Robust Design of Low Emittance Rings

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Outline

- Equations of Motion.
- The Quadratic Hamiltonian.
- Colliders: The FODO Cell.
- Chasman-Green Lattices: The OFODOFO Cell.
- Global Properties.
- Scaling Properties.
- LEGO[1] - "Rapid Prototyping" of Straw Man Lattices:
  10-BA×37, C=1,100 m, 7-BA×20, C=500 m, 4-BA×20, C=300 m.
- The Minimum Emittance (ME) Cell: A Self-Consistent Approach.
- An ME Cell for a 3°, L_b=1.0 m Bend: The MAX-IV Unit Cell Unfolded.

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1. LEGO: "LEg GOdt" -> "play well". By chance, Latin: "to put together". -> Synthesis vs. reductionism. The company was founded by a Danish Master Carpenter & Joiner Ole Christiensen 1932. Note, strong focusing was invented by a Greek Elevator Engineer: N. Christofilos 1950; US Patent 2,736,799.

References

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[3] Farvacque, N. Carmignani, J. Chavanne, A. Franchi, G. Le Bec, S. Liuzzo, B. Nash, T. Perron, P. Raimondi "A Low Emittance Lattice for the ESRF" IPAC 2013.

[4] S. Leeman, Å. Andersson, M. Eriksson, L.-J. Lindgren, E. Wallén, J. Bengtsson, A. Streun "Beam Dynamics and Expected Performance of Sweden's New Storage-Ring Light Source: MAX-IV" PR STAB 12, 120701 (2009).

[5] ICFA Beam Dynamics Newsletter 57, Ed. Y.-H. Cai (2012).

[6] L. Teng "Minimum Emittance Lattice for Synchrotron Radiation Storage Rings" ANL LS-17 (1985).

[7] R. Helm, H. Wiedemann "Emittance in a FODO-Cell Lattice SLAC-PEP Note 303 (1979).

[8] R. Helm, M. Lee, P. Morton "Evaluation of Synchrotron Radiation Integrals" PAC 1973.

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[10] M. Sands "The Physics of Electron Storage Rings" SLAC-121 (1970).

[11] R. Chasman, G. Green "Preliminary Design of a Dedicated Synchrotron Radiation Facility" PAC 1975.

[12] E. Courant, M. Livingston, H. Snyder, J. Blewett "Origin of the 'Strong-Focusing' Principle" Phys. Rev. 91, 202-203 (1953).

Equations of Motion

The relativistic Hamiltonian for a charged particle in and external e-m field in the comoving frame used for modeling particle accelerator is

$$H(\bar{x};s) = -\left(\frac{q}{p_o}A_s(s) + \sqrt{1 - \frac{p_\tau}{\beta} + p_\tau^2 - \left(p_x - \frac{q}{p_o}A_x(s)\right)^2 - \left(p_y - \frac{q}{p_o}A_y(s)\right)^2}\right)$$

with the phase space coordinates

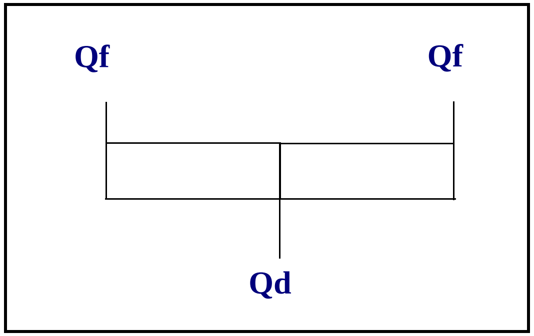
$$\bar{x} \equiv [x, p_x, y, p_y, c_0 t, p_\tau], \qquad p_\tau \equiv \frac{E - E_0}{E_0} \to \delta, v \to c_0.$$

The expanded (small angle) quadratic Hamiltonian describing the linear optics is

$$H_2(\bar{x};s) = \frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{b_2(s)}{2}(x^2 - y^2) + \frac{1}{2\rho^2(s)}x^2 - \frac{1}{\rho(s)}x\delta.$$

Colliders: The FODO Cell

The optics for a FODO cell



is well known (for $k_{\mathrm{Qf}} = -k_{\mathrm{Qd}}$)

$$\beta_{x\max} = \frac{2\rho\sin(\phi)(1 + k\rho\sin(\phi))}{\sin(\mu_x)} \to \frac{2L_{\mathrm{b}}(1 + kL_{\mathrm{b}})}{\sin(\mu_x)} + O(\phi^3) = \frac{2L_{\mathrm{b}}\left(1 + \sin\left(\frac{\mu_x}{2}\right)\right)}{\sin(\mu_x)} + O(\phi^3),$$

$$\eta_{x\max} = \frac{2\rho\sin^2\left(\frac{\phi}{2}\right)(2 + k\rho(2\sin(\phi)))}{\sin^2\left(\frac{\mu_x}{2}\right)} \to \frac{\rho\phi^2\left(1 + \frac{1}{2}kL_{\mathrm{b}}\right)}{\sin^2\left(\frac{\mu_x}{2}\right)} + O(\phi^4) = \frac{L_{\mathrm{b}}^2\left(1 + \frac{1}{2}\sin\left(\frac{\mu_x}{2}\right)\right)}{\rho\sin^2\left(\frac{\mu_x}{2}\right)} + O(\phi^4)$$

where we have used $\sin\left(\frac{\mu_x}{2}\right) = \pm\frac{kL_{\mathrm{b}}}{2}$.

