Higher order QCD corrections via local subtraction

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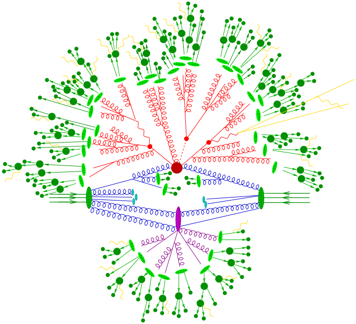
QCD at the LHC

Complicated environment, QCD must be understood/modeled as best as feasible

- parton model - beams of partons
- radiation off incoming partons
- primary hard scattering ($\mu \simeq Q \gg \Lambda_{\mathrm{QCD}}$)
- radiation off outgoing partons ($Q > \mu > \Lambda_{\mathrm{QCD}}$)
- hadronization and heavy hadron decay ($\mu \simeq \Lambda_{\mathrm{QCD}}$)
- multiple parton interactions, underlying event



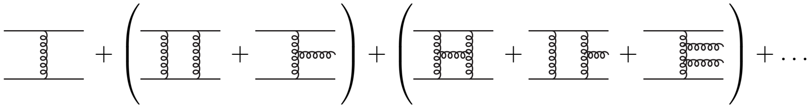
The hard process in perturbation theory

The scale of the hard scattering is $\mu \gg \Lambda_{\mathrm{QCD}}$, so by asymptotic freedom, it can be treated in perturbation theory, i.e., by expansion in powers of the strong coupling, $\alpha_{\mathrm{S}}(\mu)$.

Consider a generic cross section for producing *m* jets

$$\sigma_m = \alpha_{\mathrm{S}}^p \Big(\sigma_m^{\mathrm{LO}} + \alpha_{\mathrm{S}} \sigma_m^{\mathrm{NLO}} + \alpha_{\mathrm{S}}^2 \sigma_m^{\mathrm{NNLO}} + \dots \Big)$$

Representative Feynman-diagrams



How many terms to compute?

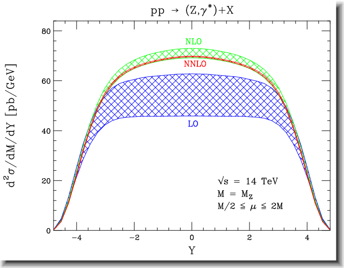
Why NNLO?

LO prediction: order of magnitude estimate, rough shapes of distributions

NLO is mandatory for meaningful normalization and shape predictions

NNLO may be relevant

- NLO corrections are large:
  - Higgs production from gluon fusion in hadron collisions
- for benchmark processes measured with high experimental accuracy:
  - α_s measurements form e^+e^- event shapes
  - W, Z production
  - heavy quark hadroproduction
- reliable error estimate is needed:
  - processes relevant for PDF determination
  - important background processes

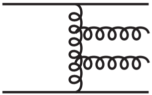


(Anastasiou, Dixon, Melnikov, Petriello, Phys. Rev. **D69** (2004) 094008.)

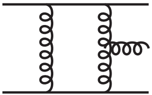
NNLO ingredients

A generic m-jet cross section at NNLO involves

- Tree-level squared matrix elements
  - with $m+2$ parton kinematics
  - known from LO calculations
  - ‘doubly-real’ contribution (RR)



- One-loop squared matrix elements
  - with $m+1$ parton kinematics
  - usually known from NLO calculations
  - ‘real-virtual’ contribution (RV)



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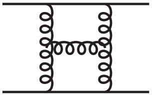

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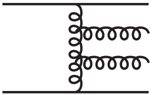

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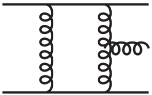

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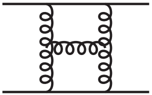

Assuming we know the relevant matrix elements, can we use those matrix elements to compute cross sections?

The problem - IR singularities

Consider the NNLO correction to a generic m-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m} \,.$$

Doubly-real

- $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2}$
- Tree MEs with $m+2$-parton kinematics
- kin. singularities as one or two partons unresolved: up to $\mathrm{O}(\epsilon^{-4})$ poles from PS integration
- no explicit ϵ poles

Real-virtual

- $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1}$
- One-loop MEs with $m+1$-parton kinematics
- kin. singularities as one parton unresolved: up to $\mathrm{O}(\epsilon^{-2})$ poles from PS integration
- explicit ϵ poles up to $\mathrm{O}(\epsilon^{-2})$

Doubly-virtual

- $\mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$
- One- and two-loop MEs with m-parton kinematics
- kin. singularities screened by jet function: PS integration finite
- explicit ϵ poles up to $\mathrm{O}(\epsilon^{-4})$

The problem - IR singularities

Consider the NNLO correction to a generic *m*-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m} \, .$$

THE KLN THEOREM

Infrared singularities cancel between real and virtual quantum corrections at the same order in perturbation theory, for sufficiently inclusive (i.e. IR safe) observables.

HOWEVER

How to make this cancellation explicit, so that the various contributions can be computed numerically? Need a method to deal with implicit poles.

Approaches

Sector decomposition

(Binoth, Heinrich; Anastasiou, Melnikov, Petriello; Czakon)

- extract ϵ poles of each contribution (RR, RV, VV) separately by expanding the integrand in distributions
- resulting expansion coefficients are finite multi-dimensional integrals, integrate numerically
- cancellation of poles numerical, depends on observable
- first method to yield physical results, but can it handle complicated final states?

Subtraction

(Catani, Grazzini; Cieri, Ferrera, de Florian; Gehrmann, Gehrmann-De Ridder, Glover; Weinzierl; Del Duca, Trócsányi, GS)

- rearrange the poles between real and virtual contributions by subtracting and adding back suitable approximate cross sections
- cancellation of explicit ϵ poles achieved analytically, remaining PS integrals are finite
- nice properties (generality, efficiency) expected form experience at NLO
- definition of subtraction terms is not unique, hence several approaches: $q_\perp$, antenna, local

Approaches

Sector decomposition

(Binoth, Heinrich, Anastasiou, Dixon, Melnikov, Petriello, Czakon)

- ✔ first method to yield physical cross sections
- ✔ cancellation of divergences fully numerical
- ✘ cancellation of poles also, and depends on jet function
- ✘ can it handle complicated final states?

