Averaging, passages through resonances, and captures into resonance in dynamics of charged particles

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Based on joint works with:

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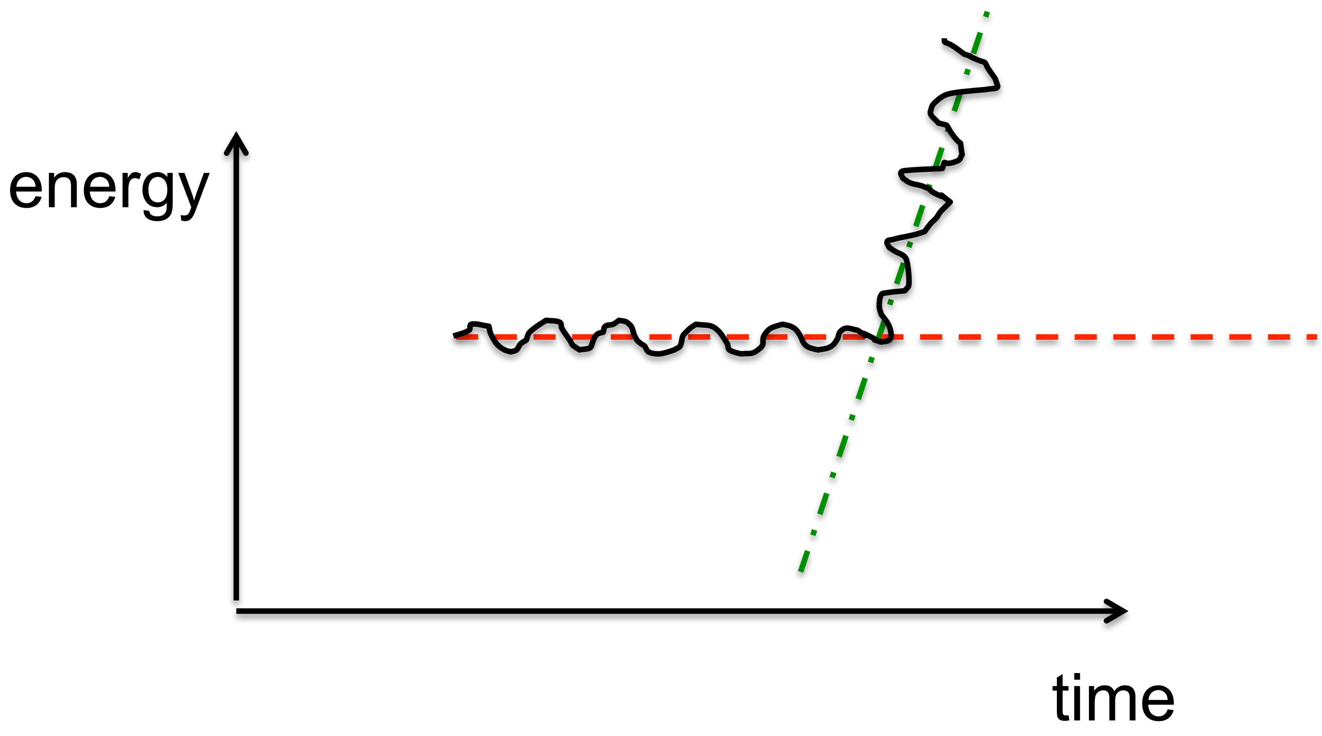
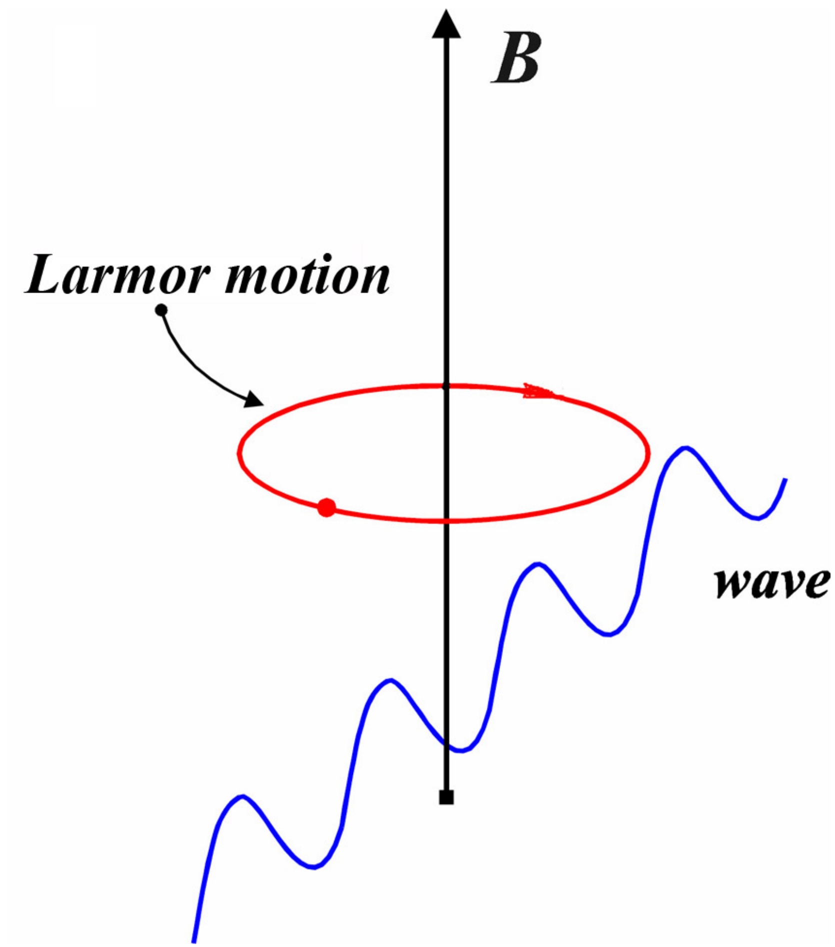
L.Zelenyi

This is how a capture into resonance looks like:



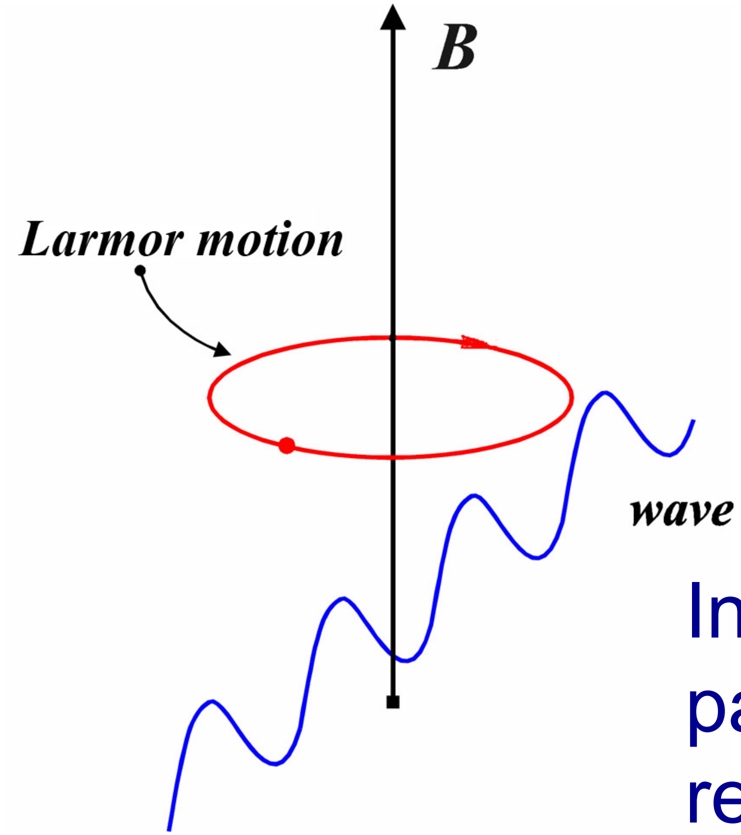
We consider motion of a charged particle in a constant background magnetic field and a wave exited in a plasma



This is a natural problem for application of the averaging method:

- for strong magnetic field: averaging over Larmor rotation,

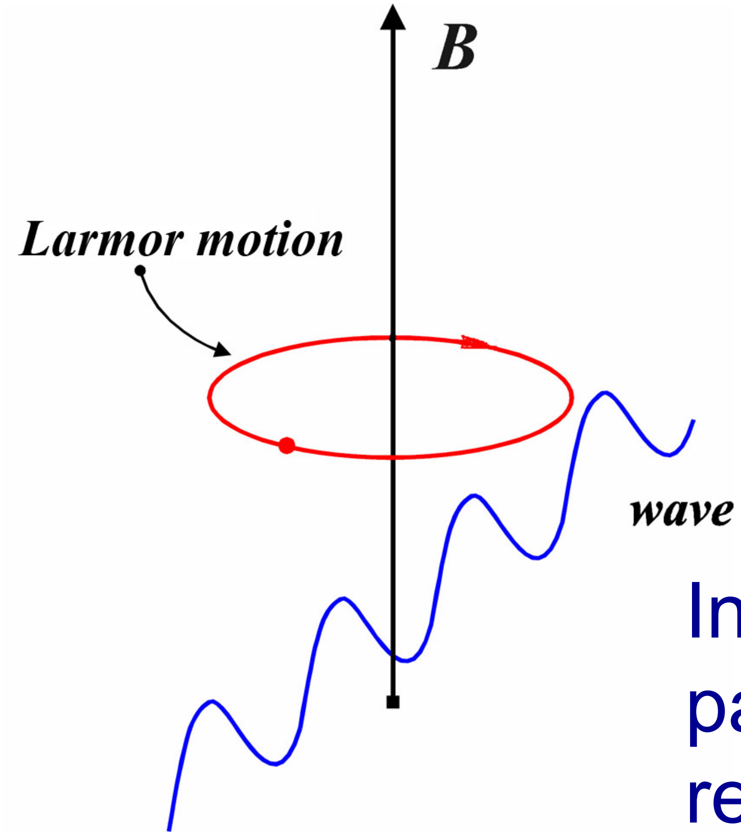
- for high frequency wave: averaging over the phase of the wave – this is the case which we consider.



Averaging over the phase of the wave washes out the effect of the wave. The averaged motion is just a Larmor motion.

In the process of Larmor motion the particle may approach Cherenkov's resonance with the wave:

projection of the particle's velocity onto direction of the wave propagation is equal to the phase velocity of the wave.



Averaging over the phase of the wave washes out the effect of the wave. The averaged motion is just a Larmor motion.

In the process of Larmor motion the particle may approach Cherenkov's resonance with the wave:

projection of the particle's velocity onto direction of the wave propagation is equal to the phase velocity of the wave.

This is a particular case of a general problem of passage through resonances in slow-fast Hamiltonian systems.

Equation of motion

$$\frac{d}{dt}(m\vec{V}) = \frac{e}{c}\vec{V} \times (\vec{B}_0 + \vec{B}) + e\vec{E}$$

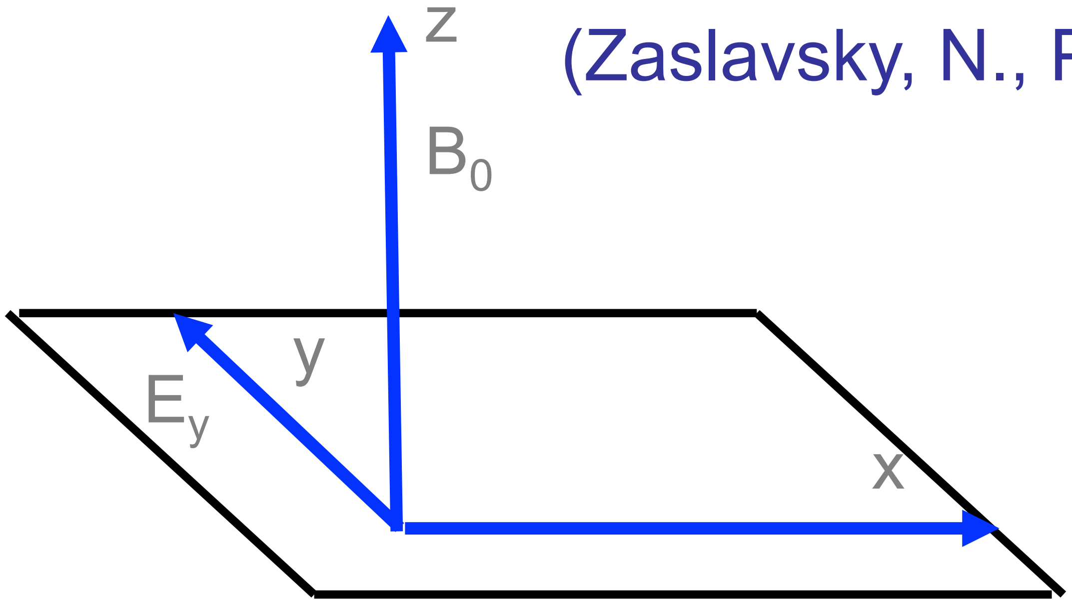
$\vec{B}_0$ - constant background magnetic field

$\vec{E}$ - electric field of the wave

$\vec{B}$ - magnetic field of the wave

1. Electrostatic wave perpendicular to constant magnetic field,   non-relativistic particle

(Zaslavsky, N., Petrovichev, Sagdeev, 1989)