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Colliders: The FODO Cell (cont.)

If it was considered being used for e.g. a damping ring, the hor. emittance would be

$$\varepsilon_x = C_q \gamma^2 \frac{\langle H/|\rho|^3 \rangle}{J_x \langle 1/\rho^2 \rangle},$$

where $\mathcal{H}$ is the linear dispersion action

$$H \equiv \|\tilde{\eta}\| = \tilde{\eta}^T \tilde{\eta}, \qquad \tilde{\eta} \equiv A^{-1} \bar{\eta}, \qquad A^{-1} = \begin{bmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{bmatrix},$$

which to leading order is

$$\langle H(s) \rangle = \frac{\rho \phi_b^3}{\sin^3\left(\frac{\mu_x}{2}\right)} \left[1 - \frac{(kL_b)^2}{16}\right] + O(\phi_b^4)$$

and has a min for

$$\Delta\mu_x = 180°, \qquad kL_b = 2, \qquad \langle H(s) \rangle = \rho \phi_b^3.$$

Colliders: The FODO Cell (cont.)

However, this is a leading order result. An exercise in algebra leads to the rather neat result [6-7]

$$\langle \mathcal{H}(s) \rangle = \frac{\rho\phi^3}{\sin^3\left(\frac{\mu_x}{2}\right)\cos\left(\frac{\mu_x}{2}\right)}\left(1 - \frac{3}{4}\sin^2\left(\frac{\mu_x}{2}\right) + \frac{1}{60}\sin^4\left(\frac{\mu_x}{2}\right)\right) + O(\phi^5)$$

which has a min for

$$\mu_x = 2\,\mathrm{atan}\left(\frac{1}{2}\sqrt{\frac{75 + 3\sqrt{1905}}{8}}\right) \approx 0.38 \cdot 2\pi, \qquad \langle \mathcal{H}(s) \rangle_{\min} = \frac{1}{60}\sqrt{\frac{16075 + 381\sqrt{1905}}{6}} \approx 1.23 \cdot \rho\phi_b^3 .$$

However, a formula is (to our knowledge) not provided for the linear chromaticity. One can show that [1]

$$\xi_x = -\frac{1}{\pi}\tan\left(\frac{\mu_x}{2}\right)$$

and it follows that

$$\xi_x = \begin{cases} -1/\pi, & \mu_x = 90° \\ -0.81, & \mu_x = 137° \end{cases} .$$

Chasman-Green Lattices: The OFODOFO Cell

After the pioneering papers from 1970-1985 [6-10], there is no lack of analysis, and papers, on how to control the linear dispersion action $\mathcal{H}$ for Chasman-Green type lattices; aka TME ("Theoretical" Minimum Emittance) Cell.

However, the concept/theory is fundamentally flawed because it ignores control of linear chromaticity; a leading order effect.

Hence, the analysis is inconsistent and their results incomplete. A reflection, perhaps, of the fact that a consistent treatment is nontrivial.

In fact, already Teng noted that [6] (p. 18):

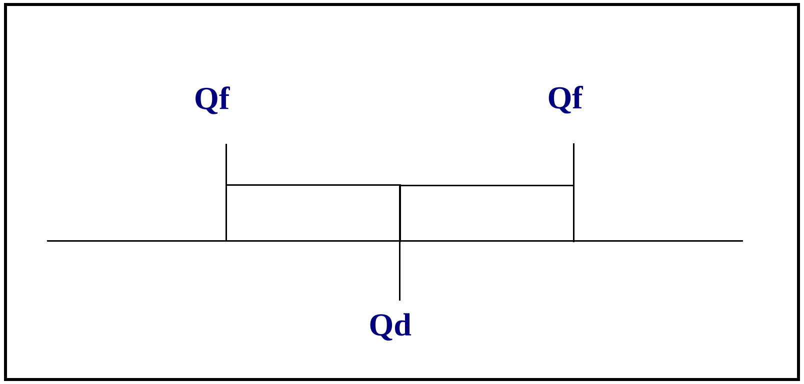
"This theoretical minimum should be at least a factor 2 smaller than the desired emittance because when one gets to the later steps, it is unlikely that one can attain and then maintain optimum values for all the parameters."

To remedy the situation we will introduce the OFODOFO cell. Conceptually, the "atom" (reductionist) for a Chasman-Green type lattice such that it is:

1. **simple enough to provide useful formula from analytic methods,**
2. **and, yet, complex enough, so that when systematically synthesized, it yields realistic designs/lattices, like molecules,**

although the latter are complex dynamical *systems*.

The OFODOFO Cell (cont.)



The linear dispersion action has a min for

$$\eta_{xc} = \frac{L_b\phi}{24}, \qquad \eta'_{xc} = 0,$$

$$\alpha_{xc} = 0, \qquad \beta_{xc} = \frac{L_b}{2\sqrt{15}\sqrt{1-\frac{3}{8}k_dL_b+\frac{3}{80}k_d^2L_b^2}},$$

$$\langle\mathcal{H}(s)\rangle_{min} = \frac{L_b\phi^2}{12\sqrt{15}}\sqrt{1-\frac{3}{8}k_dL_b+\frac{3}{80}k_d^2L_b^2}$$

The OFODOFO Cell (cont.)