$q_\perp$ subtraction

(Catani, Grazzini, Cieri, Ferrera, de Florian, Tramontano)

- ✔ exploits universal behavior of $q_\perp$ distribution at small $q_\perp$
- ✔ efficient and fully exclusive calculation
- ✘ limited scope: applicable only to production of massive colorless final states in hadron collisions

Antenna subtraction

(Gehrmann, Gehrmann-De Ridder, Glover, Heinrich, Weinzierl)

- ✔ successfully applied to $e^+e^- \to 2, 3j$
- ✔ analytic integration of antennae over unresolved phase space is understood
- ✘ counterterms are nonlocal
- ✘ treatment of color is implicit
- ✘ cannot cut factorized phase space

Approaches - developments

Refinement of the sector decomposition algorithm

(Anastasiou, Lazopoulos, Herzog 2010)

- uses non-linear mappings to disentangle overlapping singularities
- the aim is to increase efficiency by reducing the large number of sectors/terms generated during decomposition
- first application: fully exclusive $H \to b\bar{b}$ decay at NNLO

(Anastasiou, Lazopoulos, Herzog 2011)

Refinement of phase space integration via sector decomposition

(Czakon 2010; Boughezal, Melnikov, Petriello 2011)

- FKS-like approach to double real radiation in $t\bar{t}$ production
- sector decomposition used to make singular contributions explicit, guided by known universal IR structure
- first NNLO computation of $pp \to t\bar{t}$ total cross section

(Baernreuther, Czakon, Mitov 2012)

Why a new scheme?

Goal: devise a subtraction scheme with

- general and explicit expressions, including color
  (view towards automation, color space notation is used)
- fully local counterterms, taking account of all color and spin correlations
  (mathematical rigor, efficiency)
- option to constrain subtractions to near singular regions
  (efficiency, important check)
- very algorithmic construction
  (valid at any order in perturbation theory)

Basics of subtraction

Strategy: rearrange IR singularities between various contributions by subtracting and adding back suitably defined approximate cross sections.

- subtraction terms match the singularity structure of real emission point wise (in d dimensions) $\Rightarrow$ phase space integrals over real radiation rendered convergent
- integrated forms of subtraction terms have the same pole structure as virtual contribution $\Rightarrow$ explicit ϵ-poles cancel point by point

The construction of a general (i.e. process- and observable-independent) subtraction algorithm

- made possible by the universal structure of IR singularities, embodied in so-called IR factorization formulae
- is not unique, hence several approaches (FKS, dipole, antenna,...)

Subtraction - a caricature

Want to evaluate (at $\epsilon \to 0$)

$$\sigma = \int_0^1 \mathrm{d}\sigma^{\mathrm{R}}(x) + \sigma^{\mathrm{V}} \qquad \text{where} \qquad \begin{aligned} \mathrm{d}\sigma^{\mathrm{R}}(x) &= x^{-1-\epsilon} R(x) \\ R(0) &= R_0 < \infty \\ \sigma^{\mathrm{V}} &= R_0/\epsilon + V \end{aligned}$$

- define the counterterm

$$\mathrm{d}\sigma^{\mathrm{R,A}}(x) = x^{-1-\epsilon} R_0$$

- use it to reshuffle singularities between R and V contributions

$$\begin{aligned} \sigma &= \int_0^1 \left[\mathrm{d}\sigma^{\mathrm{R}}(x) - \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} + \left[\sigma^{\mathrm{V}} + \int_0^1 \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} \\ &= \int_0^1 \left[\frac{R(x) - R_0}{x^{1+\epsilon}}\right]_{\epsilon=0} + \left[\frac{R_0}{\epsilon} + V - \frac{R_0}{\epsilon}\right]_{\epsilon=0} \\ &= \int_0^1 \frac{R(x) - R_0}{x} + V \end{aligned}$$

The last integral is finite, computable with standard numerical methods.

Structure of the NNLO correction

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

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4. $\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$
5. $\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}\right)^{\mathrm{A_1}}$ regularizes the singly-unresolved limit of $\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$

Defining a subtraction scheme

Two issues must be addressed

1. What to subtract?
2. How to add it back?

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1. What to subtract?
2. How to add it back?

Strategy: IR limits are process independent and known

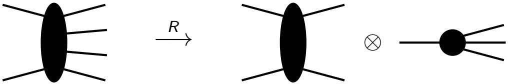
1. Start by defining subtraction terms based on IR limit formulae $\Rightarrow$ the result is trivially general and explicit ✔

2. Worry about integrating them later, since this is *in principle* a very narrowly defined problem, given 1., but in practice turns out to be very cumbersome ✘

IR factorization formulae

The structure of IR singularities is universal, i.e., does not depend on the process. The general structure of these formulae is the same in all limits

$$|\mathcal{M}_{n+p}^{(0)}(\{p\}_{n+p})|^2 \xrightarrow{R} (8\pi\alpha_s\mu^{2\epsilon})^p \,\mathrm{Sing}_R(\{p\}_p) \otimes |\mathcal{M}_n^{(0)}(\{p\}_n)|^2$$



IR factorization formulae

Collinear and soft currents at NNLO are known

- Tree level 3-parton splitting functions and double soft gg and $q\bar{q}$ currents

(Campbell, Glover 1997; Catani, Grazzini 1998;
Del Duca, Frizzo, Maltoni 1999; Kosower 2002)

- One-loop 2-parton splitting functions and soft gluon current

(Bern, Dixon, Dunbar, Kosower 1994; Bern, Del Duca, Kilgore, Schmidt 1998-9; Kosower, Uwer 1999; Catani, Grazzini 2000; Kosower 2003)

IR factorization formulae

Collinear and soft currents at NNLO are known

- Tree level 3-parton splitting functions and double soft gg and $q\bar{q}$ currents

(Campbell, Glover 1997; Catani, Grazzini 1998; Del Duca, Frizzo, Maltoni 1999; Kosower 2002)

- One-loop 2-parton splitting functions and soft gluon current

(Bern, Dixon, Dunbar, Kosower 1994; Bern, Del Duca, Kilgore, Schmidt 1998-9; Kosower, Uwer 1999; Catani, Grazzini 2000; Kosower 2003)

Are these useful for NNLO?

IR factorization formulae

Limit formulae cannot be used as subtraction terms as they stand. Consider

- collinear: $p_i || p_r$

$$|\mathcal{M}_{n+1}^{(0)}(p_i, p_r, \ldots)|^2 \xrightarrow{i||r} 8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon} \frac{1}{s_{ir}} \hat{P}_{f_i f_r}^{(0)}(z_i, z_r, k_\perp; \epsilon) \otimes |\mathcal{M}_n^{(0)}(p_i + p_r, \ldots)|^2$$

- soft: $p_r \to 0$

$$|\mathcal{M}_{n+1}^{(0)}(p_r, \ldots)|^2 \xrightarrow{r\to 0} -8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon} \sum_{i,k} \frac{s_{ik}}{s_{ir} s_{kr}} |\mathcal{M}_{n,(i,k)}^{(0)}(\cancel{p_r}, \ldots)|^2$$

Issues

- the singular function associated to a specific limit, $\mathrm{Sing}_R(\{p\}_p)$ may be singular in other IR limits as well
- both $\mathrm{Sing}_R(\{p\}_p)$ and $|\mathcal{M}_n^{(0)}(\{p\}_n)|^2$ are only well-defined in the strict R limit