The wave propagates along y-axis

B_0 - the uniform magnetic field
E_y - the electric field of the wave

$$E_y = E_0 \sin(ky - \omega t).$$

$$\dot{v}_x = \frac{eB_0}{mc} v_y,$$

$$\dot{v}_y = -\frac{eB_0}{mc} v_x + \frac{e}{m} E_y$$

$$\ddot{y} = -\left(\frac{eB_0}{mc}\right)^2 y + \frac{eE_0}{m} \sin(ky - \omega t)$$

Slow-fast form of equations

$\phi = ky - \omega t$ - the phase of the wave,

$v_\phi = \omega / k$ - the phase velocity of the wave,

$v_x = \dot{x}, v_y = \dot{y}, b_n = eB_0/(mc), \delta = eE_0/m$

Assume that

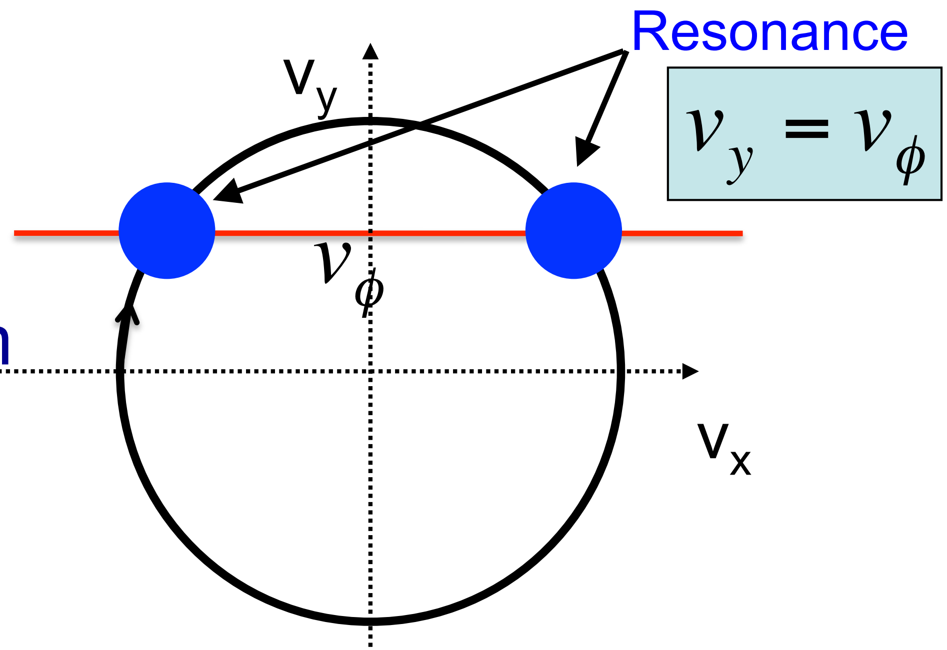
$v_\phi, v_x, v_y, b_n, \delta \propto 1,$

$\omega >> 1, k >> 1.$

$$\begin{cases} \dot{v}_x = b_n v_y, \\ \dot{v}_y = -b_n v_x + \delta \sin(\phi) \\ \dot{\phi} = k v_y - \omega \end{cases}$$

v_x, v_y - slow variables,

ϕ - a fast variable.



The averaged over ϕ system

$$\dot{v}_x = b_n v_y, \dot{v}_y = -b_n v_x.$$

describes Larmor rotation.

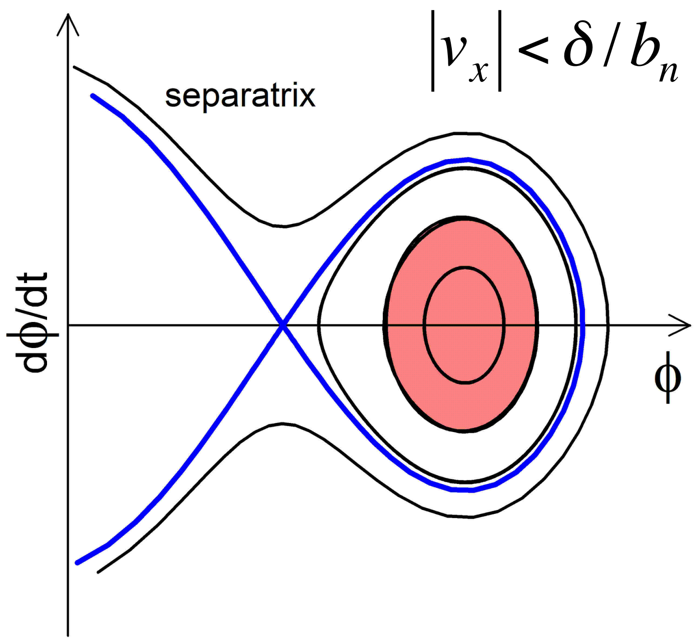
Near the resonance in the principal approximation

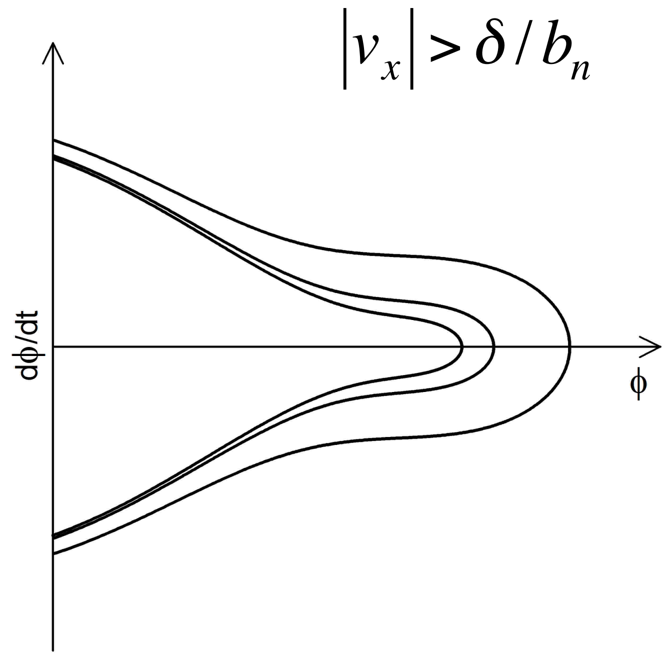
$$\ddot{\phi} = k\delta\sin(\phi) - kb_n v_x, \quad \dot{v}_x = b_n v_\phi$$

$$k >> 1$$

In this system v_x is a slow variable, and $\phi, \dot{\phi}$ are fast variables. This is the equation of a pendulum with an external torque and slowly varying parameter.

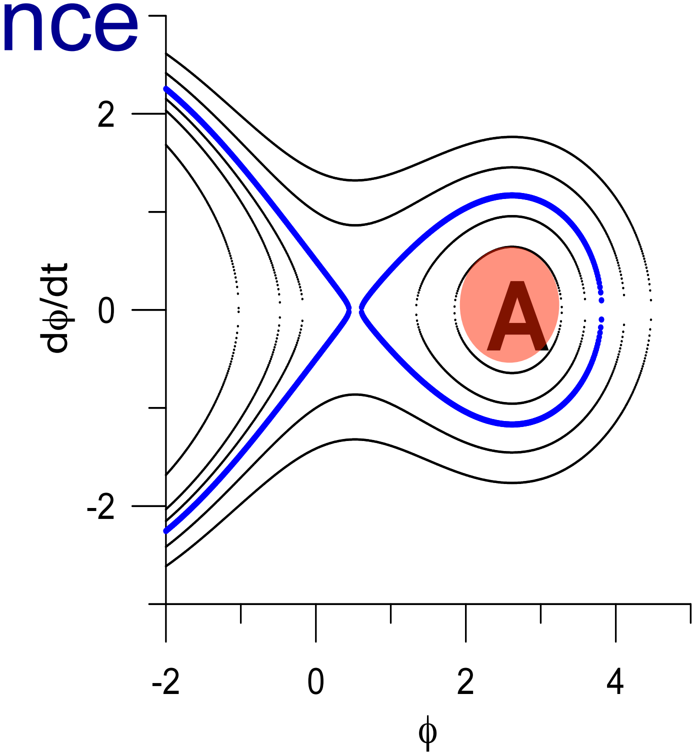
Two types of phase portraits for frozen v_x :



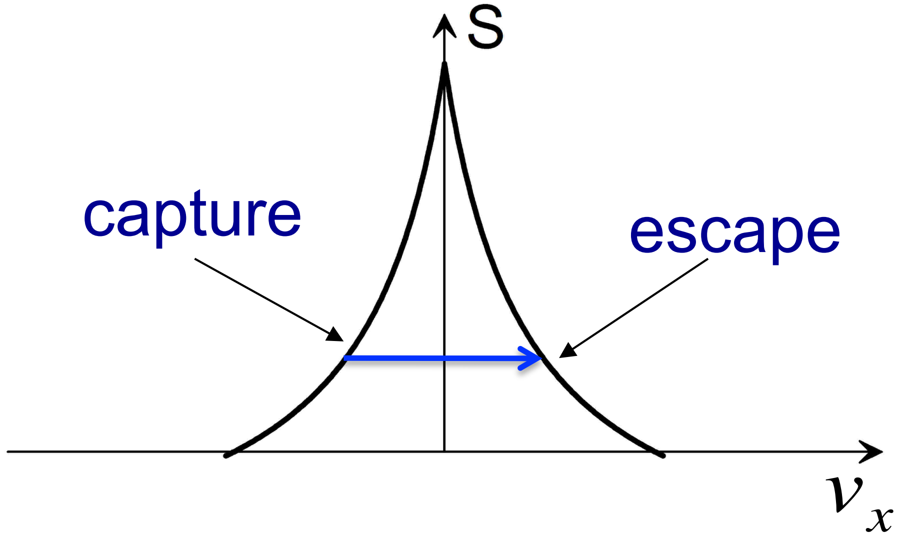


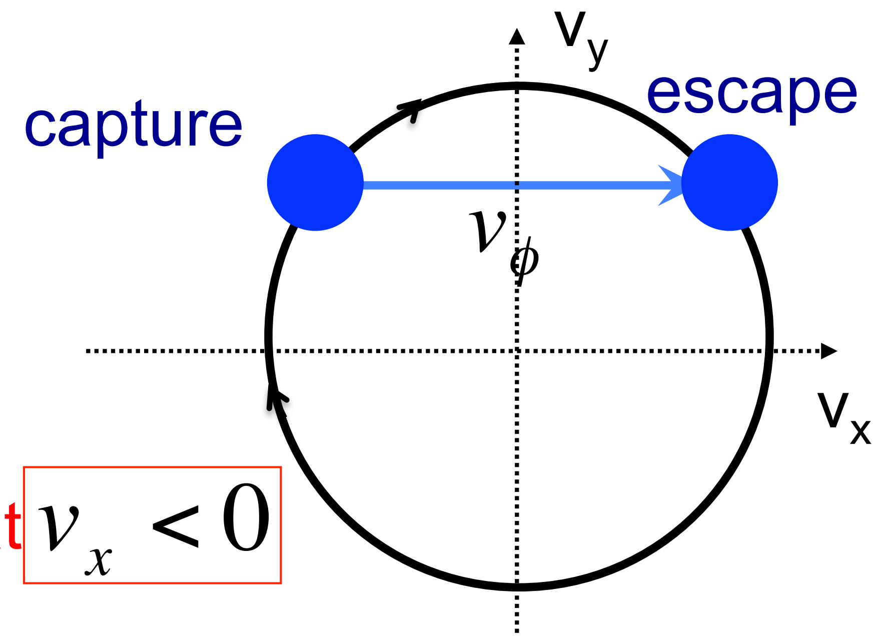
Capture into resonance

The area **A** surrounded by the trajectory is an adiabatic invariant: its value is approximately conserved in the evolution.