One can also show that

$$\mu_x = 2\,\mathrm{atan}\left(-\frac{3}{\sqrt{15}}\frac{1-\frac{1}{4}k_d L_b}{\sqrt{1-\frac{3}{8}k_d L_b+\frac{3}{80}k_d^2 L_b^2}}\right)$$

and

$$\xi_x = -\frac{12}{8\pi\sqrt{15}}\frac{1-\frac{1464}{3072}k_d L_b+\frac{254}{3072}k_d^2 L_b^2-\frac{13}{3072}k_d^3 L_b^3}{\left(1-\frac{1}{8}k_d L_b\right)\sqrt{1-\frac{30}{80}k_d L_b+\frac{3}{80}k_d^2 L_b^2}},$$

$$\xi_y = (43008 + 24672 k_d L_b - 25200 k_d^2 L_b^2 + 6330 k_d^3 L_b^3 - 609 k_d^4 L_b^4 + 20 k_d^5 L_b^5)$$

$$/(4\pi \mathrm{sqrt}(-242810880 - 91594752 k_d L_b + 163349504 k_d^2 L_b^2 - 15035392 k_d^3 L_b^3 - 34034880 k_d^4 L_b^4$$

$$17085840 k_d^5 L_b^5 - 3894596 k_d^6 L_b^6 + 500032 k_d^7 L_b^7 - 37141 k_d^8 L_b^8 + 1490 k_d^9 L_b^9 - 25 k_d^{10} L_b^{10})$$

The OFODOFO Cell: Scaling Laws (cont.)

The hor linear chromaticity is essentially flat and vertical has a min for

$$k_d L_b \approx -1.24088$$

which gives

$$k_d \approx \frac{-1.24088}{L_b}, \qquad L_1 \approx 0.88121 \cdot L_b, \qquad k_f \approx \frac{2.86572}{L_b}$$

with

$$\beta_{yc} \approx 13.1 \cdot L_b, \qquad \nu_x \approx 0.781, \qquad \nu_y \approx 0.220, \qquad \xi_x \approx -1.195, \qquad \xi_y \approx -1.718$$

and the total cell length is

$$L = (1 + 2L_1)L_b \approx 2.76242 \cdot L_b.$$

For an example, we may choose

$$\phi_b = 3°, \qquad L_b = 1.0 \text{ m}, \qquad \rho_b = \frac{L_b}{\phi_b} \approx 19.1 \text{ m}$$

and scale it by a factor 0.1.

Interestingly, the linear chromaticity is not increased.

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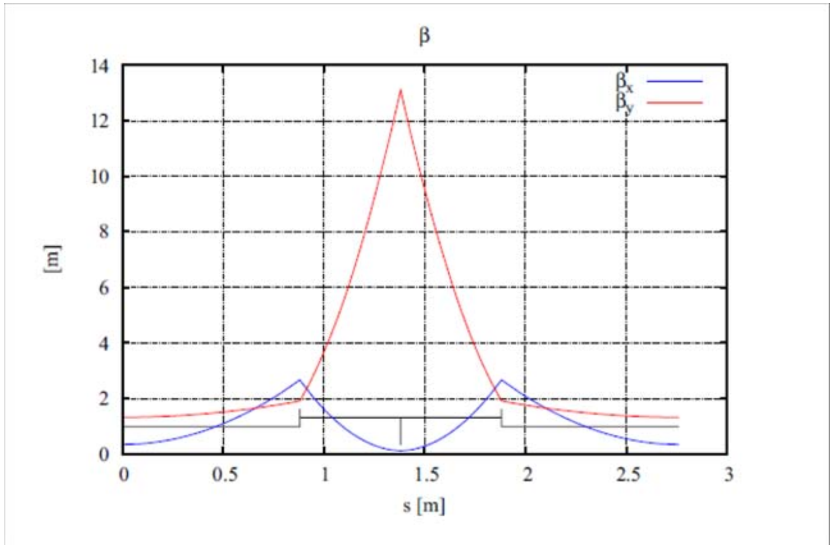
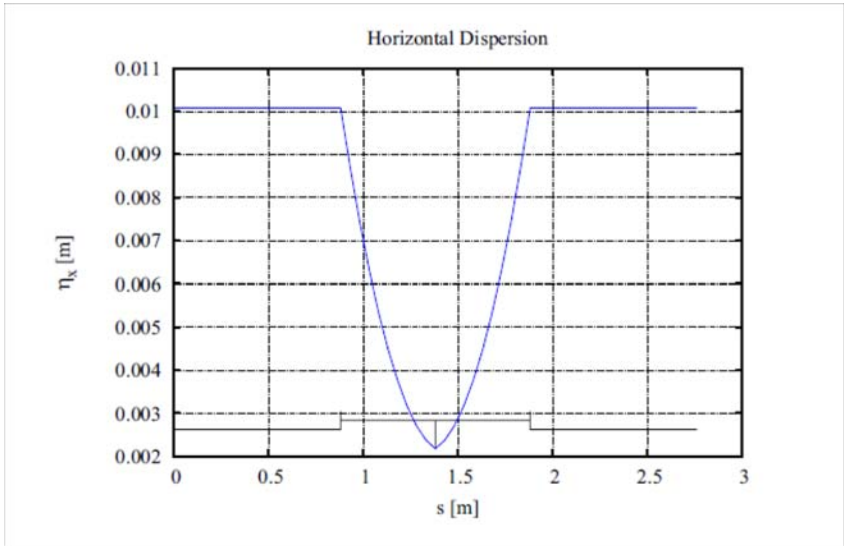
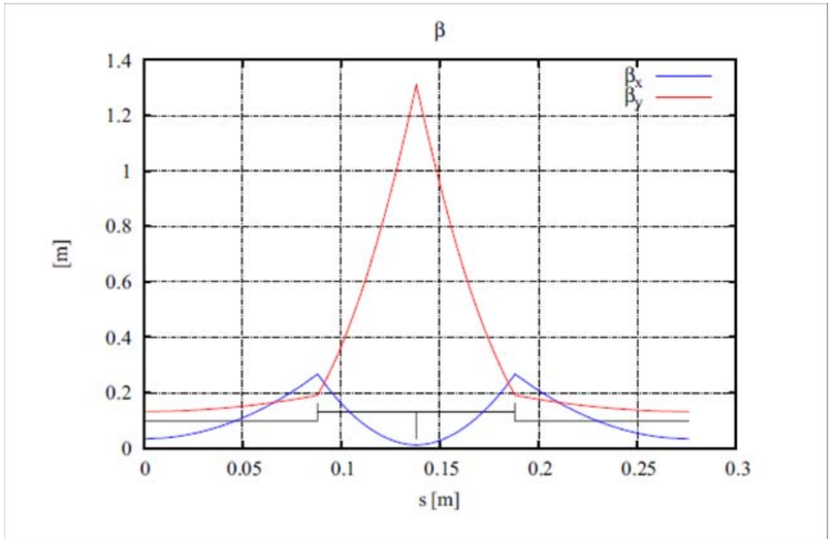
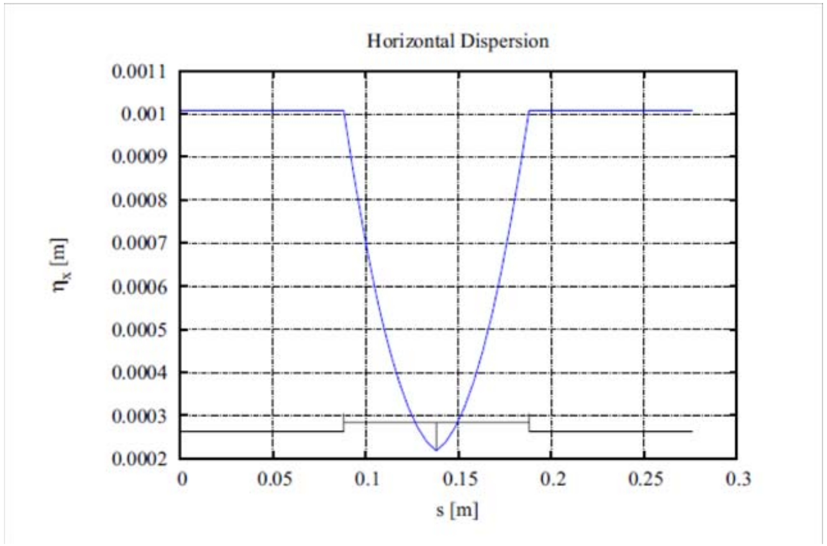
The OFODOFO Cell (cont.)