Defining a subtraction scheme

The following three problems must be addressed

1. Matching of limits to avoid multiple subtraction in overlapping singular regions of PS. Easy at NLO: collinear limit + soft limit - collinear limit of soft limit.

$$\mathbf{A}_1|\mathcal{M}^{(0)}_{m+1}|^2 = \sum_i \Big[\sum_{i\neq r}\frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i\neq r}\mathbf{C}_{ir}\mathbf{S}_r\Big]|\mathcal{M}^{(0)}_{m+1}|^2$$

2. Extension of IR factorization formulae over full PS using momentum mappings that respect factorization and delicate structure of cancellations in all limits.

$$\{p\}_{m+1} \xrightarrow{r} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+1}(\{p\}_{m+1};Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m;Q)[\mathrm{d}p_{1,m}]$$
$$\{p\}_{m+2} \xrightarrow{r,s} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+2}(\{p\}_{m+2};Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m;Q)[\mathrm{d}p_{2,m}]$$

3. Integration of the counterterms over the phase space of the unresolved parton(s).

Defining a subtraction scheme

Specific issues at NNLO

1. Matching is cumbersome if done in a brute force way. However, an efficient solution that works at any order in PT is known.
2. Extension is delicate. E.g., counterterms for singly-unresolved real emission (unintegrated and integrated) must have universal IR limits. This is not guaranteed by QCD factorization.
3. Choosing the counterterms such that integration is (relatively) straightforward generally conflicts with the delicate cancellation of IR singularities.

NNLO subtraction terms - general features

Based on universal IR limit formulae

- Altarelli-Parisi splitting functions, soft currents (tree and one-loop, triple AP functions)
- simple and general procedure for matching of limits using physical gauge
- extension based on momentum mappings that can be generalized to any number of unresolved partons

Fully local in color $\otimes$ spin space

- no need to consider the color decomposition of real emission ME's
- azimuthal correlations correctly taken into account in gluon splitting
- can check explicitly that the ratio of the sum of counterterms to the real emission cross section tends to unity in any IR limit

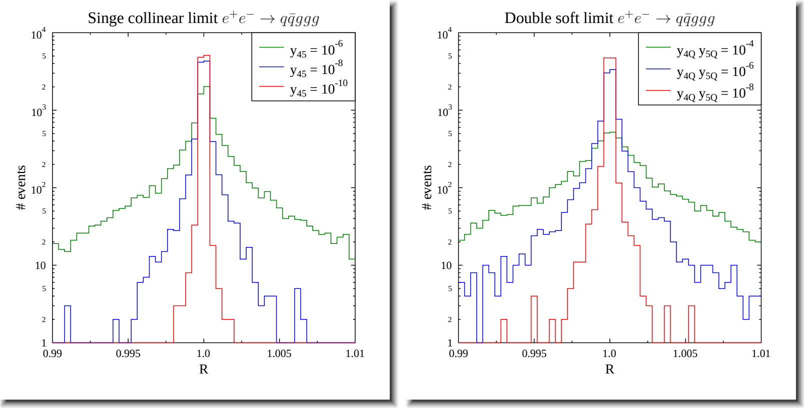
Straightforward to constrain subtractions to near singular regions

- gain in efficiency
- independence of physical results on phase space cut is strong check

Given completely explicitly for any process with non colored initial state

NNLO subtraction terms

Can check the ratio of the real emission matrix element and the sum of all subtractions for all IR limits



Integrating the subtractions

Momentum mappings used to define the counterterms

$$\{p\}_{n+p} \xrightarrow{R} \{\tilde{p}\}_n \quad \Rightarrow \quad \mathrm{d}\phi_{n+p}(\{p\};Q) = \mathrm{d}\phi_n(\{\tilde{p}\}_n^{(R)};Q)[\mathrm{d}p_{p,n}^{(R)}]$$

- implement exact momentum conservation, recoil distributed democratically (can be generalized to any p)
- different collinear and soft mappings (R labels precise limit)
- exact factorization of phase space

Counterterms are products (in color and spin space) of

- factorized ME's independent of variables in $[\mathrm{d}p_{p,n}^{(R)}]$
- singular factors (AP functions, soft currents), to be integrated over $[\mathrm{d}p_{p,n}^{(R)}]$

$$\mathcal{X}_R(\{p\}_{n+p}) = \left(8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon}\right)^p \mathit{Sing}_R(p_p^{(R)}) \otimes |\mathcal{M}_n^{(0)}(\{\tilde{p}\}_n^{(R)})|^2$$

Can compute once and for all the integral over unresolved partons

$$\int_p \mathcal{X}_R(\{p\}_{n+p}) = \left(8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon}\right)^p \left[\int_p \mathit{Sing}_R(p_p^{(R)})\right] \otimes |\mathcal{M}_n^{(0)}(\{\tilde{p}\}_n^{(R)})|^2$$

Solving the integrlas

Master integrals

- use algebraic and symmetry relations to reduce to a basic set of integrals
- note: this is not the usual notion of MIs (no IBPs used)

Strategy for computing the master integrals

1. write phase space in terms of angles and energies
2. angular integrals in terms of Mellin-Barnes representations
3. resolve the ϵ poles by analytic continuation
4. MB integrals to Euler-type integrals, pole coefficients are finite parametric integrals

1. choose explicit parametrization of phase space
2. write the parametric integral representation in chosen variables
3. resolve the ϵ poles by sector decomposition
4. pole coefficients are finite parametric integrals

5. evaluate the parametric integrals in terms of multiple polylogs
6. simplify result (optional)

List of master integrals

Int	status
$\mathcal{I}^{(k)}_{1\mathcal{C},0}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},1}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},2}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},3}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},4}$	✔
$\mathcal{I}^{(k,l)}_{1\mathcal{C},5}$	✔
$\mathcal{I}^{(k,l)}_{1\mathcal{C},6}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{C},7}$	✔
$\mathcal{I}_{1\mathcal{C},8}$	✔

Int	status
$\mathcal{I}_{1\mathcal{S},0}$	✔
$\mathcal{I}_{1\mathcal{S},1}$	✔
$\mathcal{I}_{1\mathcal{S},2}$	✘
$\mathcal{I}^{(k)}_{1\mathcal{S},3}$	✔
$\mathcal{I}_{1\mathcal{S},4}$	✔
$\mathcal{I}_{1\mathcal{S},5}$	✔
$\mathcal{I}_{1\mathcal{S},6}$	✔
$\mathcal{I}_{1\mathcal{S},7}$	✔