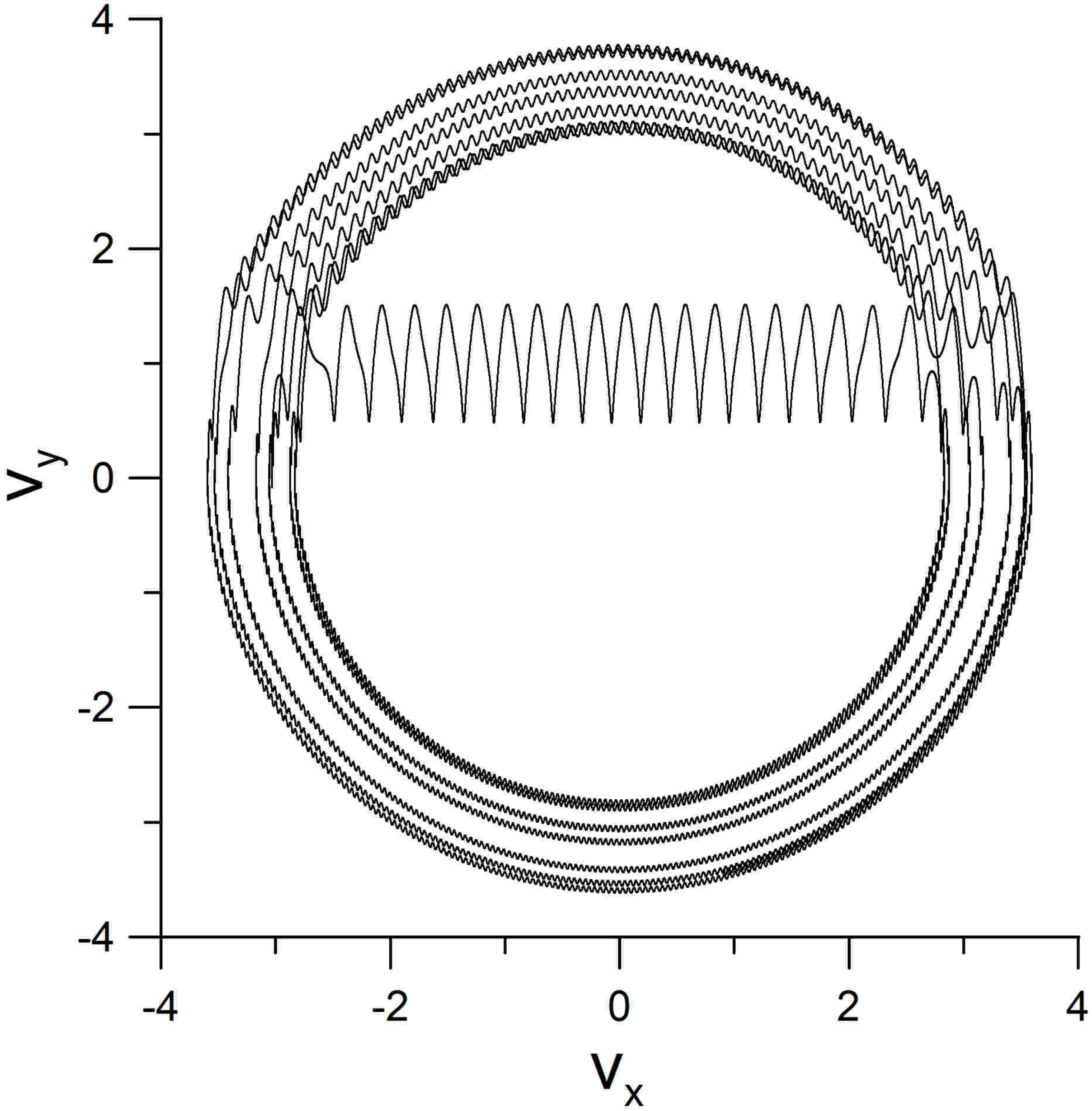
The graph of the area of the oscillatory domain:



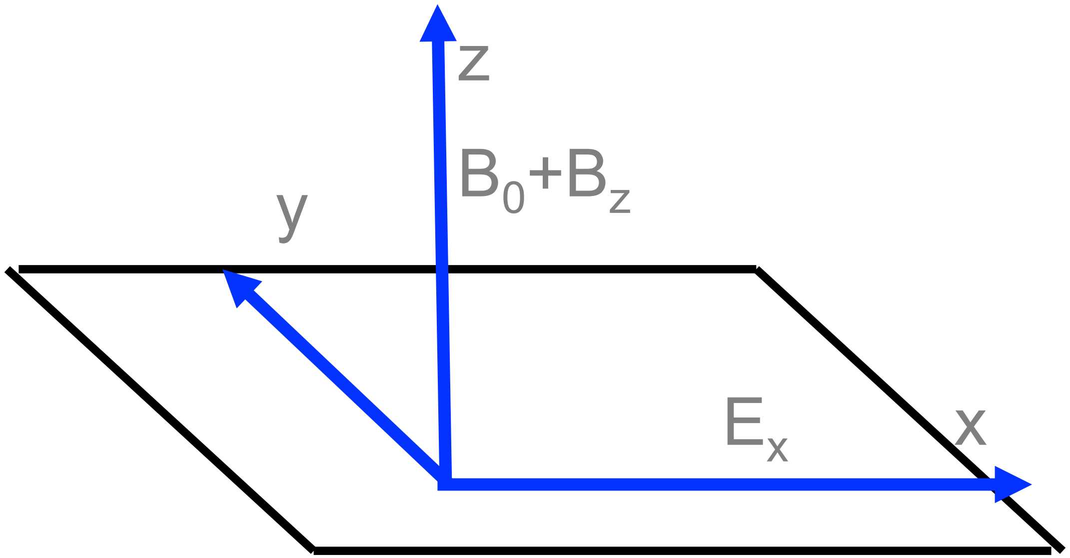


Capture into resonance is possible at $v_x < 0$

Capture into resonance and escape from the resonance



2. Electromagnetic wave perpendicular to constant magnetic field, non-relativistic particle (N., Artemyev, Zelenyi, Vainchtein, 2009)



The wave propagates along y-axis

B_0 - the uniform magnetic field
B_z - the magnetic field of the wave
E_x - the electric field of the wave

$$\dot{v}_x = \frac{e}{mc}(B_0 + B_z)v_y + \frac{e}{m}E_x,$$

$$\dot{v}_y = -\frac{e}{mc}(B_0 + B_z)v_x$$

$$B_z = -\tilde{B}\sin(ky - \omega t),$$

$$E_x = \frac{\tilde{B}\omega}{kc}\sin(ky - \omega t).$$

Slow-fast form of equations

$\phi = ky - \omega t$ - the phase of the wave,

$v_\phi = \omega / k$ - the phase velocity of the wave,

$v_x = \dot{x},\ v_y = \dot{y},\ b_n = eB_0/(mc),\ \delta = e\tilde{B}/(mc)$

Assume that

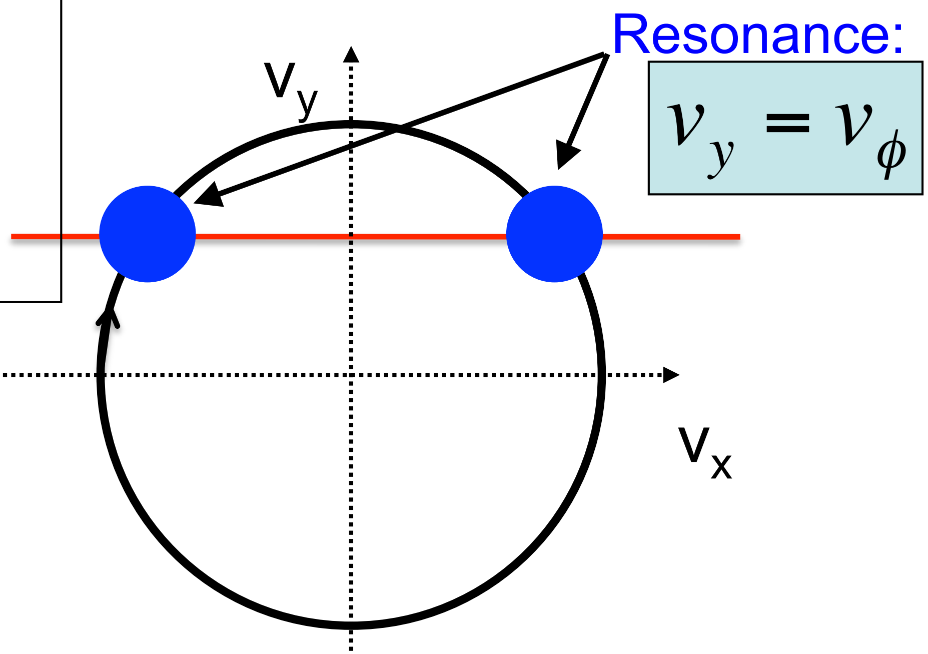
$v_\phi, v_x, v_y, b_n, \delta \propto 1,$

$k >> 1,\ \omega >> 1$

$$\begin{cases} \dot{v}_x = (b_n - \delta \sin(\phi))v_y + \delta v_\phi \sin(\phi), \\ \dot{v}_y = -(b_n - \delta \sin(\phi))v_x, \\ \dot{\phi} = kv_y - \omega \end{cases}$$

v_x, v_y - slow variables,

ϕ - a fast variable.



The averaged over ϕ system

$$\dot{v}_x = b_n v_y, \dot{v}_y = -b_n v_x$$

describes Larmor rotation.

Reduction near the resonance

$$\begin{cases} \dot{v}_x = (b_n - \delta\sin(\phi))(\dot{\phi}/k + v_\phi) + \delta v_\phi \sin(\phi), \\ \ddot{\phi} = -k(b_n - \delta\sin(\phi))v_x \end{cases}$$

Near the resonance $\dot{\phi} \approx 0$. Thus in the principal approximation

$$\begin{cases} \dot{v}_x = b_n v_\phi, \\ \ddot{\phi} = -k(b_n - \delta\sin(\phi))v_x \end{cases}$$

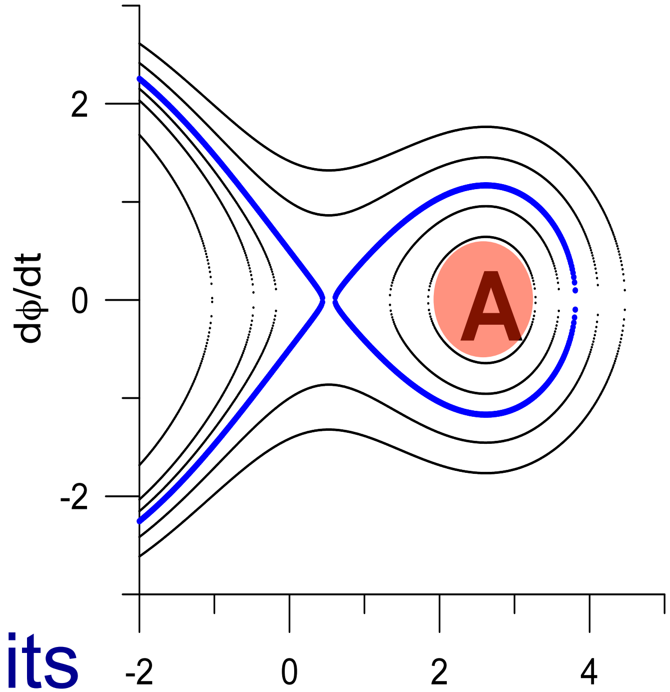
$$k >> 1$$

In this system v_x is a slow variable, and $\phi, \dot{\phi}$ are fast variables. This is the equation of a pendulum with an external torque and slowly varying parameter.

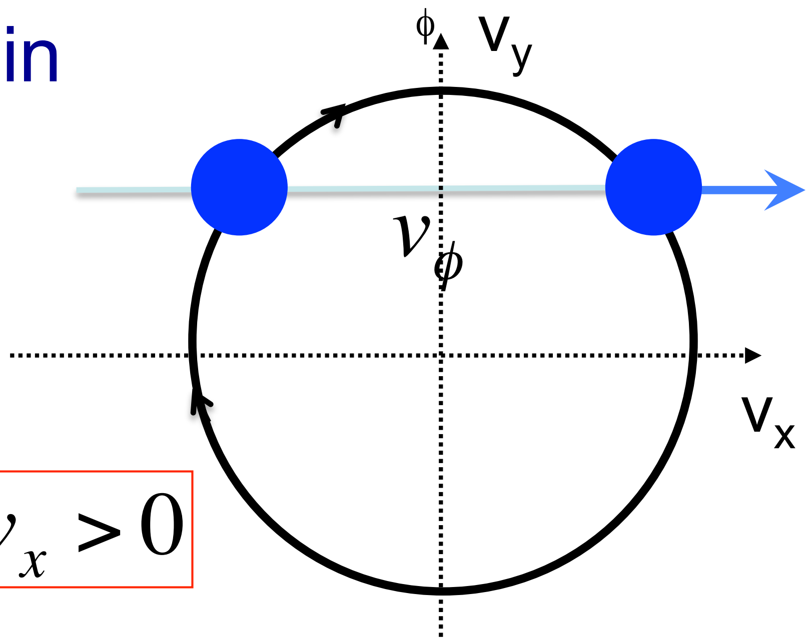
Capture into resonance

Let $\delta > b_n$.

$$\begin{cases} \dot{v}_x = b_n v_\phi , \\ \ddot{\phi} = -k v_x (b_n - \delta \sin(\phi)) \end{cases}$$



The area **A** surrounded by the trajectory is an adiabatic invariant: its value is approximately conserved in the evolution.