Synthesis of ME Cells

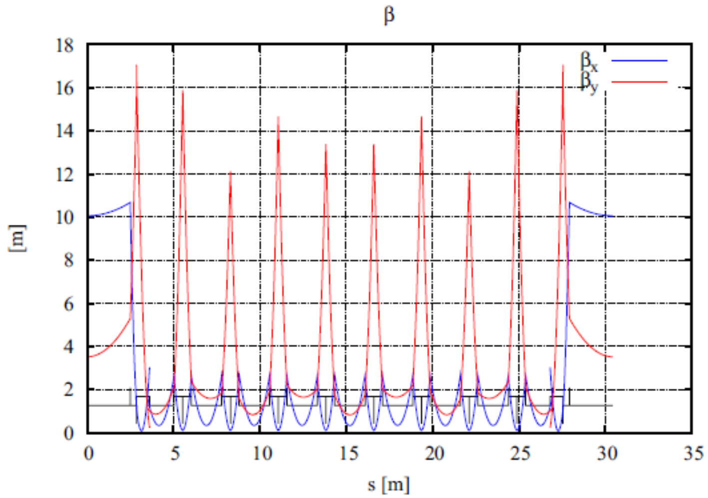
By first introducing a suitable unit and matching cell, arbitrary MBA type lattices can then be constructed with ease.

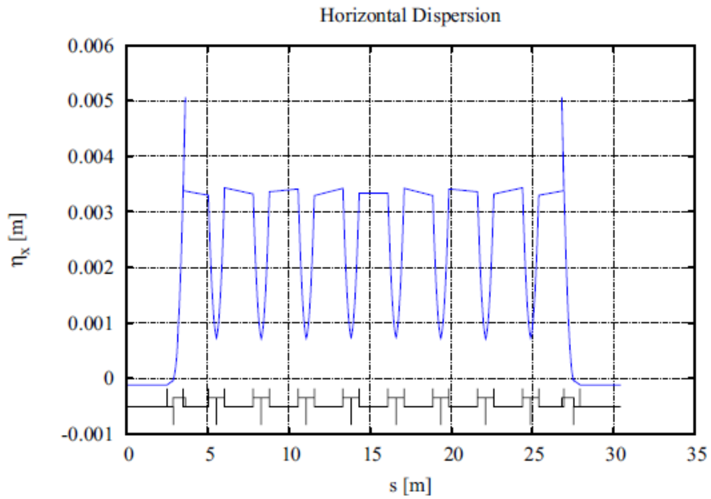
For an illustration we choose ($\mathcal{H}_{\min}$ cells):