Int	status
$\mathcal{I}_{1\mathcal{CS},0}$	✔
$\mathcal{I}_{1\mathcal{CS},1}$	✔
$\mathcal{I}^{(k)}_{1\mathcal{CS},2}$	✔
$\mathcal{I}_{1\mathcal{CS},3}$	✔
$\mathcal{I}_{1\mathcal{CS},4}$	✔

Int	status
$\mathcal{I}^{(k,l)}_{12\mathcal{C},1}$	✔
$\mathcal{I}^{(k,l)}_{12\mathcal{C},2}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},3}$	✔
$\mathcal{I}^{(k,l)}_{12\mathcal{C},4}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},5}$	✘
$\mathcal{I}^{(k)}_{12\mathcal{C},6}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},7}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},8}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},9}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{C},10}$	✔

Int	status
$\mathcal{I}_{2\mathcal{S},1}$	✔
$\mathcal{I}_{2\mathcal{S},2}$	✔
$\mathcal{I}_{2\mathcal{S},3}$	✔
$\mathcal{I}_{2\mathcal{S},4}$	✔
$\mathcal{I}_{2\mathcal{S},5}$	✔
$\mathcal{I}_{2\mathcal{S},6}$	✔
$\mathcal{I}_{2\mathcal{S},7}$	✔
$\mathcal{I}_{2\mathcal{S},8}$	✔
$\mathcal{I}_{2\mathcal{S},9}$	✔
$\mathcal{I}_{2\mathcal{S},10}$	✔
$\mathcal{I}_{2\mathcal{S},11}$	✔
$\mathcal{I}_{2\mathcal{S},12}$	✔
$\mathcal{I}_{2\mathcal{S},13}$	✔
$\mathcal{I}_{2\mathcal{S},14}$	✔
$\mathcal{I}_{2\mathcal{S},15}$	✔
$\mathcal{I}_{2\mathcal{S},16}$	✔
$\mathcal{I}_{2\mathcal{S},17}$	✔
$\mathcal{I}_{2\mathcal{S},18}$	✔
$\mathcal{I}_{2\mathcal{S},19}$	✘
$\mathcal{I}_{2\mathcal{S},20}$	✔
$\mathcal{I}_{2\mathcal{S},21}$	✔
$\mathcal{I}_{2\mathcal{S},22}$	✔
$\mathcal{I}_{2\mathcal{S},23}$	✔

Int	status
$\mathcal{I}^{(k)}_{12\mathcal{S},1}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{S},2}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{S},3}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{S},4}$	✔
$\mathcal{I}^{(k)}_{12\mathcal{S},5}$	✔
$\mathcal{I}_{12\mathcal{S},6}$	✔
$\mathcal{I}_{12\mathcal{S},7}$	✔
$\mathcal{I}_{12\mathcal{S},8}$	✔
$\mathcal{I}_{12\mathcal{S},9}$	✔
$\mathcal{I}_{12\mathcal{S},10}$	✘
$\mathcal{I}_{12\mathcal{S},11}$	✘
$\mathcal{I}_{12\mathcal{S},12}$	✔
$\mathcal{I}_{12\mathcal{S},13}$	✔

Int	status
$\mathcal{I}^{(k)}_{12\mathcal{CS},1}$	✔
$\mathcal{I}_{12\mathcal{CS},2}$	✔
$\mathcal{I}_{12\mathcal{CS},3}$	✔

Int	status
$\mathcal{I}^{(j,k,l,m)}_{2\mathcal{C},1}$	✔
$\mathcal{I}^{(j,k,l,m)}_{2\mathcal{C},2}$	✔
$\mathcal{I}^{(j,k,l,m)}_{2\mathcal{C},3}$	✔
$\mathcal{I}^{(j,k,l,m)}_{2\mathcal{C},4}$	✘
$\mathcal{I}^{(j,k,l,m)}_{2\mathcal{C},5}$	✘
$\mathcal{I}^{(k,l)}_{2\mathcal{C},6}$	✔

Int	status
$\mathcal{I}^{(k)}_{2\mathcal{CS},1}$	✘
$\mathcal{I}^{(k)}_{2\mathcal{CS},2}$	✘
$\mathcal{I}^{(k)}_{2\mathcal{CS},3}$	✔
$\mathcal{I}^{(k)}_{2\mathcal{CS},4}$	✔
$\mathcal{I}^{(k)}_{2\mathcal{CS},5}$	✔

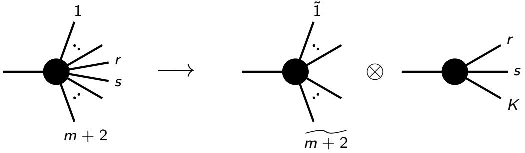
Phase space integrals - an example

Abelian double soft counterterm: among many others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2}$ we find

$$\left(\mathcal{S}_{rs}^{(0,0)}\right)^{\mathrm{ab}} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \sum_{i,k,j,l} \frac{1}{4} \frac{s_{ik}}{s_{ir}s_{kr}} \frac{s_{jl}}{s_{js}s_{ls}} |\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$
$$\times (1 - y_{rQ} - y_{sQ} + y_{rs})^{d_0' - m(1-\epsilon)} \Theta(y_0 - y_{rQ} - y_{sQ} + y_{rs})$$

The set of m momenta, $\{\tilde{p}\}$, is obtained by a momentum mapping which leads to an exact factorization of phase space

$$\{p\}_{m+2} \xrightarrow{\mathsf{S}_{rs}} \{\tilde{p}\} : \quad \mathrm{d}\phi_{m+2}(\{p\}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}; Q)[\mathrm{d}p_{2,m}^{(rs)}]$$



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$$\times (1 - y_{rQ} - y_{sQ} + y_{rs})^{d_0' - m(1-\epsilon)} \Theta(y_0 - y_{rQ} - y_{sQ} + y_{rs})$$

The set of m momenta, $\{\tilde{p}\}$, is obtained by a momentum mapping which leads to an exact factorization of phase space

$$\{p\}_{m+2} \xrightarrow{\mathsf{S}_{rs}} \{\tilde{p}\} : \quad \mathrm{d}\phi_{m+2}(\{p\}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}; Q)[\mathrm{d}p_{2,m}^{(rs)}]$$

Then we must compute

$$\int [\mathrm{d}p_{2,m}^{(rs)}] \left(\mathcal{S}_{rs}^{(0,0)}\right)^{\mathrm{ab}} \equiv \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i,k,j,l} [\mathrm{S}_{rs}^{(0)}]^{(i,k),(j,l)} |\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$

where $[\mathrm{S}_{rs}^{(0)}]^{(i,k),(j,l)} \equiv [\mathrm{S}_{rs}^{(0)}]^{(i,k),(j,l)}(p_i, p_k, p_j, p_l, \epsilon, y_0, d_0')$ is a kinematics dependent function.