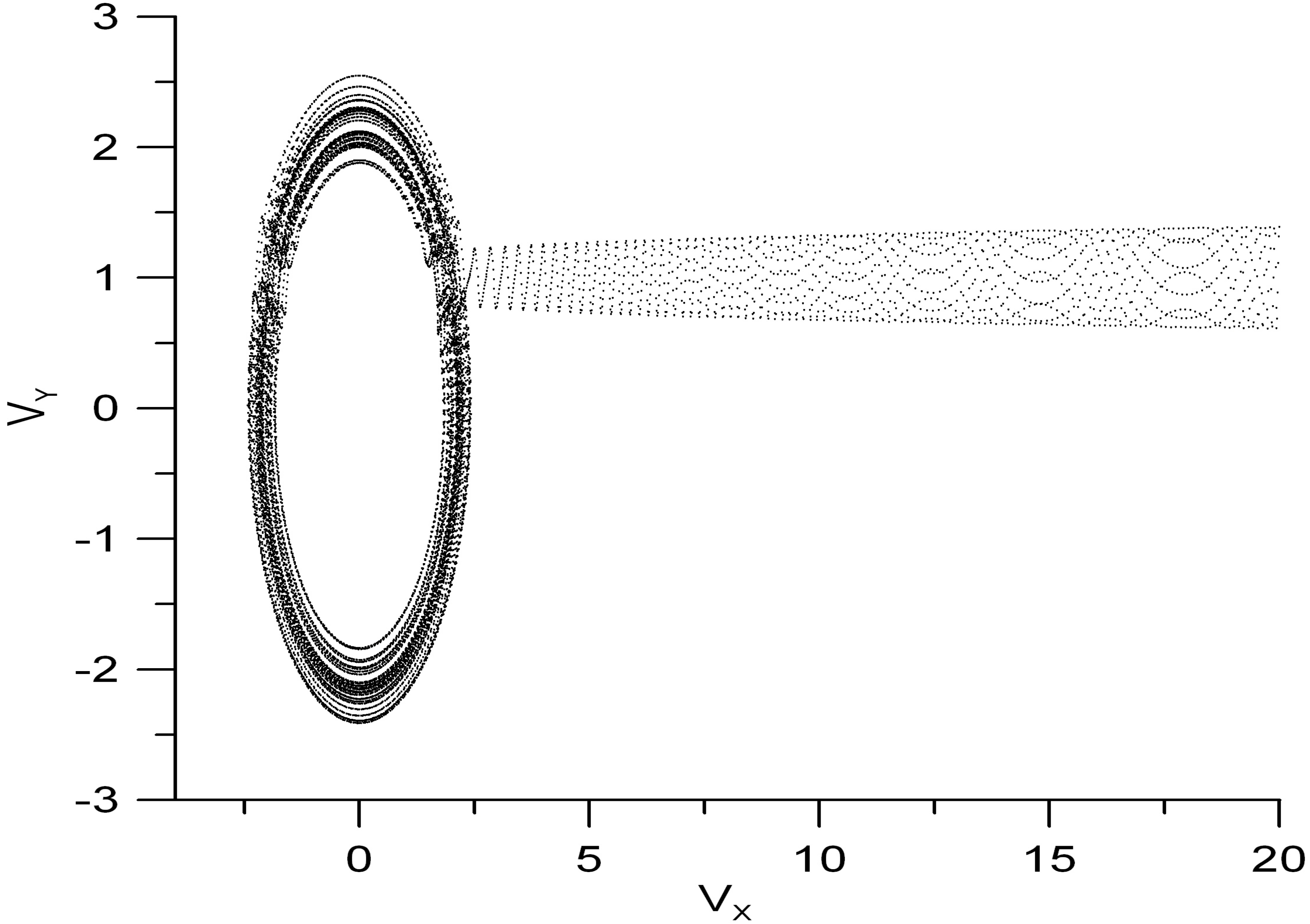


The area of the oscillatory domain

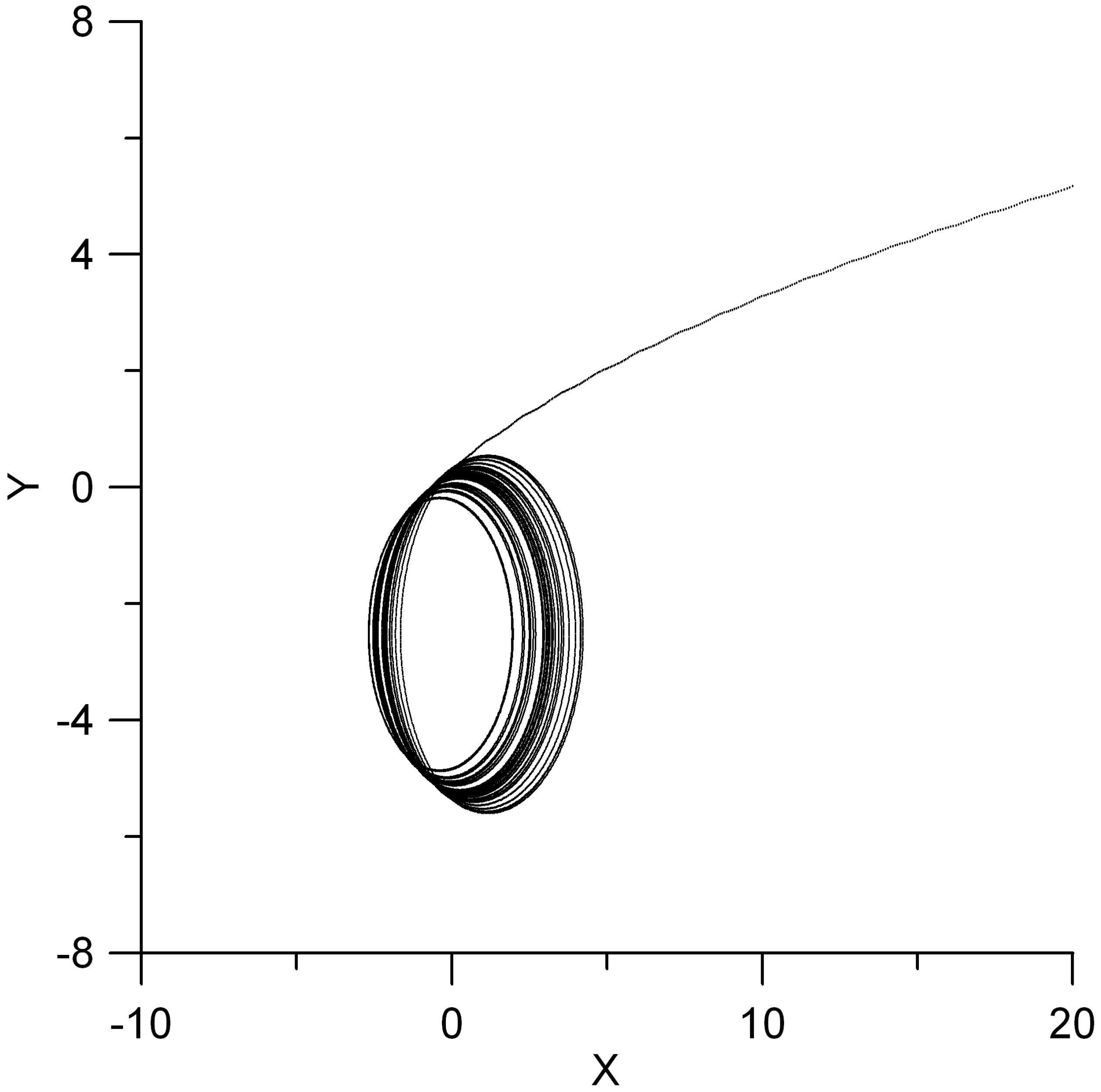
$$S \propto \sqrt{k |v_x|}$$

Capture into resonance is possible at $v_x > 0$

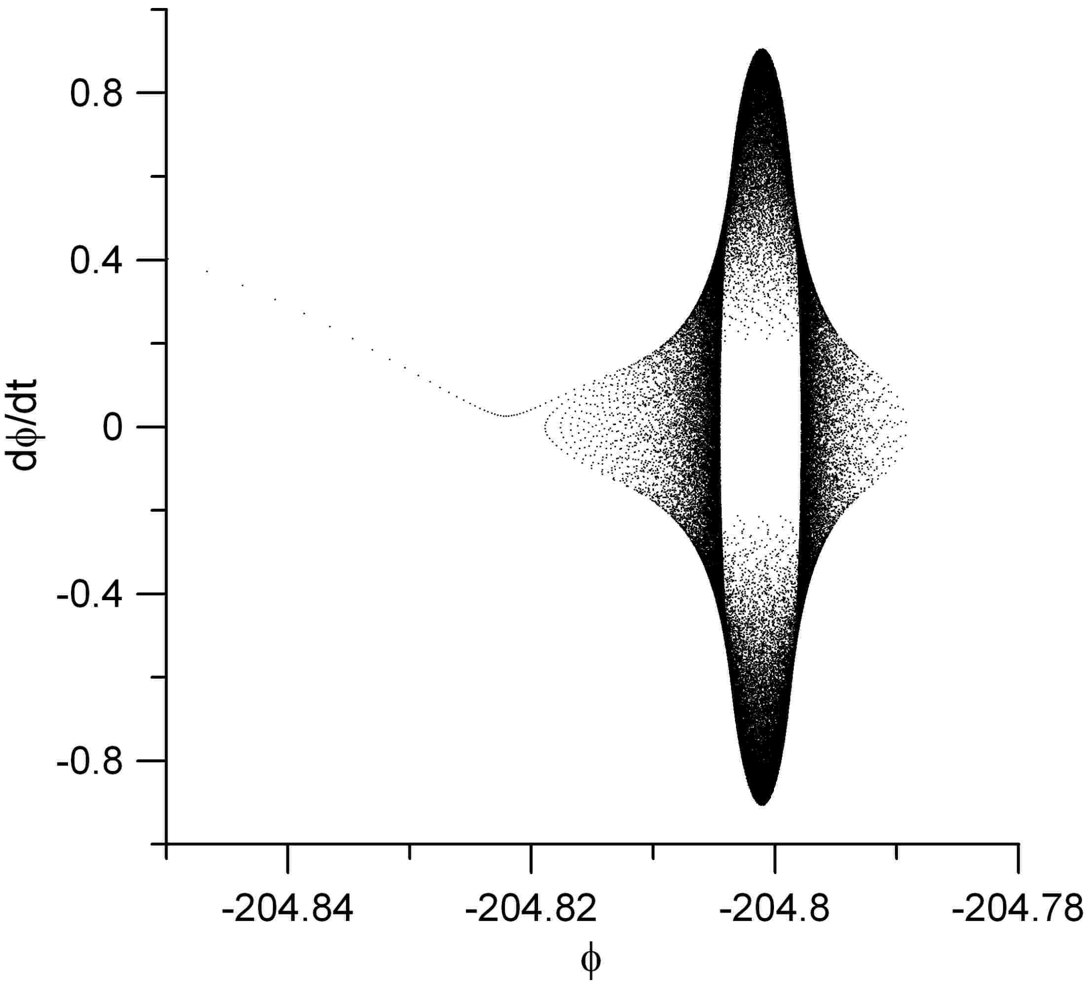
Capture into resonance (velocity plane)



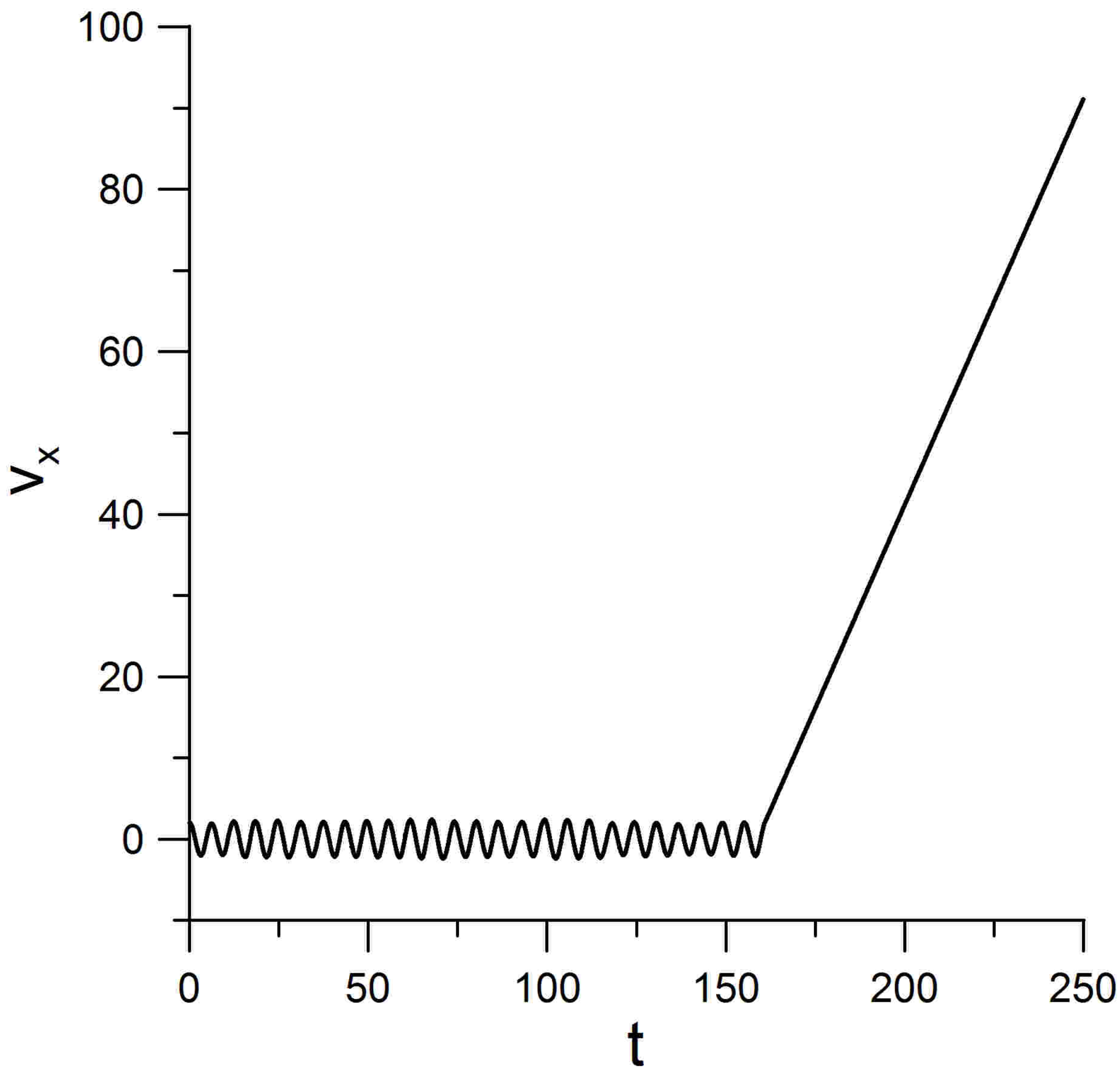
Capture into resonance (coordinate plane)