1. **10-BA×37, C = 1,100 m, ε_x = 8 pm·rad @6 GeV,**
2. **7-BA×20, C = 500 m, ε_x = 50 pm·rad @3 GeV,**
3. **4-BA×20, C = 500 m, ε_x = 180 pm·rad @3 GeV,**

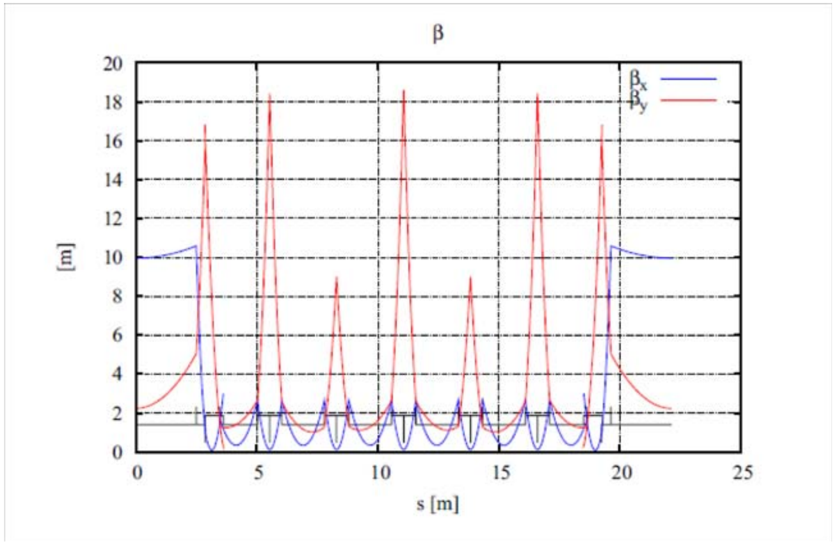
where we have ignored the impact of IBS.

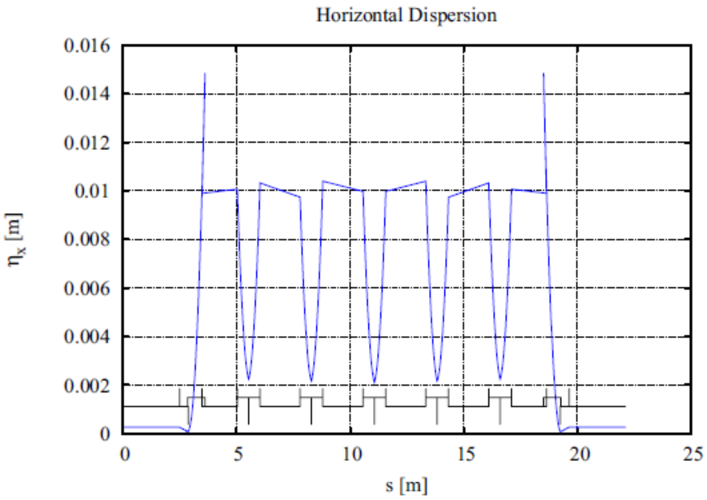
10-BA×37, C = 1,100 m, ε_x = 8 pm·rad @6 GeV





7-BA×20, C = 500 m, ε_x = 180 pm·rad @3 GeV





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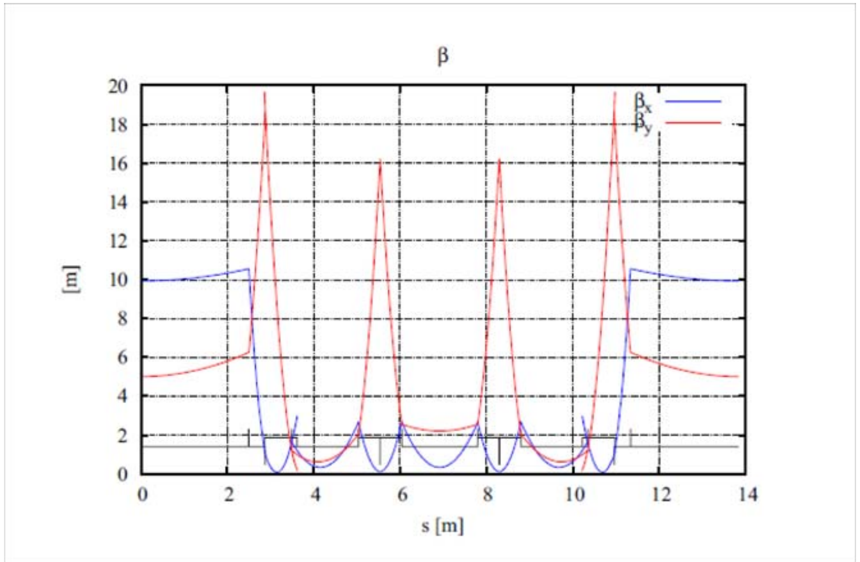
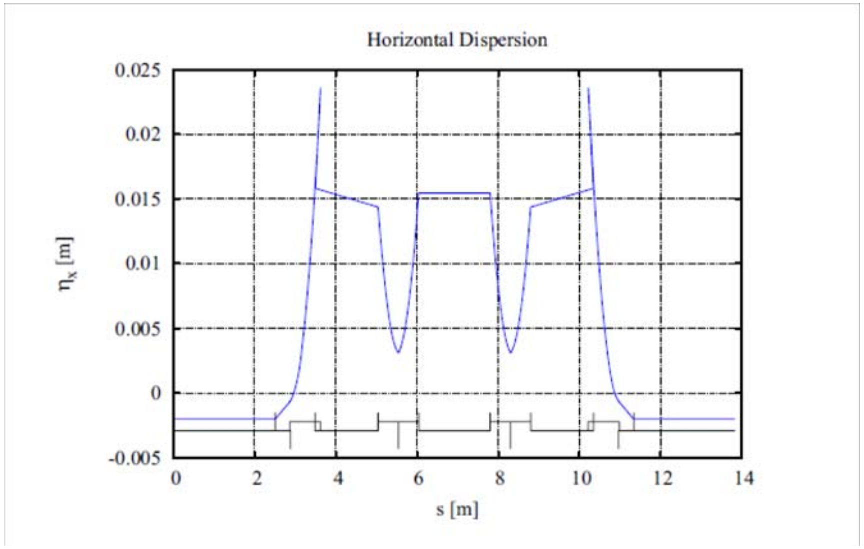
4-BA×20, C = 300 m, ε_x = 50 pm·rad @3 GeV