Abelian double soft integral

For simplicity, consider the terms in the sum where $j = i$ and $l = k$: $[\mathrm{S}_{rs}^{(0)}]^{(i,k),(i,k)}$. Kinematical dependence is through $\cos\chi_{ik} = \measuredangle(p_i, p_k)$, we set $\cos\chi_{ik} = 1 - 2Y_{ik,Q}$, i.e., $Y_{ik,Q}$ is between zero and one.

Using angles and energies in the Q rest frame with some specific orientation to parametrize the factorized phase space measure, $[\mathrm{d}p_{2,m}^{(rs)}]$, we find that $[\mathrm{S}_{rs}^{(0)}]^{(i,k),(i,k)}$ is proportional to

$$\mathcal{I}_{2\mathcal{S},2}(Y_{ik,Q};\epsilon,y_0,d_0') = -\frac{4\Gamma^4(1-\epsilon)}{\pi\Gamma^2(1-\epsilon)}\frac{B_{y_0}(-2\epsilon,d_0')}{\epsilon}Y_{ik,Q}\int_0^{y_0}\mathrm{d}y\,y^{-1-2\epsilon}(1-y)^{d_0'-1+\epsilon}$$
$$\times\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1}\mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon}\big[f(\vartheta,\varphi;0)\big]^{-1}\big[f(\vartheta,\varphi;Y_{ik,Q})\big]^{-1}$$
$$\times\big[Y(y,\vartheta,\varphi;Y_{ik,Q})\big]^{-\epsilon}{}_2F_1\big(-\epsilon,-\epsilon,1-\epsilon,1-Y(y,\vartheta,\varphi;Y_{ik,Q})\big)$$

where

$$f(\vartheta,\varphi;Y_{ik,Q}) = 1 - 2\sqrt{Y_{ik,Q}(1-Y_{ik,Q})}\sin\vartheta\cos\varphi - (1-2Y_{ik,Q})\chi\cos\vartheta$$

$$Y(y,\vartheta,\varphi;\chi) = \frac{4(1-y)Y_{ik,Q}}{[2(1-y)+y\,f(\vartheta,\varphi;0)][2(1-y)+y\,f(\vartheta,\varphi;Y_{ik,Q})]}$$

Abelian double soft integral

This integral is equal to $(y_0 = 1,\ d_0' = 3 - 3\epsilon)$

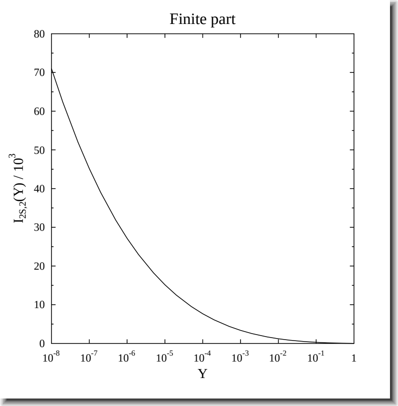
$$
\begin{aligned}
&\mathcal{I}_{2\mathcal{S},2}(Y;\epsilon,1,3-3\epsilon) = \\
&\quad = \frac{1}{2\epsilon^4} - \frac{1}{\epsilon^3}\Big[\ln(Y) - 3\Big] + \frac{1}{\epsilon^2}\Bigg[2\,\mathrm{Li}_2(1-Y) + \ln^2(Y) - \pi^2 - \bigg(\frac{2}{1-Y} \\
&\quad - \frac{1}{2(1-Y)^2} + \frac{9}{2}\bigg)\ln(Y) + \frac{1}{2(1-Y)} + 16\Bigg] + \frac{1}{\epsilon}\Bigg[\frac{5}{3}\bigg(\frac{18\,\mathrm{Li}_3(1-Y)}{5} + \frac{6\,\mathrm{Li}_3(Y)}{5} \\
&\quad - \frac{6\,\mathrm{Li}_2(1-Y)\ln(Y)}{5} - \frac{2}{5}\ln^3(Y) + \frac{3}{5}\ln(1-Y)\ln^2(Y) + \pi^2\ln(Y) - \frac{78\zeta(3)}{5}\bigg) \\
&\quad + \bigg(\frac{3}{1-Y} - \frac{3}{4(1-Y)^2} + \frac{15}{4}\bigg)\Big(2\,\mathrm{Li}_2(1-Y) + \ln^2(Y)\Big) - 6\pi^2 - \bigg(\frac{27}{2(1-Y)} \\
&\quad - \frac{13}{4(1-Y)^2} + \frac{91}{4}\bigg)\ln(Y) + \frac{19}{4(1-Y)} + \frac{163}{2}\Bigg] + \mathrm{O}(\epsilon^0)
\end{aligned}
$$

Note the $Y \to 1$ limit is finite

$$
\lim_{Y\to 1}\mathcal{I}_{2\mathcal{S},2}(Y;\epsilon,1,3-3\epsilon) = \frac{1}{2\epsilon^4} + \frac{3}{\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{71}{4} - \pi^2\right) + \frac{1}{\epsilon}\left(\frac{393}{4} - 6\pi^2 - 24\zeta(3)\right) + \mathrm{O}(\epsilon^0)
$$

Abelian double soft integral

Finite term is computed numerically $(y_0 = 1,\ d_0' = 3 - 3\epsilon)$



Analytic vs. numeric

As a matter of principle

- A rigorous proof of cancellation of IR poles requires the poles of integrated counterterms in analytic form.

However

- An actual implementation needs numbers for the finite parts of the integrated counterterms.
- These finite parts are smooth functions of kinematic variables.

Hence

- Numerical forms of the finite parts are sufficient for practical purposes. The final results can be conveniently given by interpolating tables or approximating functions computed once and for all.
- In particular, suitable approximating functions may be obtained by fitting.

Example of approximation by fitting

Doubly-unresolved double-collinear master integral $\mathcal{I}_{2\mathcal{C},6}(x_{ir}, x_{js}; \epsilon, 1, 3-3\epsilon, k, l)$

$$\mathcal{I}_{2\mathcal{C},6}(x_{ir}, x_{js}; \epsilon, \alpha_0, d_0; k, l) = x_{ir} x_{js} \int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta \int_0^1 \mathrm{d}v\,\mathrm{d}u\,\alpha^{-1-\epsilon}\beta^{-1-\epsilon}(1-\alpha-\beta)^{2d_0-2(1-\epsilon)}$$
$$\times [\alpha+(1-\alpha-\beta)x_{ir}]^{-1-\epsilon}[\beta+(1-\alpha-\beta)x_{js}]^{-1-\epsilon} v^{-\epsilon}(1-v)^{-\epsilon} u^{-\epsilon}(1-u)^{-\epsilon}$$
$$\times \left(\frac{\alpha+(1-\alpha-\beta)x_{ir} v}{2\alpha+(1-\alpha-\beta)x_{ir}}\right)^k \left(\frac{\beta+(1-\alpha-\beta)x_{js} u}{2\beta+(1-\alpha-\beta)x_{js}}\right)^l \Theta(\alpha_0-\alpha-\beta)$$