Capture into resonance (pendulum phase plane)



Capture into resonance (velocity vs. time)

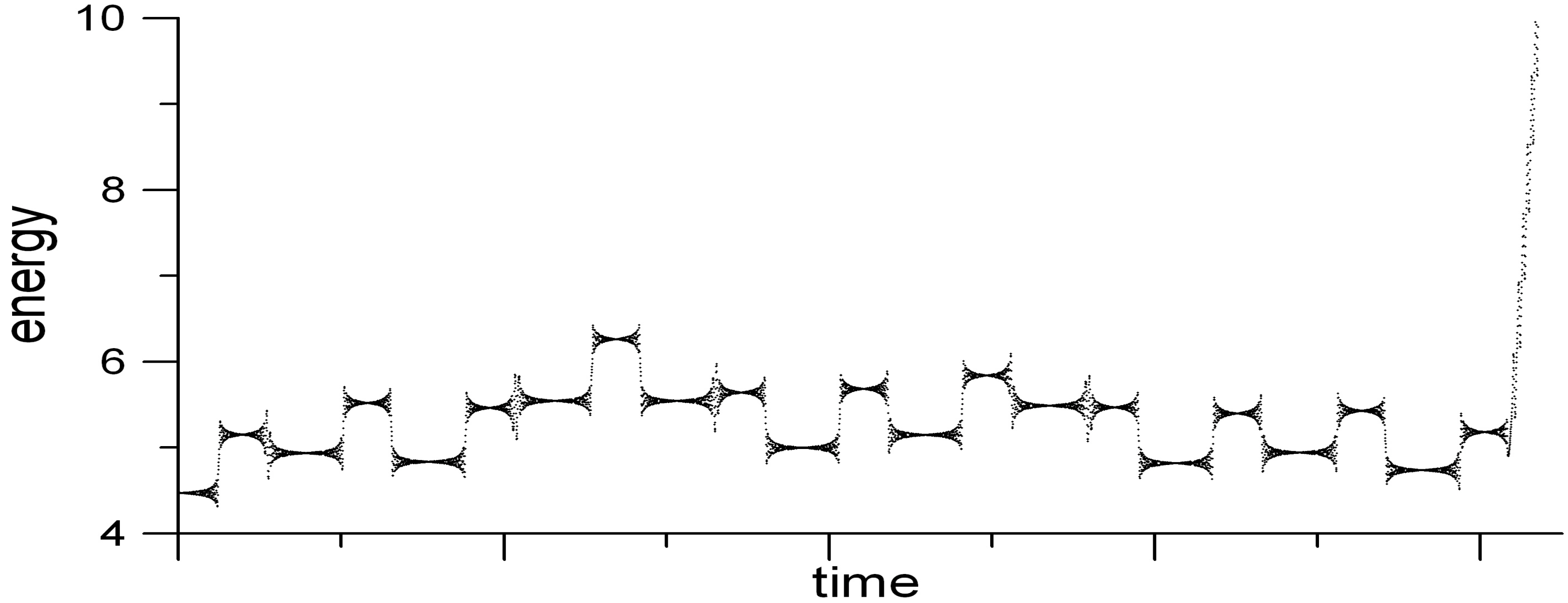


Capture into resonance is a quasi-random phenomenon.

The probability of capture for one passage through the resonance (on one Larmor round) is

$$P = Sv_{\phi}/(4\pi k v_x^2) \propto 1/\sqrt{k}.$$

The probability of capture for $\propto\sqrt{k}$ passages through the resonance (on $\propto\sqrt{k}$ Larmor rounds) is of order 1 (cf. D.Dolgopyat, 2005).



Remarks.

The particle moves with the wave and accelerates in the direction along the wave front. Such acceleration is called a *surfatron acceleration*.

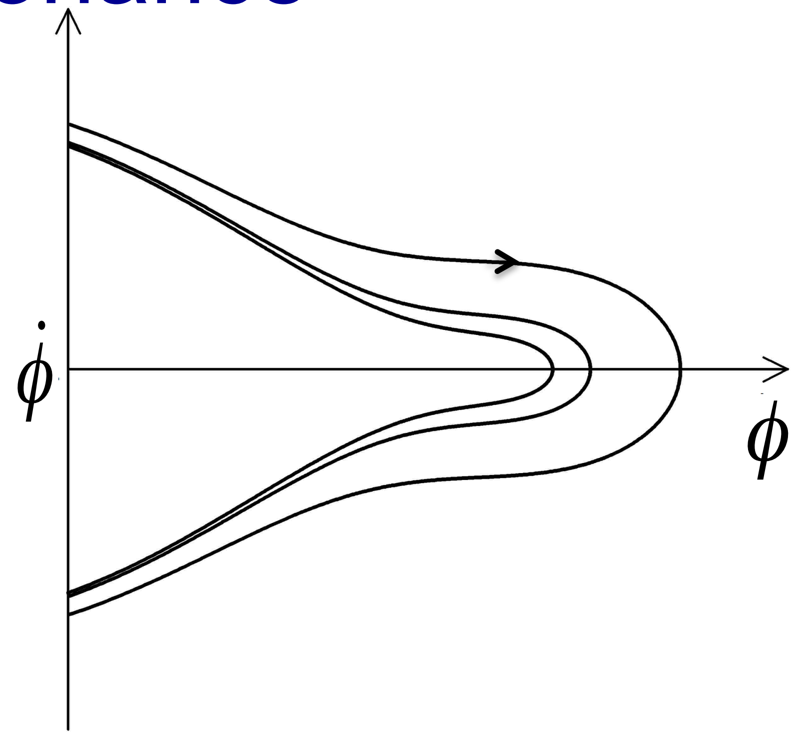
The surfatron acceleration was first discussed by R.Z.Sagdeev (1964) for non-relativistic particles and electrostatic waves. In this case a unlimited acceleration does not exist.

The possibility of a unlimited surfatron acceleration of ultra-relativistic particles by electrostatic waves was discovered by T.Katsouleas and J.Dawson (1983).

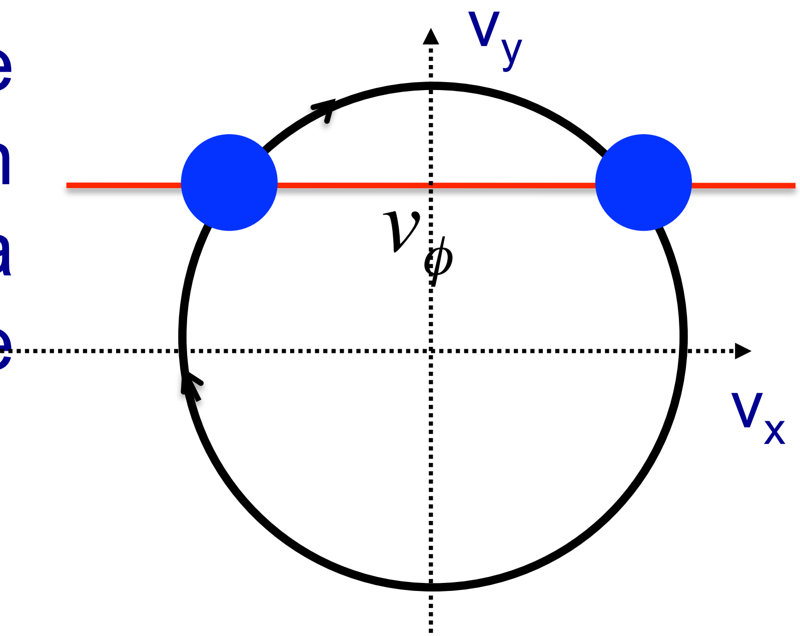
Scattering on resonance

Let $\delta < b_n$.

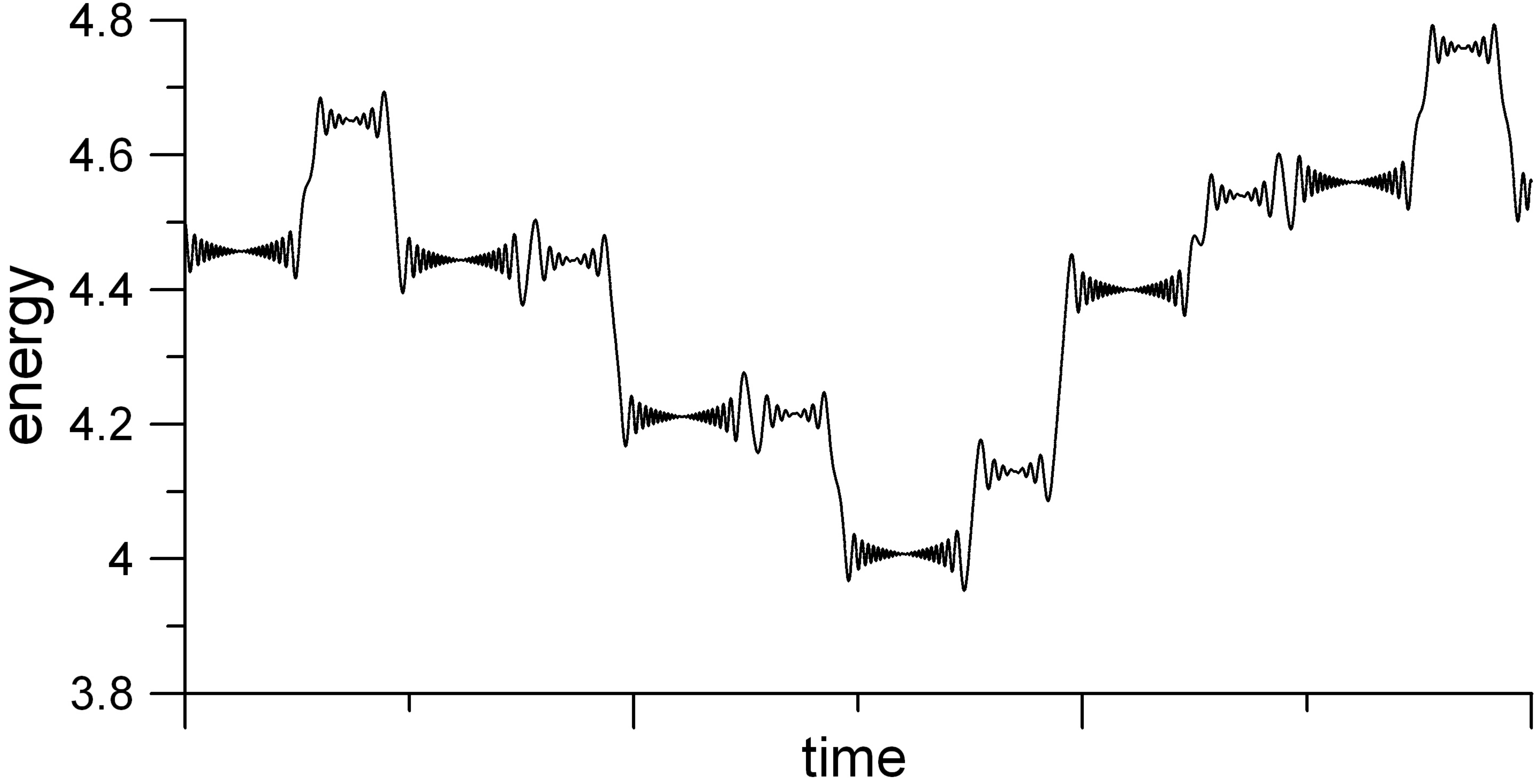
$$\begin{cases} \dot{v}_x = b_n v_\phi , \\ \ddot{\phi} = -k v_x (b_n - \delta \sin(\phi)) \end{cases}$$



There are no captures into the resonance. Each passage through resonance leads to a scattering: a small change of the energy of the particle occurs.