An ME Cell for a 3°, L_b=1.0 m Bend

We will now return to our initial example:

$$\phi_b = 3°, \qquad L_b = 1.0\ \mathrm{m}, \qquad \rho_b = \frac{L_b}{\phi_b} \approx 19.1\ \mathrm{m},$$

reduce ε_x by a factor of $\varepsilon_r = 13$ to $\varepsilon_x = 0.615$ nm·rad @3 GeV for minimum hor/ver linear chromaticity, and compare it with the MAX-IV unit cell, i.e., for the same linear dispersion action $\mathcal{H}$.

In particular, the MAX-IV unit cell has $\varepsilon_x = 0.334$ but $J_x \approx 2$ because the Qd gradient is integrated into the dipole.

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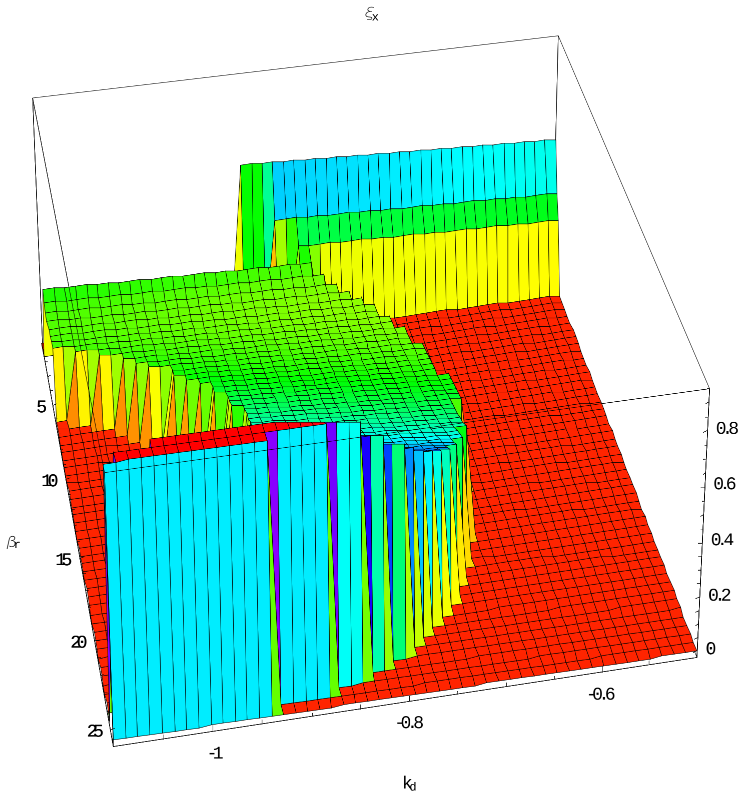
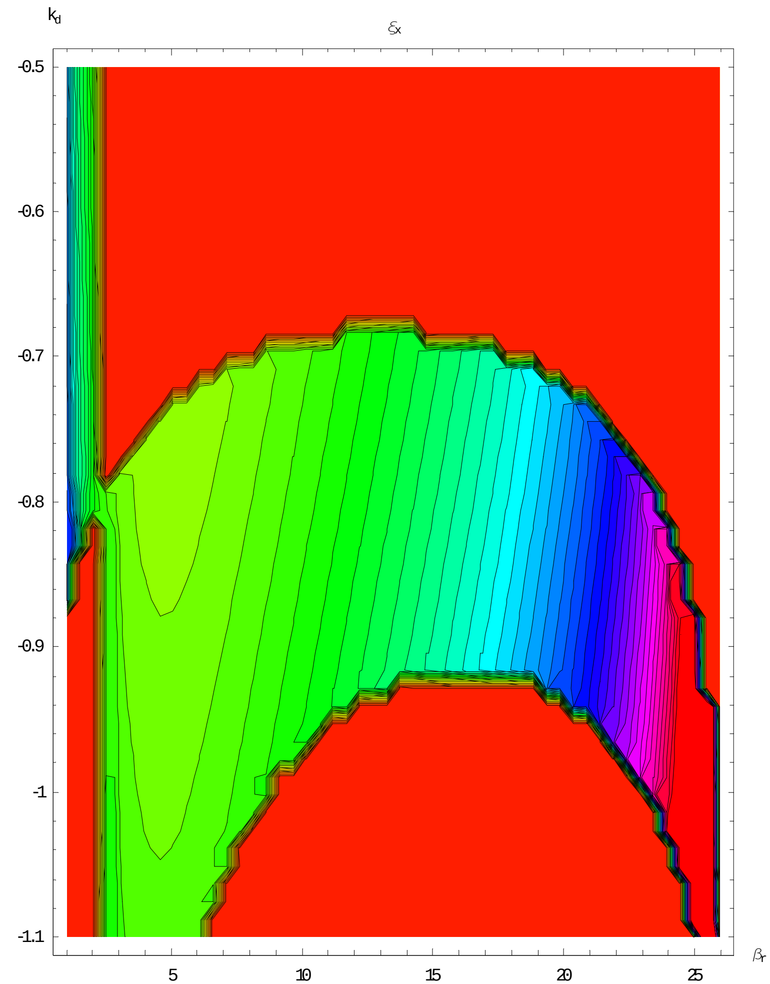
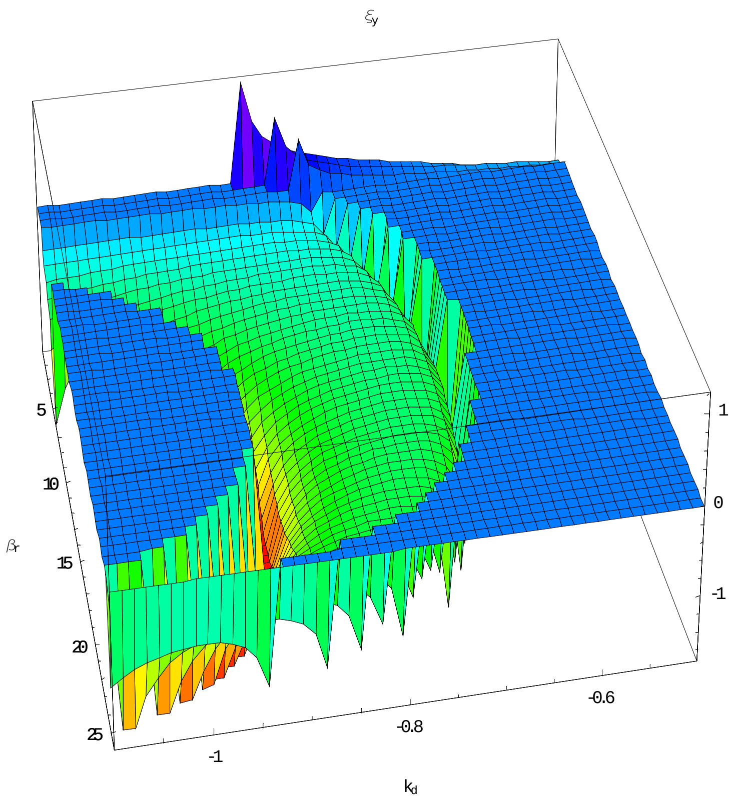
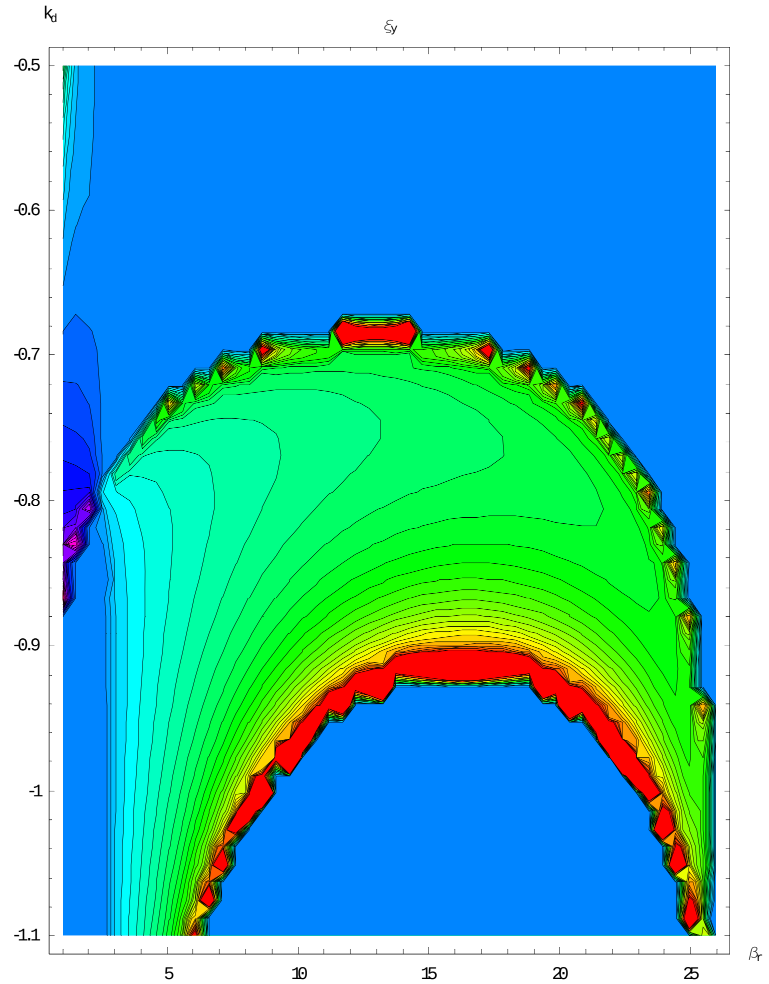
3D Parametric Plot of Hor Linear Chromaticity



