mitra-Giamarchi '10....

- Transverse field quench in Ising/XY model

  Barouch-McCoy '70, Igloi-Rieger '00-13, Sengupta et al '04, Rossini et al. '10, PC, Essler, Fagotti '11-13
  Foini-Gambassi-Cugliandolo'12, Bucciantini, Kormos, PC '14........

- Quench to the Tonks-Girardeau model

  Rostunov, Gritsev, Demler '10, Collura, Sotiriadis, PC '13, Kormos, Collura, PC '14............

- Few more.....

The GGE always turned out to work

How generic this is?

The importance of the initial state

If we take a linear superposition of a finite number of eigenstates, the system will obviously oscillate forever

Can we find some conditions for the initial state/Hamiltonian guaranteeing steady state and GGE/thermalization?

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A simple general condition

Sotiriadis, PC 2014

For a free theory, the steady state is described by the GGE if the initial state satisfy the cluster decomposition property

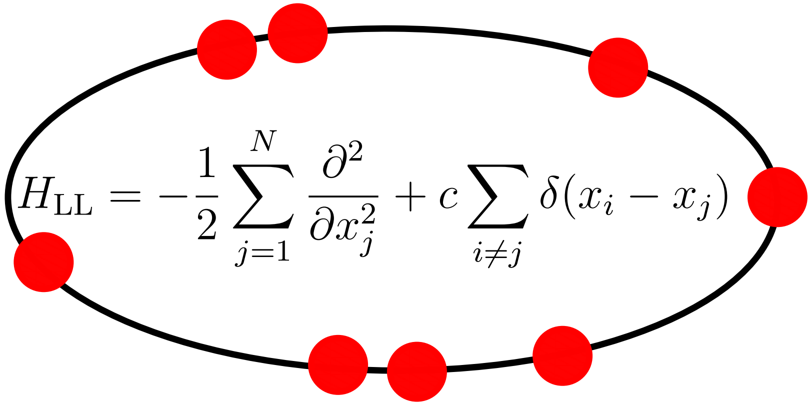
$$\lim_{R\to\infty}\left\langle \prod_i \phi(x_i) \prod_j \phi(x_j+R)\right\rangle = \left\langle \prod_i \phi(x_i)\right\rangle \left\langle \prod_j \phi(x_j)\right\rangle.$$

see also Cramer & Eisert 2011

What about interacting integrable systems?

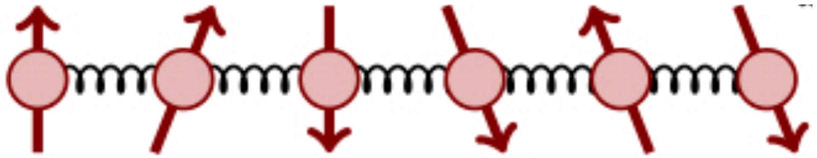
Two paradigmatic models:

Lieb-Liniger gas



$$H_{\mathrm{LL}} = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + c\sum_{i\neq j}\delta(x_i - x_j)$$

XXZ Spin chain



$$H = J\sum_{i=1}^{L}\left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right]$$

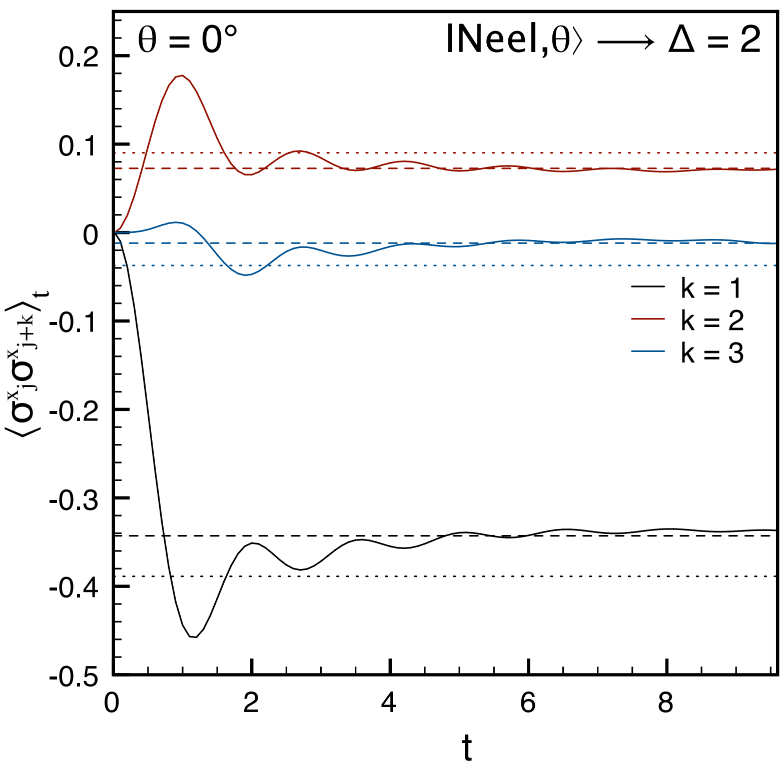
The calculations become immensely more complicated