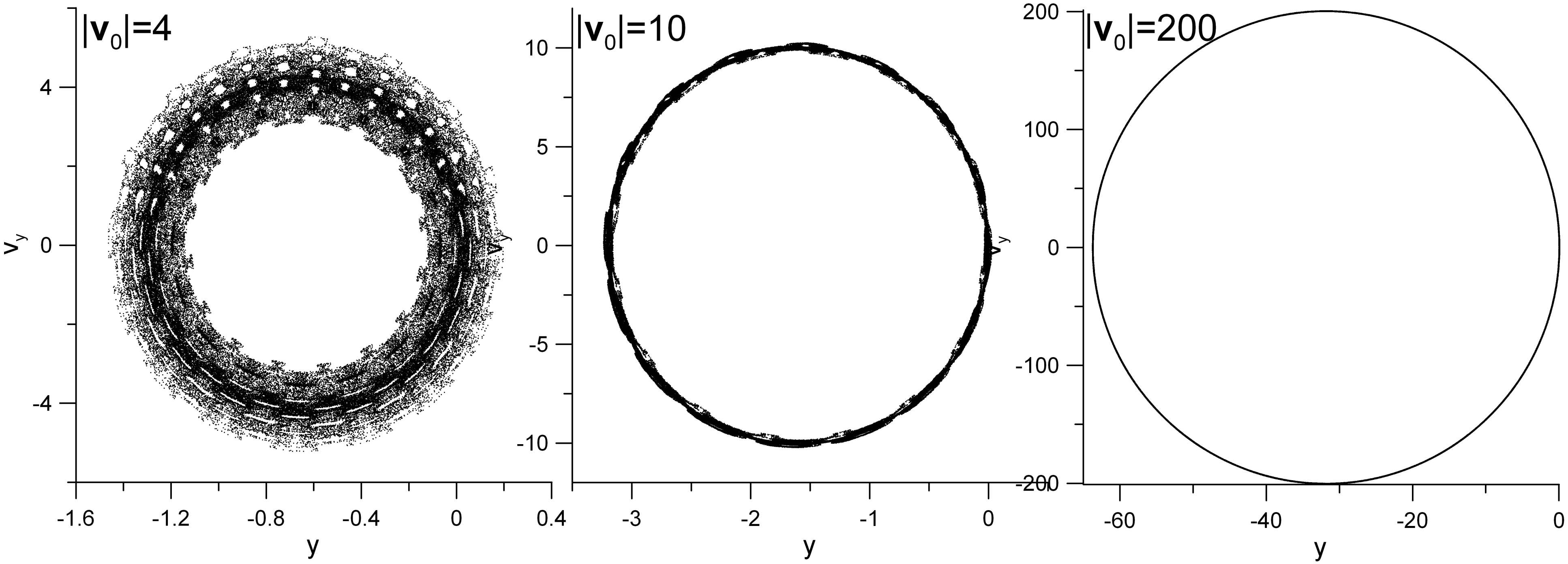


Scatterings on resonance



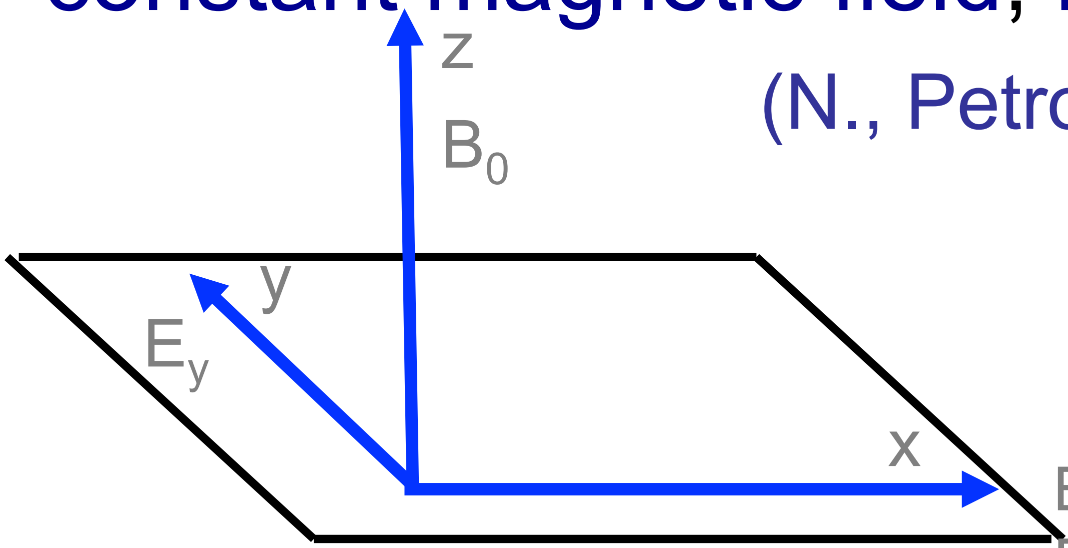
The diffusion due to scattering does not lead to a unlimited grow of the energy (this follows from KAM-theory)

Poincaré sections for the period $2\pi/\omega$:



3. Electrostatic wave perpendicular to constant magnetic field, relativistic particle

(N., Petrovichev, Chernikov, 1989)



The wave propagates along y-axis

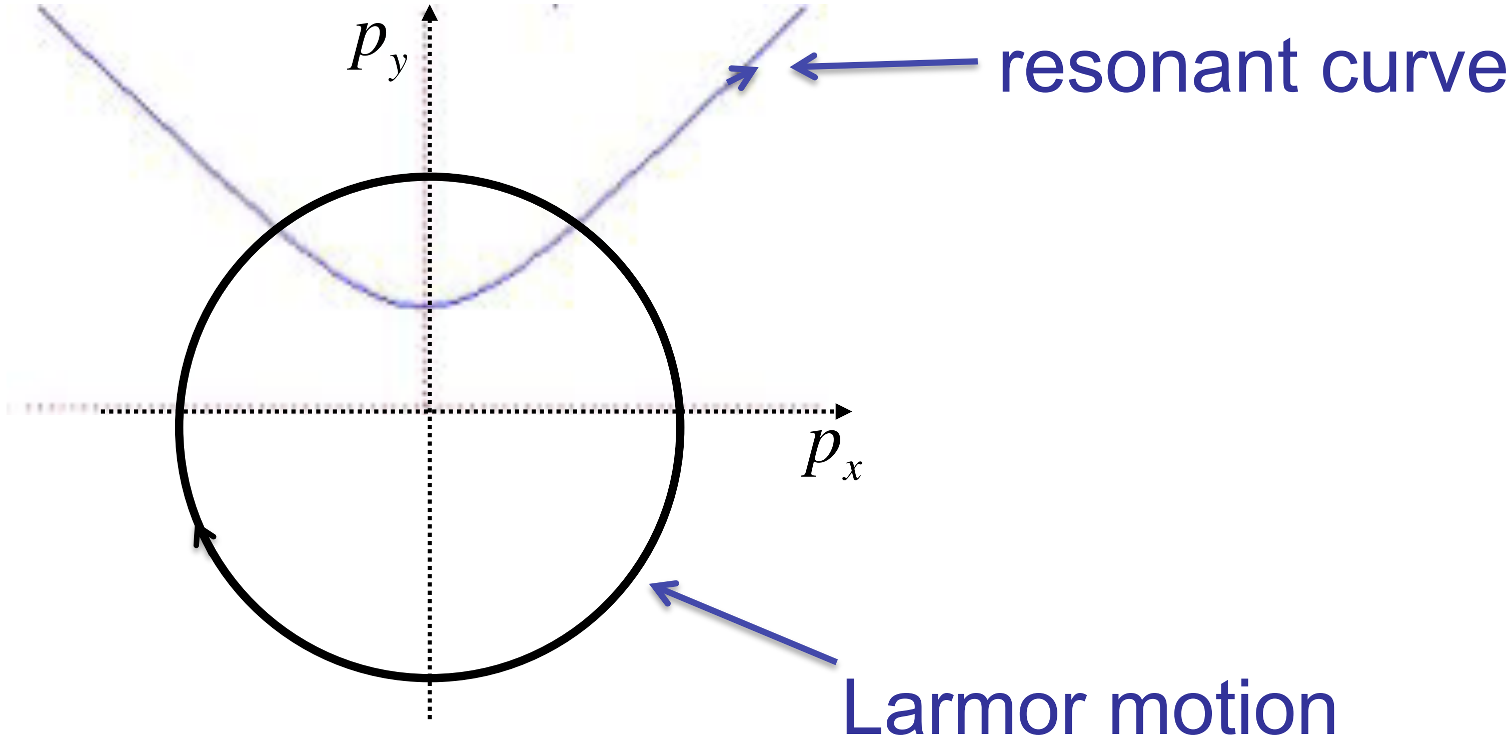
B_0 - the uniform magnetic field
E_y - the electric field of the wave

$$\dot{p}_x = \frac{eB_0}{c} v_y,$$

$$\dot{p}_y = -\frac{eB_0}{c} v_x + eE_y$$

$$E_y = E_1 \sin(ky - \omega t).$$

$$p_x = m_0 \gamma v_x,\ \ p_y = m_0 \gamma v_y,\ \ \gamma = 1/\sqrt{1-(v_x^2+v_y^2)/c^2}$$



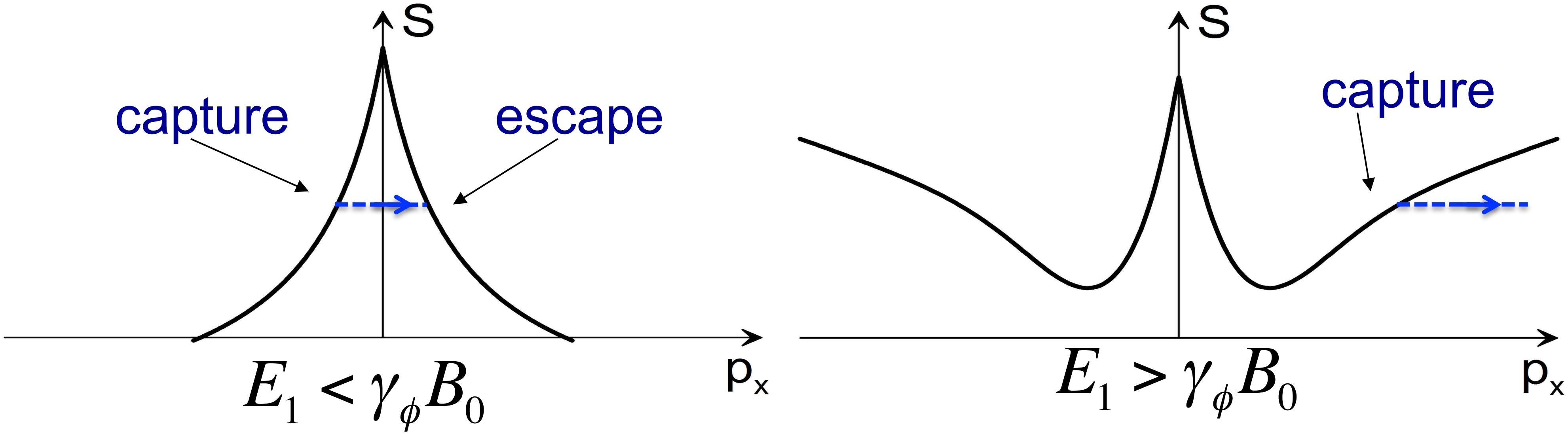
Resonant flow: $\dot{p}_x = (eB_0 / c) v_\phi$.

Reduction near the resonance:

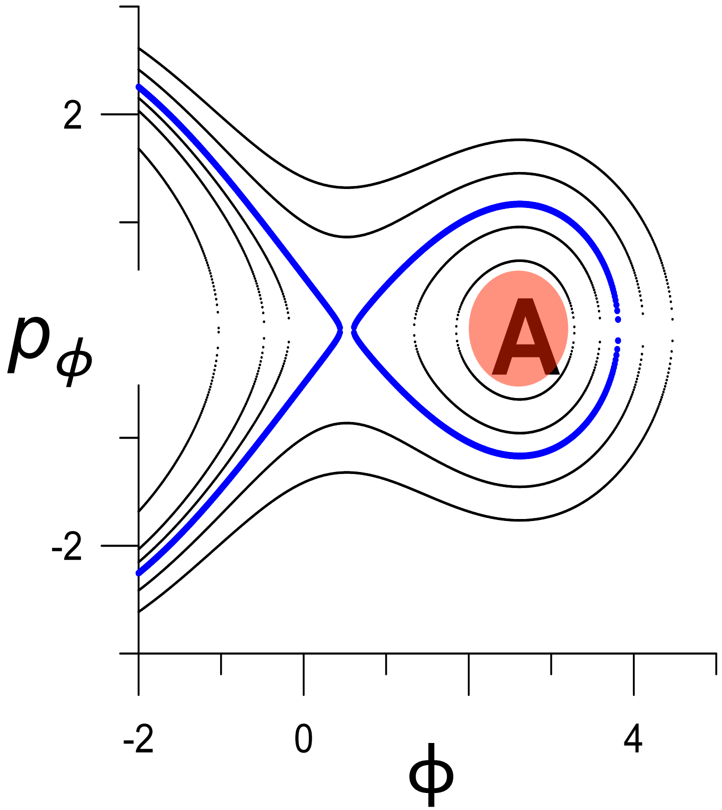
- the motion is described by pendulum-like system depending on slowly varying parameter p_x :

$$\dot{p}_x = (eB_0 / c) v_\phi \,.$$

- the graph of the area *S* of oscillatory domain has one of the following forms (here $\gamma_\phi = 1/\sqrt{1-(v_\phi / c)^2}$) :