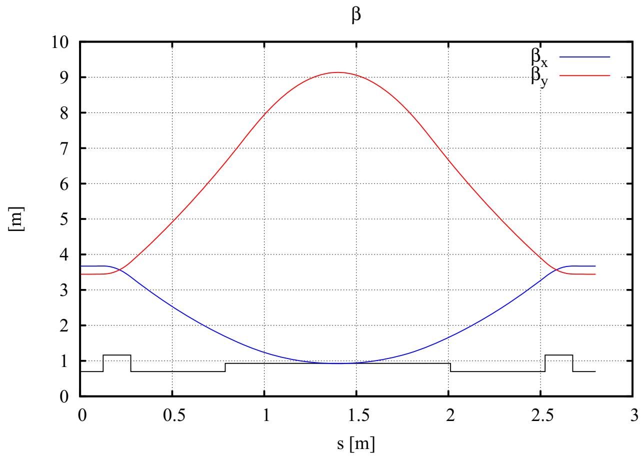


3D Parametric Plot of Ver Linear Chromaticity

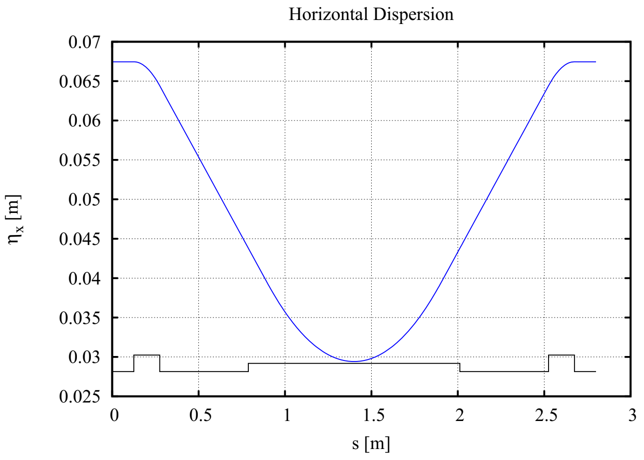


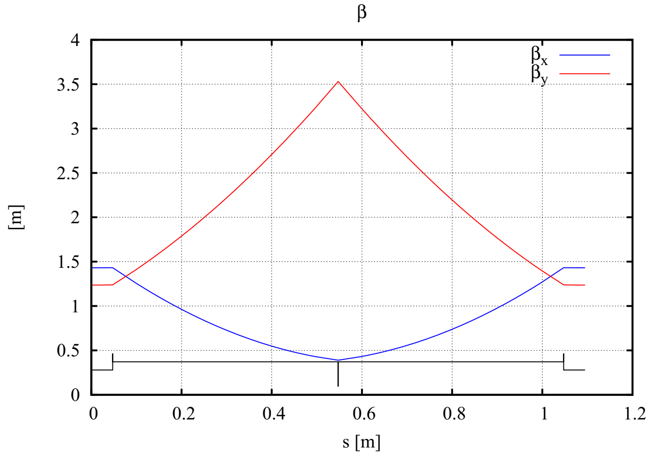


An ME Cell for a 3°, L_b=1.0 m Bend (cont.)

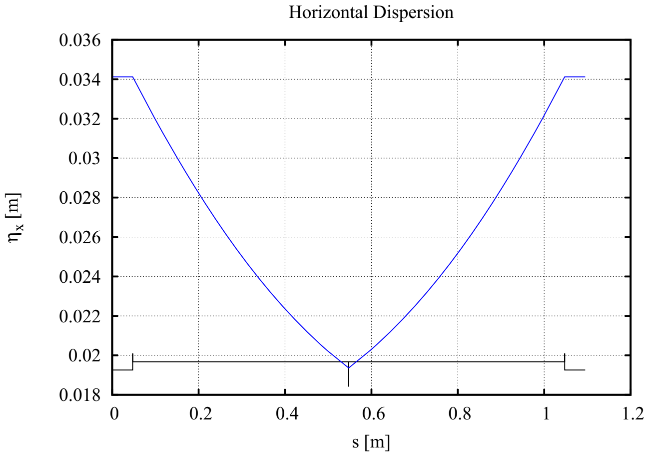


MAX-IV Unit Cell





ME Cell



An ME Cell for a 3°, L_b=1.0 m Bend (cont.)

The tune and linear chromaticity are

$$\nu_x \approx 0.244, \qquad \nu_y \approx 0.089, \qquad \xi_x \approx -0.232, \qquad \xi_y \approx -0.232$$

whereas the MAX-IV unit cell has

$$\nu_x \approx 0.265, \qquad \nu_y \approx 0.082, \qquad \xi_x \approx -0.270, \qquad \xi_y \approx -0.241.$$

To summarize, it is well optimized for the given parameters.

Note, it can be scaled according to the scaling properties summarized on slide 11, without affecting the linear chromaticity; but the peak beta functions and hor linear will change.

Conclusions

- We have outlined a self-consistent control theory for the design of Minimum Emittance (ME) cells.
- In particular, we have included the parametric dependance of the hor/ver linear chromaticity into the framework.
- We have also worked out the scaling properties for such a cell, i.e., without affecting the dispersion action and linear chromaticity.
- We have illustrated how to generate arbitrary MBA straw man lattices by pursuing a LEGO approach.
- For a "reality check" we have applied the theory to the MAX-IV unit cell, for which it reproduced the tune and hor/ver linear chromaticity.
- A control theory approach for the Dynamic Aperture problem has been outlined in SLS Tech Note 9/97 (1997).