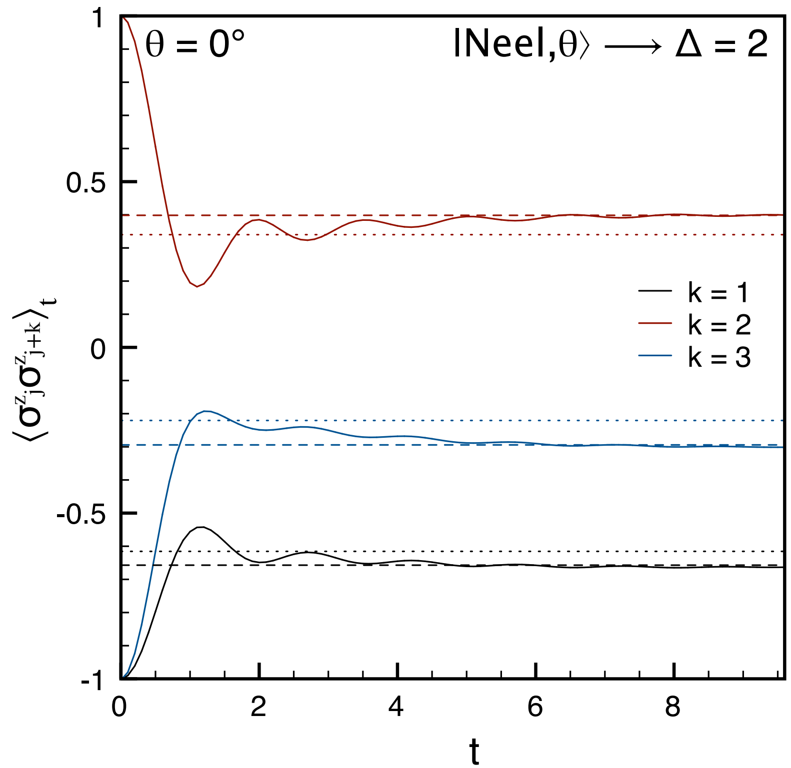
Quench dynamics of XXZ chain

We developed a method to calculate expectation values in the GGE

Fagotti, Collura, Essler, PC 2013

Analytics vs Numerics for the Neel →Δ quench:





Similar agreement with other initial states and final H

This is not the end :(

A new method to compute the exact time evolution developed

Essler, Caux '13

Particularly effective to compute the long-time limit

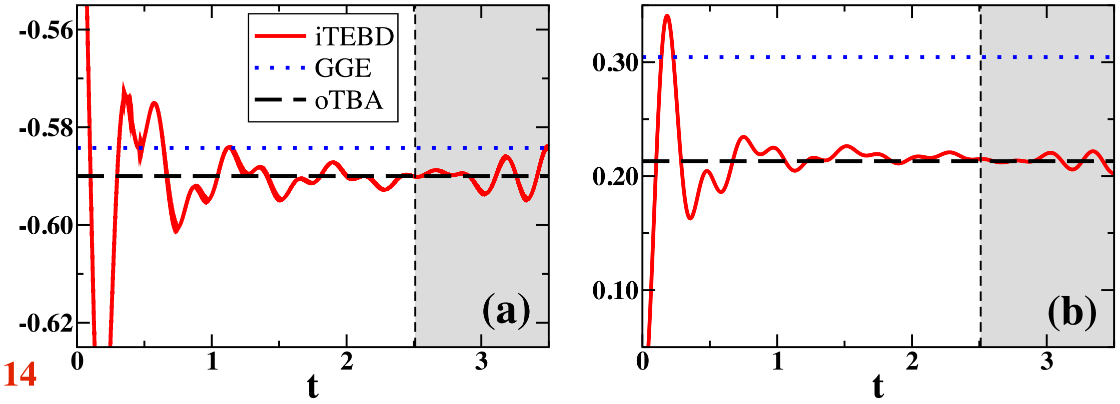
Applied to XXZ chain for the Neel quench:

Brockmann, Wouters, Fioretto, De Nardis, Vlijm, Caux '14

$$\langle \sigma_1^z \sigma_2^z \rangle_{\text{sp}} = -1 + \frac{2}{\Delta^2} - \frac{7}{2\Delta^4} + \frac{77}{16\Delta^6} -$$

$$\langle \sigma_1^z \sigma_2^z \rangle_{\text{GGE}} = -1 + \frac{2}{\Delta^2} - \frac{7}{2\Delta^4} + \frac{43}{8\Delta^6}$$

The difference is more evident starting from the dimer state



Pozsgay, Mestyan, Werner, Kormos, Zarand, Takacs '14

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The GGE does not work?? :(
more work to be done!