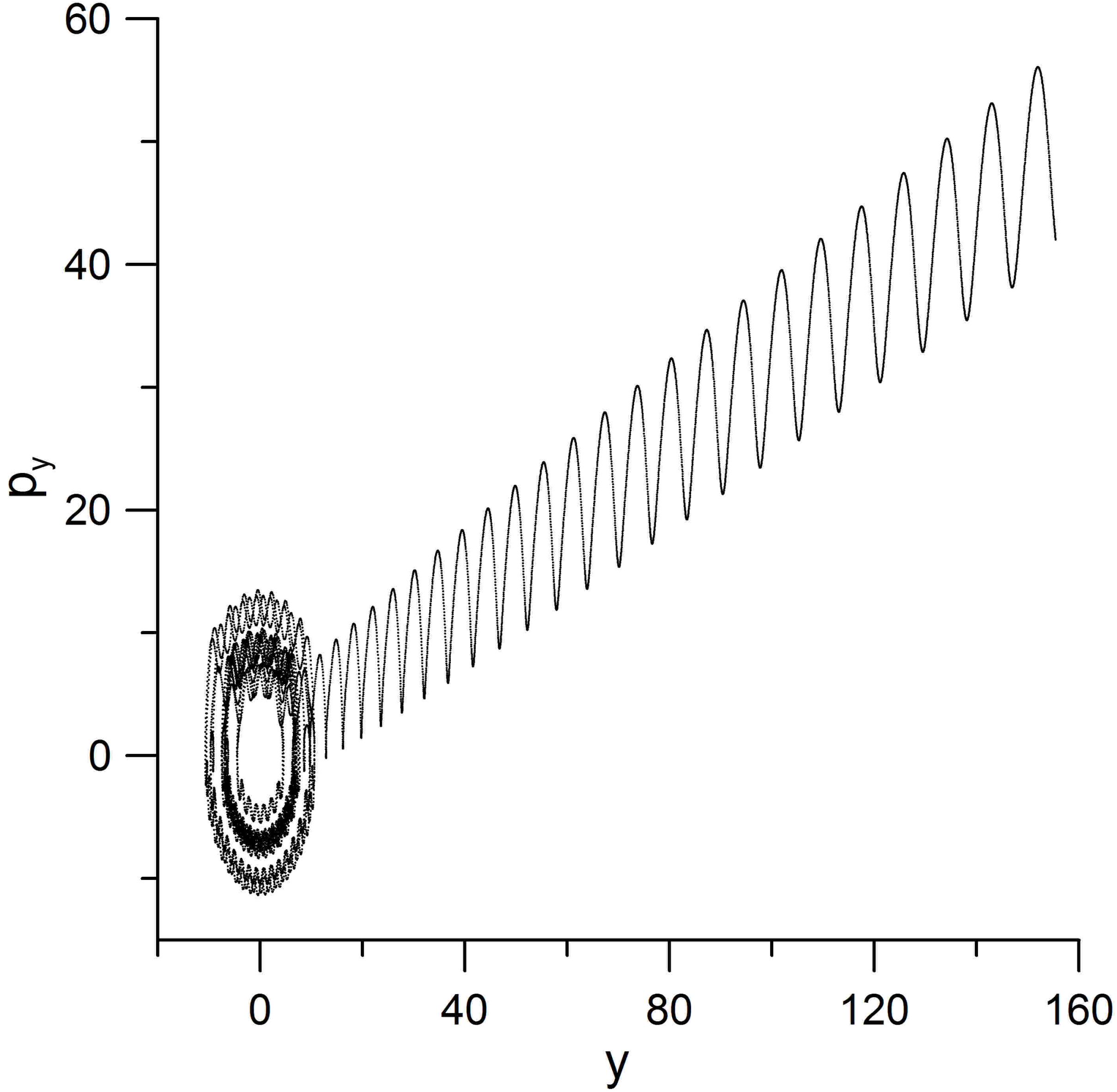


Condition of surfatron acceleration $E_1 > \gamma_\phi B_0$ was introduced by T.Katsouleas and J.Dawson (1983).

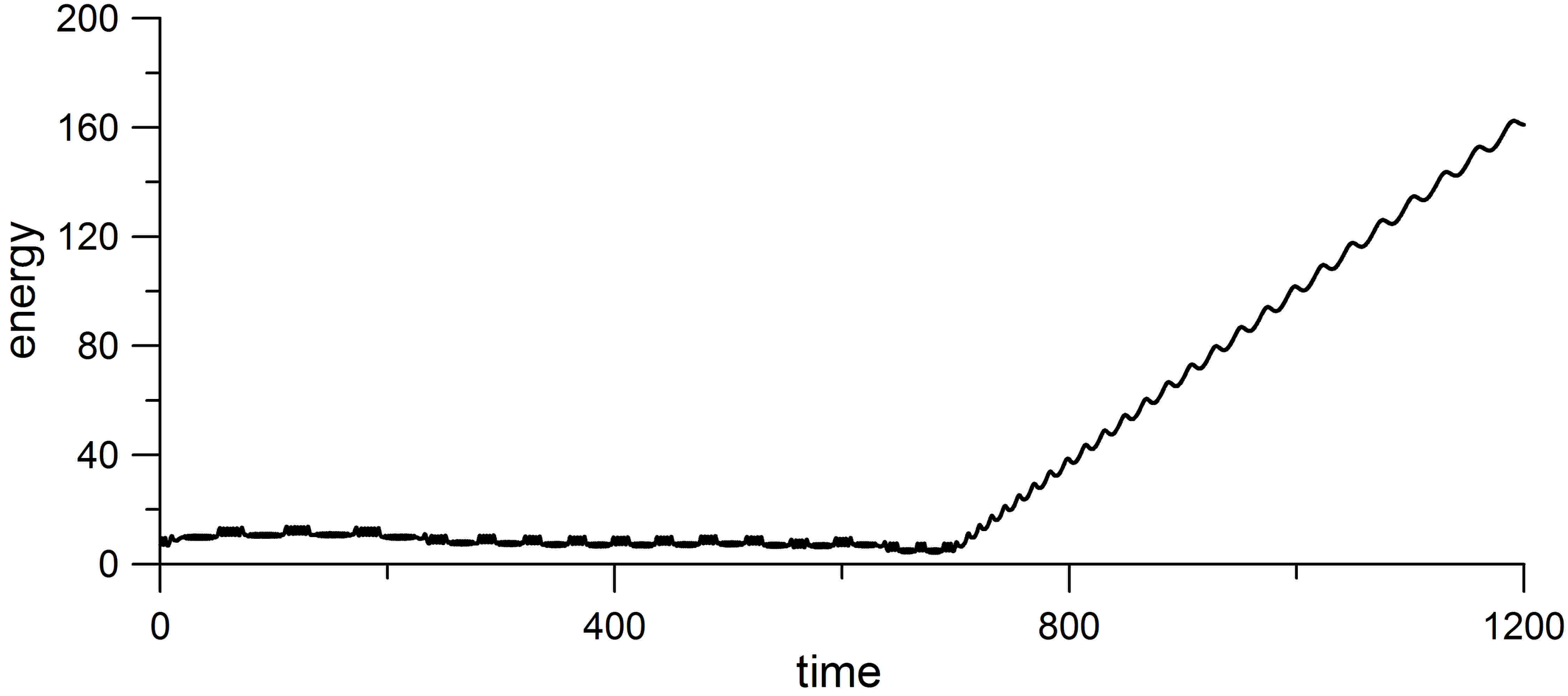


2
-2
p_ϕ
-2
0
4
ϕ
A

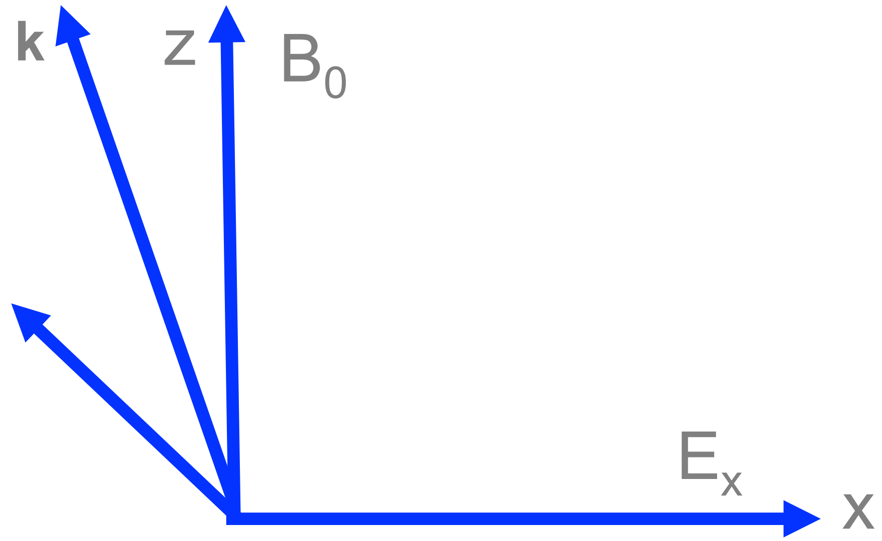
Capture into resonance, electrostatic wave, relativistic particle



Capture into resonance, electrostatic wave, relativistic particle, continued



4. Electromagnetic wave inclined to constant magnetic field, non-relativistic particle



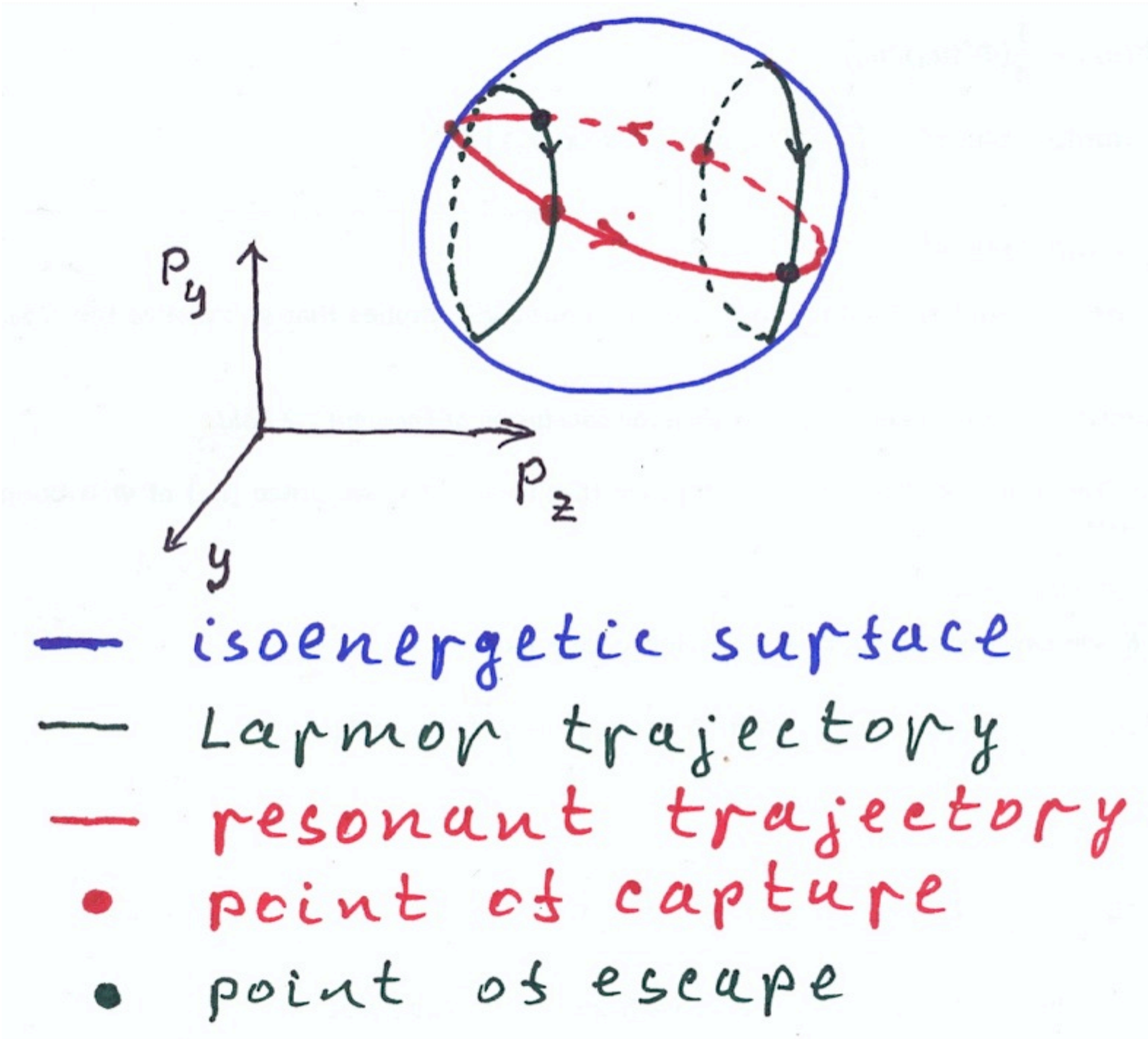
The wave propagates along **k**-direction

B_0 - the uniform magnetic field
E_x - the electric field of the wave
B_z, B_y- magnetic field of the wave