Example of approximation by fitting

Doubly-unresolved double-collinear master integral $\mathcal{I}_{2\mathcal{C},6}(x_{ir}, x_{js}; \epsilon, 1, 3-3\epsilon, k, l)$

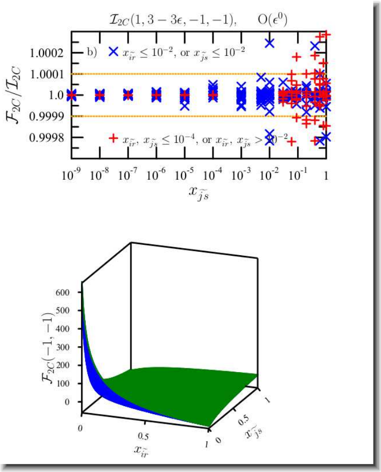
- poles (up to $\mathrm{O}(\epsilon^{-4})$) extracted via sector decomposition
- numerical values of pole coefficients computed for a 17×17 grid with precision of $\sim 10^{-7}$
- define three regions (note: result is symmetric in x_{ir}, x_{js})
  - asymptotic: $x_{ir}, x_{js} < 10^{-4}$
  - non-asymptotic: $x_{ir}, x_{js} > 10^{-2}$
  - border: $x_{ir} < 10^{-2}$ or $x_{js} < 10^{-2}$
- in each region, fit with ansatz

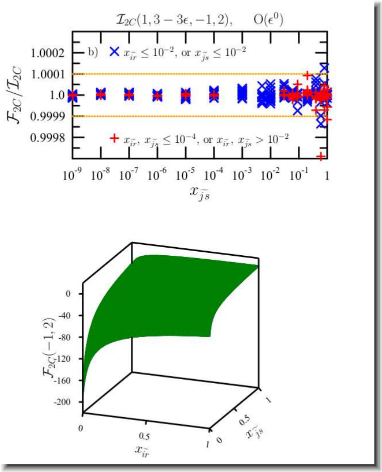
$$\mathcal{F}(x_{ir}, x_{js}) = \sum_{p_i, l_i} C_{m;p_1,p_2;l_1,l_2}(x_{ir}^{p_1} x_{js}^{p_2})(\log^{l_1}(x_{ir}) \log^{l_2}(x_{js}))$$

where $p_1 + p_2 \leq m$ with m a free parameter, while $l_1 + l_2 \leq n$ and n is predicted

Example of approximation by fitting

Doubly-unresolved double-collinear master integral $\mathcal{I}_{2\mathcal{C},6}(x_{ir}, x_{js}; \epsilon, 1, 3-3\epsilon, k, l)$





Methods of integration - angular integrals

Consider the d dimensional angular integral with n denominators (GS 2011)

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1\cdot q)^{j_1}\cdots(p_n\cdot q)^{j_n}}$$

This admits the following Mellin-Barnes representation $(j = j_1 + \ldots + j_n)$

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^n \Gamma(j_k)\Gamma(2-j-2\epsilon)}$$
$$\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^n\prod_{l=k}^n \frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})\,(v_{kl})^{z_{kl}}\right]\left[\prod_{k=1}^n \Gamma(j_k+z_k)\right]\Gamma(1-j-\epsilon-z)\,.$$

where $v_{kl} = \frac{p_k\cdot p_l}{2}$ for $k \neq l$ and $v_{kk} = \frac{p_k^2}{4}$ while

$$z = \sum_{k=1}^n\sum_{l=k}^n z_{kl} \qquad \text{and} \qquad z_k = \sum_{l=1}^k z_{lk} + \sum_{l=k}^n z_{kl}\,.$$

Methods of integration - MB to parametric integrals

Basic idea is to express products of gamma functions as real integrals

$$
\begin{aligned}
I &= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \cdots \Gamma[a+z_1+z_2]\Gamma[b-z_1-z_2] \cdots v_1^{z_1} v_2^{z_2} \\
&= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \cdots \Gamma[a+b] \int_0^1 \mathrm{d}t\, t^{a-1+z_1+z_2} (1-t)^{b-1-z_1-z_2} \cdots v_1^{z_1} v_2^{z_2}
\end{aligned}
$$

if $\Re(a+z_1+z_2) > 0$ and $\Re(b-z_1-z_2) > 0$ so the t integral converges

Eliminate enough gamma functions to be able to perform the MB integrals

- can eliminate all gamma functions for real integrals, then use

$$
\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z}{2\pi\mathrm{i}} v^z = \delta(1-v)\,, \qquad v > 0
$$

- For multidimensional MB integrals, sometimes it is more useful to eliminate just the gamma functions that couple the MB integrations. This turns the multidimensional MB integral into products of 1d MB integrals.

After solving the remaining MB integrals, we get the desired parametric representation.

Methods of integration - symbolic integration

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\ldots\,\frac{P(x,t_1,t_2,t_3,\ldots)}{Q(x,t_1,t_2,t_3,\ldots)}$$

The t_1 integration

- assuming the denominator is a product of factors all linear in t_1, after partial fractioning, we will need to compute

$$\int_0^1 \frac{\mathrm{d}t_1}{t_1^n}\,, \qquad \int_0^1 \frac{\mathrm{d}t_1}{[t_1 - a(x,t_2,\ldots)]^n}\,,$$

- $n = 1$ is non-trivial

$$\int \frac{\mathrm{d}t_1}{t_1} = \ln t_1\,, \qquad \int \frac{\mathrm{d}t_1}{t_1 - a(x,t_2,\ldots)} = \ln[t_1 - a(x,t_2,\ldots)]$$

- e.g., we have

$$\int_0^1 \frac{\mathrm{d}t_1}{t_1 - a(x,t_2,\ldots)} = \ln\left[1 - \frac{1}{a(x,t_2,\ldots)}\right]$$

- this is elementary, although there is some fine print for definite integrals

Methods of integration - symbolic integration

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\ldots\,\frac{P(x,t_1,t_2,t_3,\ldots)}{Q(x,t_1,t_2,t_3,\ldots)}$$

The t_2 integration

- assuming the new denominator is a product of factors all linear in t_2, after partial fractioning — aside from the integrals we already encountered — we will have to compute

$$\int_0^1 \frac{\mathrm{d}t_2}{t_2^n} \ln\left[1-\frac{1}{a(x,t_2,\ldots)}\right], \qquad \int_0^1 \frac{\mathrm{d}t_2}{[t_2-b(x,t_3,\ldots)]^n} \ln\left[1-\frac{1}{a(x,t_2,\ldots)}\right],$$