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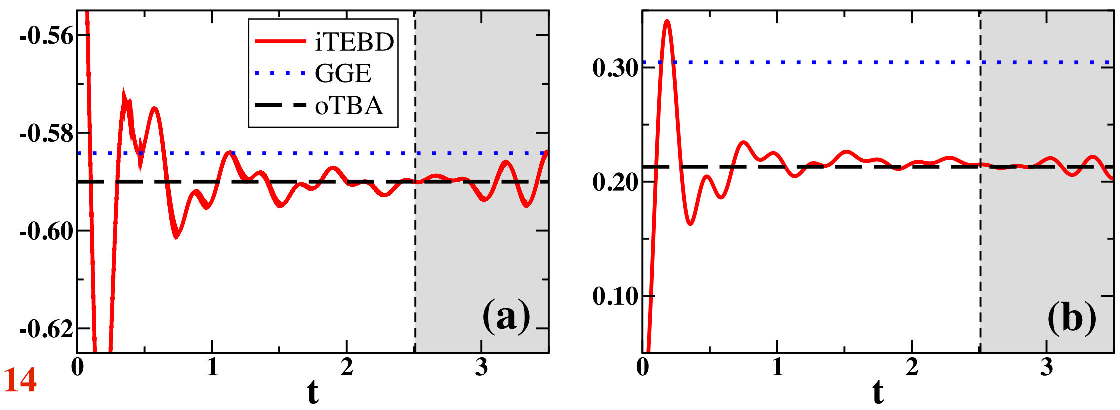

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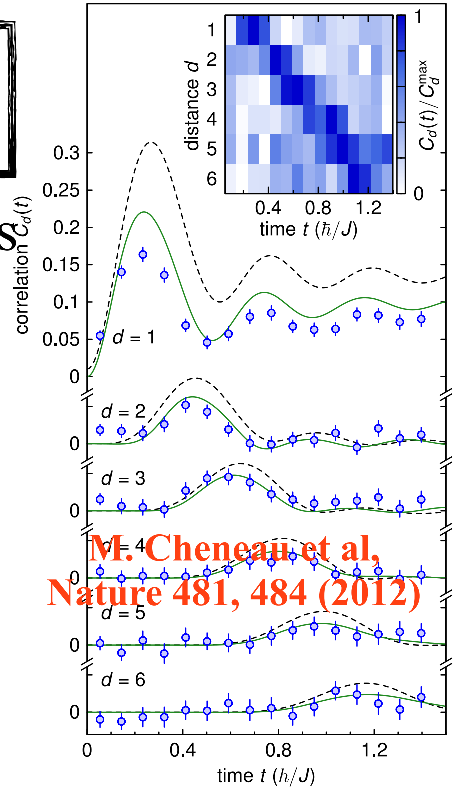
What is missing here

The approach to the steady state shows very interesting features
(light-cone spreading of correlations)

PC, J Cardy 2006/07



M. Cheneau et al, Nature 481, 484 (2012)

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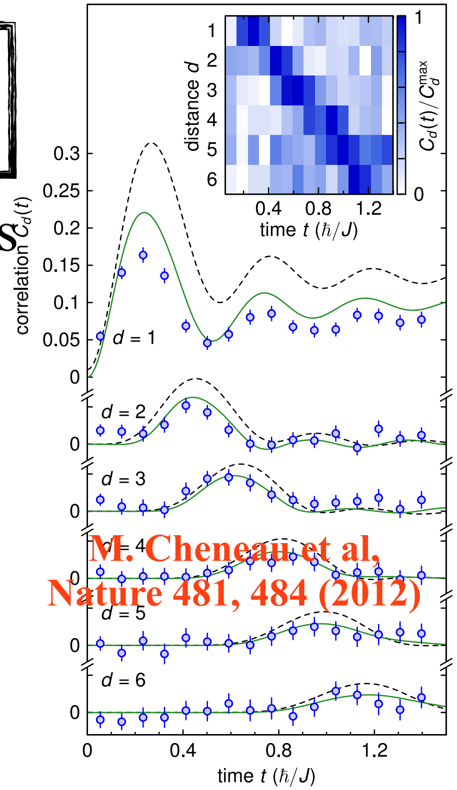

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Conclusions

Non-equilibrium dynamics of isolated systems represent a theoretical and experimental challenge raising many fundamental questions in many-body quantum mechanics

Thank you for your attention