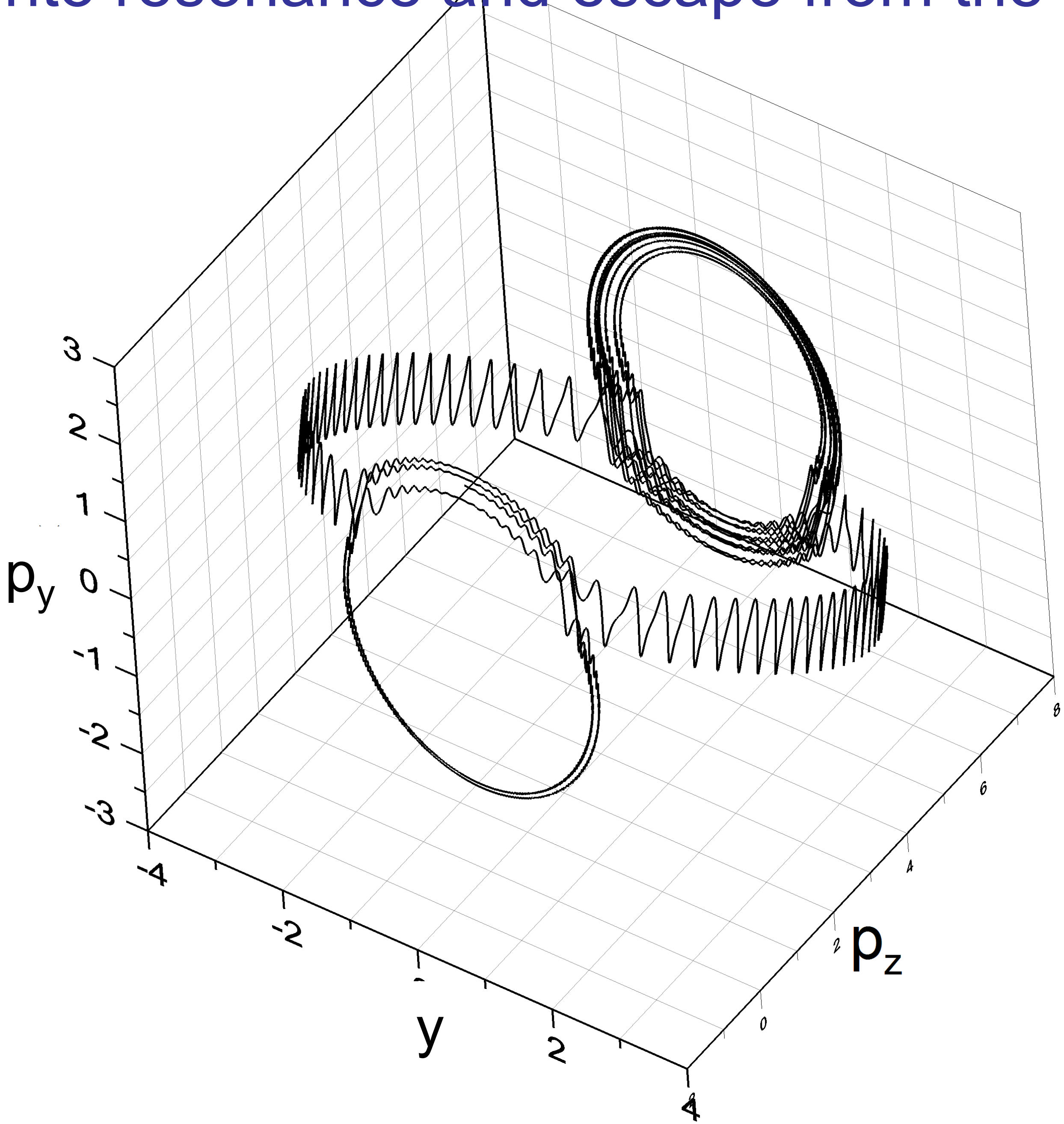
$$B_y = \frac{Bk_3}{k}\sin(\varphi),\ B_z = -\frac{Bk_2}{k}\sin(\varphi),\ E_x = \frac{B\omega}{kc}\sin(\varphi),$$

$$\varphi = k_2 y + k_3 z - \omega t,\ k = \sqrt{k_2^2 + k_3^2},\ k >> 1,\ \omega >> 1$$

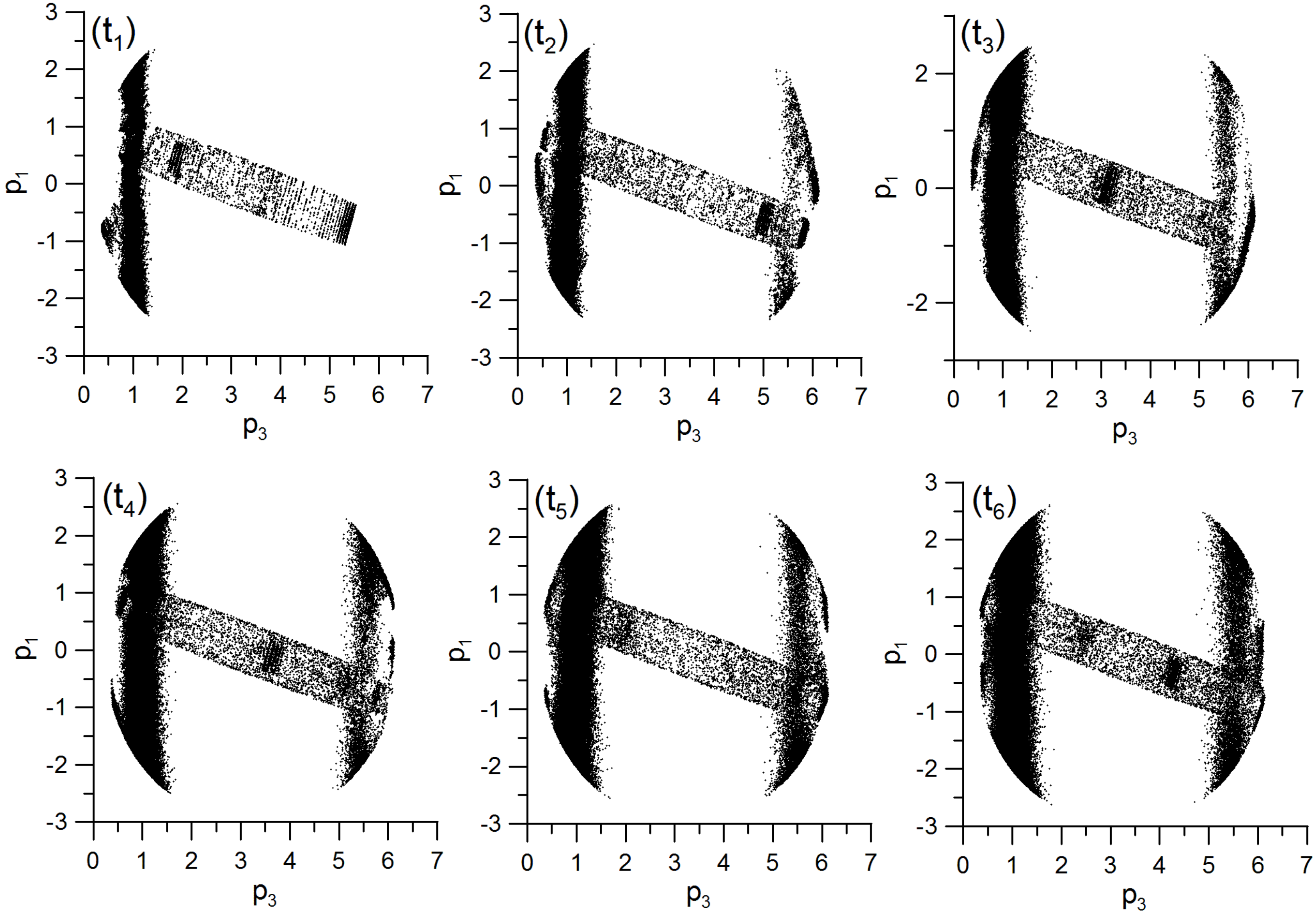
Schematic description of capture into resonance and escape from resonance in (y, p_y, p_z)-space



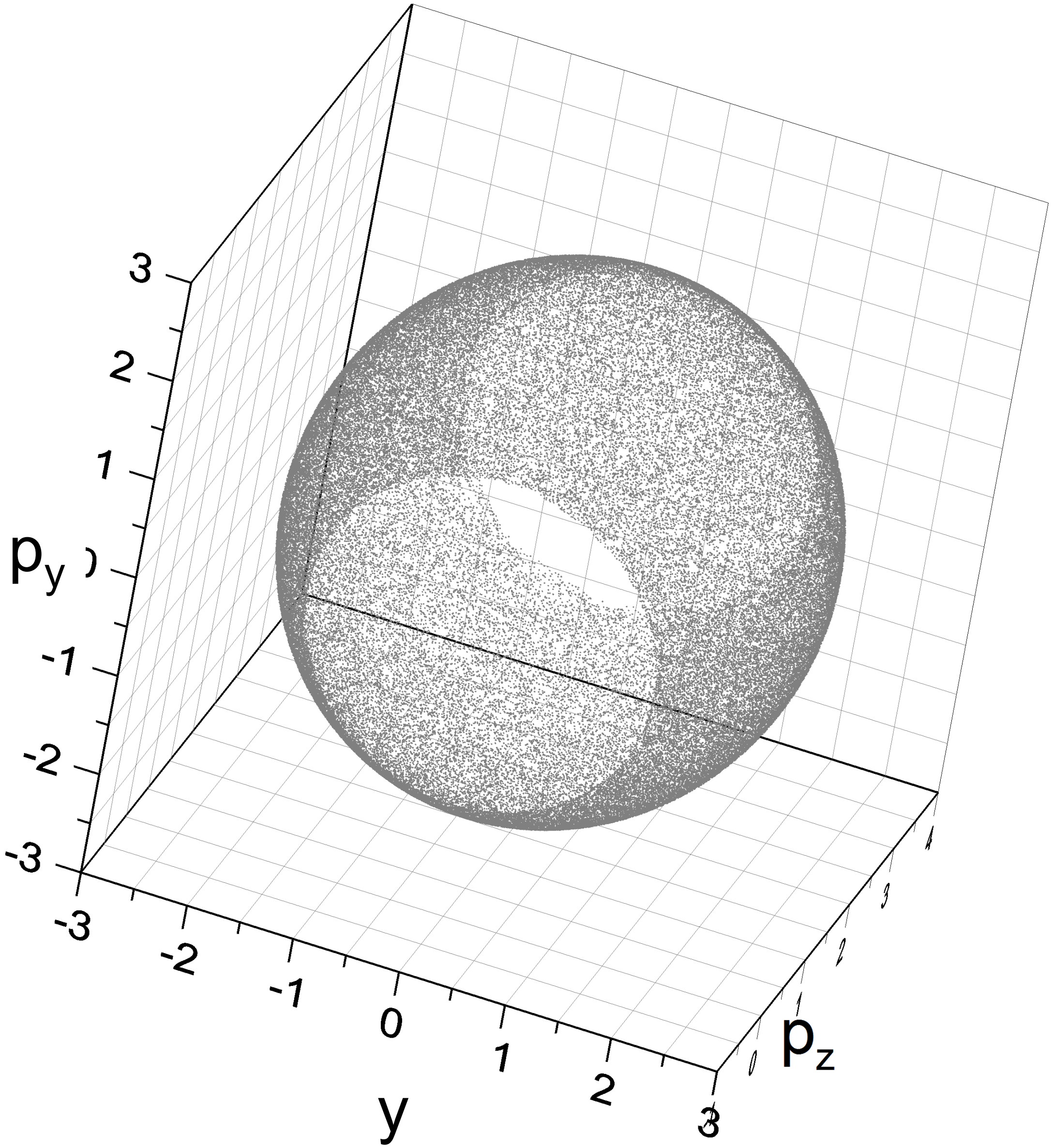
Capture into resonance and escape from the resonance



Evolution of an ensemble of particles



Scattering on resonance



Other examples:

- electrostatic wave inclined to constant magnetic field, relativistic particle (Itin, N., Vasiliev, 2000)

- electromagnetic wave perpendicular to constant magnetic field, relativistic particle (Itin, 2002, N., Artemyev, Vailiev, 2011)

References

G.M. Zaslavsky, A.I. Neishtadt, B.A. Pertovichev, R.Z.Sagdeev, Sov.J. Plasma Physics, **15** (1989), 368

A.I.Neishtadt, B.A. Petrovichev, A.A. Chernikov, Sov. J. Plasma Physics, **15** (1989), 1021.

A.I. Neishtadt, A. V. Artemyev, L. M. Zelenyi, and D. L. Vainchtein, JETP Letters, **89** (2009), 528.

A.V. Artemyev, A.I. Neishtadt, L M. Zelenyi, and D.L. Vainchtein, Chaos, **20** (2010), 043128.

A.A. Vasiliev, A. V. Artemyev, A.I. Neishtadt, Physics Letters A, **375** (2011), 3075.

Non-relativistic particle, electromagnetic wave, capture

$$b_n = \pi/4, \delta = \pi, v_\phi = 1, k = 100$$

Non-relativistic particle, electromagnetic wave, scattering

$$b_n = 2\pi, \delta = \pi/2, v_\phi = 1, k = 100$$

Relativistic particle, electromagnetic wave, capture

$$\delta/b_n = 4, \omega/b_n = 100, v_\phi = 1/(52000)c$$

Relativistic particle, electromagnetic wave, scattering

$$\delta/b_n = 0.5, \omega/b_n = 100, v_\phi = 1/(52000)c$$

Non-relativistic particle, electrostatic wave, capture

$$E/B_0 = 10, \omega_{wave}/\omega_c = 100, v_\phi = 1$$

Relativistic particle, electrostatic wave, capture

$$E/B_0 = 13, \omega_{wave}/\omega_c = 1, v_\phi/c = 0.3$$