- if $a(x,t_2,\ldots)$ is also linear in t_2, we can use the functional identities for the logarithm $[\ln(ab)=\ln a+\ln b,\ \ln(1/a)=-\ln a]$ to write

$$\ln\left[1-\frac{1}{a(x,t_2,\ldots)}\right] = \ln[a_1(x,t_3,\ldots)-t_2] - \ln[a_2(x,t_3,\ldots)-t_2]$$

- again, $n=1$ is non-trivial

$$\int \frac{\mathrm{d}t_2}{t_2}\ln t_2 = \frac{1}{2}\ln^2(t_2), \qquad \int \frac{\mathrm{d}t_2}{t_2}\ln(1-t_2) = -\,\mathrm{Li}_2(t_2)$$

Methods of integration - symbolic integration

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\ldots\,\frac{P(x,t_1,t_2,t_3,\ldots)}{Q(x,t_1,t_2,t_3,\ldots)}$$

The t_2 integration (cont.)

- e.g., we have

$$\int_0^1 \frac{\mathrm{d}t_2}{t_2 - b(x,t_3,\ldots)} \ln(t_2) = \mathrm{Li}_2\left[\frac{1}{b(x,t_3,\ldots)}\right]$$

Methods of integration - symbolic integration

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1 \, \mathrm{d}t_2 \, \mathrm{d}t_3 \, \ldots \frac{P(x, t_1, t_2, t_3, \ldots)}{Q(x, t_1, t_2, t_3, \ldots)}$$

Before going to the t_3 integration, notice

1. at each step, we needed to introduce a new transcendental function, ln, Li_2
2. we needed to know the functional identities for ln to proceed

Methods of integration - symbolic integration

Assume P and Q are polynomials and the following integral converges

$$I(x) = \int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\ldots\,\frac{P(x,t_1,t_2,t_3,\ldots)}{Q(x,t_1,t_2,t_3,\ldots)}$$

The t_3 integration

- assuming the new denominator is a product of factors all linear in t_3, after partial fractioning — aside from the integrals we already encountered — we will have to compute

$$\int_0^1 \frac{\mathrm{d}t_3}{t_3^n}\,\mathrm{Li}_2\left[\frac{p(x,t_3,\ldots)}{q(x,t_3,\ldots)}\right], \qquad \int_0^1 \frac{\mathrm{d}t_3}{[t_3 - c(x,t_4,\ldots)]^n}\,\mathrm{Li}_2\left[\frac{p(x,t_3,\ldots)}{q(x,t_3,\ldots)}\right],$$

- will need to introduce new transcendental functions $\Rightarrow$ multiple polylogs
- will need to use the functional identities for Li_2 to reduce to some standard form $\Rightarrow$ symbols, coporoducts, Hopf algebra of multiple polylogs

Multiple polylogarithms

The appropriate generalization of log and classical polylogs (Goncharov 1998, 2001)

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{\mathrm{d}t}{t-a_1} G(a_2,\ldots,a_n;t) \qquad \text{with} \qquad G(z)=1$$

$$G(\underbrace{0,\ldots,0}_{n};z) = \frac{1}{n!}\ln^n(z)$$

Logarithms and classical polylogs are special cases, e.g.,

$$G(\underbrace{a,\ldots,a}_{n};z) = \frac{1}{n!}\ln^n\left(1-\frac{z}{a}\right), \qquad G(\underbrace{0,\ldots,0}_{n-1},a;z) = -\mathrm{Li}_n\left(\frac{z}{a}\right)$$

Functional relations among Gs

- Problem: after the $(n-1)$-st step of integration, the n-th variable can appear in the a_i

$$\int \frac{\mathrm{d}t_n}{t_n-b} G(a_1(t_n,\ldots),\ldots,a_{n-1}(t_n,\ldots);z(t_n,\ldots))$$

  Must reduce to 'canonical' form, where t_n is only in the last entry.

- Unfortunately the functional equations among Gs that would be needed to do this are often unknown and need to be derived.

Symbols, coproducts

Symbols are a tool for obtaining functional equations among Gs

(Goncharov 2009; Goncharov, Spradlin, Vergu, Volovich 2010; Duhr, Gangl, Rhodes 2011)

- The symbol is a way of associating to a multiple polylog a tensor in a certain tensor space.

$$\mathcal{S}(G(a_{n-1},\ldots,a_1;a_n)) = \sum_{i=1}^{n-1} \mathcal{S}(G(a_{n-1},\ldots,a_{i-1},a_{i+1},\ldots,a_1;a_n)) \otimes \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}}\right)$$

e.g.,

$$\mathcal{S}\left(\frac{1}{n!}\ln^n(z)\right) = \underbrace{z\otimes\ldots\otimes z}_{n \text{ times}}, \qquad \mathcal{S}(\mathrm{Li}_n(z)) = -(1-z)\otimes\underbrace{z\otimes\ldots\otimes z}_{(n-1) \text{ times}}$$

- Functional equations between multiple polylogs become algebraic equations between tensors.

The idea of symbols can be refined based on the Hofp algebra structure of multiple polylogs $\Rightarrow$ coproduct (Duhr 2012)

- With these refinements one can build algorithms to reduce multiple polylogs to ‘canonical’ form.

Integrated approximate cross sections

Recall the NNLO correction is a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

Integrated approximate cross sections

Recall the NNLO correction is a sum of three terms

$$\sigma^{\rm NNLO} = \sigma^{\rm RR}_{m+2} + \sigma^{\rm RV}_{m+1} + \sigma^{\rm VV}_{m} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m}$$

each integrable in four dimensions

$$\sigma^{\rm NNLO}_{m+2} = \int_{m+2} \Big\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \Big[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \Big] \Big\}$$

$$\sigma^{\rm NNLO}_{m+1} = \int_{m+1} \Big\{ \Big[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big] J_{m+1} - \Big[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] J_m \Big\}$$

$$\sigma^{\rm NNLO}_{m} = \int_{m} \Big\{ {\rm d}\sigma^{\rm VV}_{m} + \int_2 \Big[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \Big] + \int_1 \Big[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] \Big\} J_m$$

Integrated approximate cross sections

- After summing over unobserved flavors, all integrated approximate cross sections can be written as products (in color space) of various insertion operators with lower point cross sections.
- Can be computed once and for all (though admittedly lots of tedious work).

Integrated approximate cross sections - an example

Doubly unresolved

$$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes I_2^{(0)}(\{p\}_m;\epsilon)$$

- structure of insertion operator in color $\otimes$ flavor space

$$I_2^{(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ \sum_i \left[\mathrm{C}_{2,f_i}^{(0)} T_i^2 + \sum_k \mathrm{C}_{2,f_i f_k}^{(0)} T_k^2\right] T_i^2 + \sum_{j,l} \left[\mathrm{S}_2^{(0),(j,l)} C_\mathrm{A} + \sum_i \mathrm{CS}_{2,f_i}^{(0),(j,l)} T_i^2\right] T_j T_l + \sum_{i,k,j,l} \mathrm{S}_2^{(0),(i,k)(j,l)} \{T_i T_k, T_j T_l\} \Bigg\}$$

- $\mathrm{C}_{2,f_i}^{(0)}$, $\mathrm{C}_{2,f_i f_k}^{(0)}$, $\mathrm{S}_2^{(0),(j,l)}$, $\mathrm{CS}_{2,f_i}^{(0),(j,l)}$ and $\mathrm{S}_2^{(0),(i,k)(j,l)}$ are kinematical functions with poles up to $\mathrm{O}(\epsilon^{-4})$
- kinematical dependence through

$$x_i = y_{iQ} \equiv \frac{2p_i \cdot Q}{Q^2} \qquad \text{and} \qquad Y_{ik,Q} = \frac{y_{ik}}{y_{iQ} y_{kQ}}$$

Integrated approximate cross sections - an example

Doubly unresolved

$$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes I_2^{(0)}(\{p\}_m;\epsilon)$$

- e.g., $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)

$$I_2^{(0)}(p_1,p_2,p_3;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ \frac{8C_{\mathrm{F}}^2 + 10 C_{\mathrm{A}} C_{\mathrm{F}} + 3C_{\mathrm{A}}^2}{4\epsilon^4} + \Bigg[6C_{\mathrm{F}}^2 + \frac{109 C_{\mathrm{A}} C_{\mathrm{F}}}{12}$$
$$+ \frac{77C_{\mathrm{A}}^2}{24} - \frac{7C_{\mathrm{F}} T_{\mathrm{R}} n_{\mathrm{f}}}{3} - \frac{C_{\mathrm{A}} T_{\mathrm{R}} n_{\mathrm{f}}}{2} - \left(4C_{\mathrm{F}}^2 + C_{\mathrm{A}} C_{\mathrm{F}} - \frac{3C_{\mathrm{A}}^2}{2}\right) \ln y_{12}$$
$$- \left(2C_{\mathrm{A}} C_{\mathrm{F}} + \frac{3C_{\mathrm{A}}^2}{2}\right)(\ln y_{13} + \ln y_{23})\Bigg] \frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2}) \Bigg\}$$

- notice x and Y dependence combine to produce just y_{ik} dependence, as expected

Integrated approximate cross sections - an example

Doubly unresolved

$$\int_2 \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} = \mathrm{d}\sigma^{\mathrm{B}}_{m} \otimes I_2^{(0)}(\{p\}_m;\epsilon)$$

- e.g., $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)

$$I_2^{(0)}(p_1,p_2,p_3;\epsilon) = \left[\frac{\alpha_\mathrm{s}}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2\left\{\frac{8C_\mathrm{F}^2+10C_\mathrm{A}C_\mathrm{F}+3C_\mathrm{A}^2}{4\epsilon^4}+\left[6C_\mathrm{F}^2+\frac{109C_\mathrm{A}C_\mathrm{F}}{12}\right.\right.$$
$$+\frac{77C_\mathrm{A}^2}{24}-\frac{7C_\mathrm{F}T_\mathrm{R}n_\mathrm{f}}{3}-\frac{C_\mathrm{A}T_\mathrm{R}n_\mathrm{f}}{2}-\left(4C_\mathrm{F}^2+C_\mathrm{A}C_\mathrm{F}-\frac{3C_\mathrm{A}^2}{2}\right)\ln y_{12}$$
$$\left.\left.-\left(2C_\mathrm{A}C_\mathrm{F}+\frac{3C_\mathrm{A}^2}{2}\right)(\ln y_{13}+\ln y_{23})\right]\frac{1}{\epsilon^3}+\mathrm{O}(\epsilon^{-2})\right\}$$

- notice x and Y dependence combine to produce just y_{ik} dependence, as expected

Insertion operators

Color and flavor structure of all insertion operators known

Analytic computation of poles almost complete

- all MIs known analytically up to and including $\mathrm{O}(\epsilon^{-2})$
- $1/\epsilon$ parts of ~ 10 of $\mathcal{O}(300)$ integrals to be finished

Finite parts computed numerically

- final results in the form of approximating functions obtained by fitting

Present status

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma_{m+2}^{\mathrm{RR}} + \sigma_{m+1}^{\mathrm{RV}} + \sigma_{m}^{\mathrm{VV}} = \sigma_{m+2}^{\mathrm{NNLO}} + \sigma_{m+1}^{\mathrm{NNLO}} + \sigma_{m}^{\mathrm{NNLO}}$$

Each integrable in four dimensions

$$\sigma_{m+2}^{\mathrm{NNLO}} = \int_{m+2} \left\{ \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} J_m - \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} J_{m+1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} J_m \right] \right\}$$

$$\sigma_{m+1}^{\mathrm{NNLO}} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} + \int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right] J_{m+1} - \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma_{m}^{\mathrm{NNLO}} = \int_{m} \left\{ \mathrm{d}\sigma_{m}^{\mathrm{VV}} + \int_2 \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} \right] + \int_1 \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

- ✔ unintegrated doubly real counterterms (for final state singularities)
- ✔ unintegrated real virtual counterterms (for final state singularities)
- ✔ tree-level and one-loop singly unresolved integrals [to $\mathrm{O}(\epsilon^{-1})$]
- ➠ tree-level iterated singly unresolved integrals [$\mathrm{O}(\epsilon^{-1})$ in progress]
- ➠ tree-level doubly unresolved integrals [$\mathrm{O}(\epsilon^{-1})$ in progress]

Extension to hadronic initial states

Essential elements of the construction unchanged

- write the IR factorization formulae for all limits — including initial state splittings — in such a form that their overlap structure can be disentangled
- extend the formulae relevant for initial state singularities over full phase space — requires new momentum mappings
- subtraction terms for final state splittings fine as they stand
- integrate new subtraction terms analytically and numerically

Basic steps clear, but admittedly lots of tedious work

Conclusions

NNLO is the new precision frontier

Two bottlenecks

1. can we compute the relevant (2-loop) amplitudes?
2. if yes, can we use those to compute cross sections?

Subtraction is the traditional solution to 2. We have set up

- general, explicit, local subtraction scheme for computing NNLO corrections in QCD
- construction of subtraction terms based on IR limit formulae
- analytic integration of subtraction terms is feasible with modern integration techniques and is almost finished
- the scheme is worked out in full detail for processes with no colored particles in the initial state

Extension to hadron initiated processes conceptually clear, work in progress