Revision of the Kick-map Theory And Integration of s-dependent Hamiltonian

Weiming Guo, NSLS-II

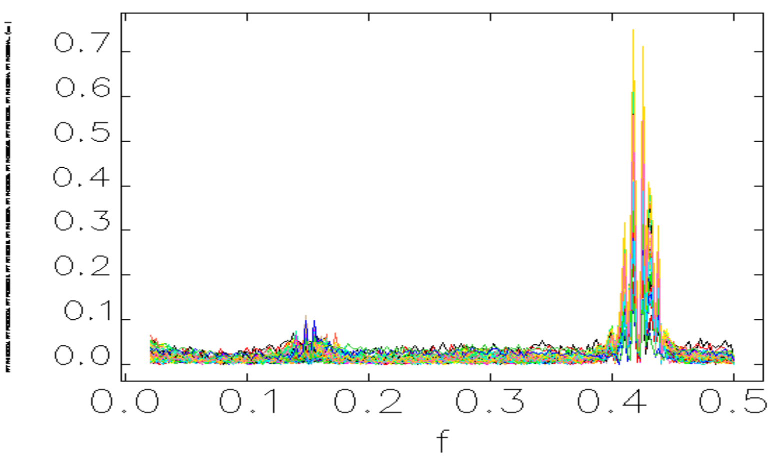
09/18/2014

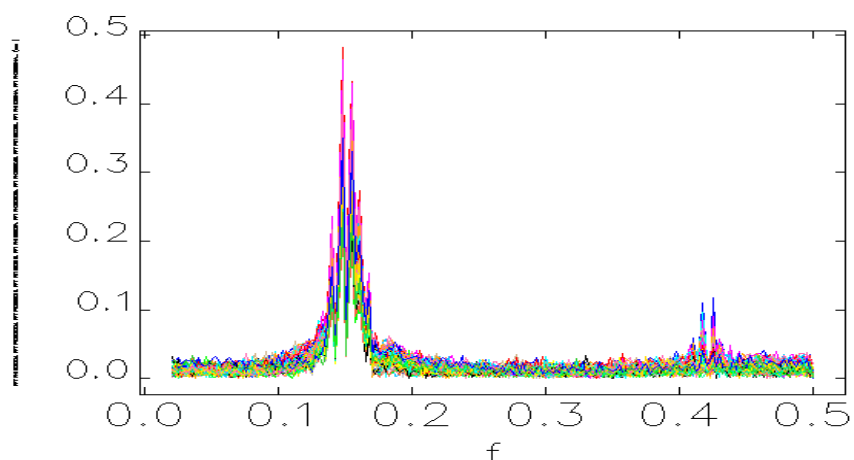
Outline

- Brief on NSLS-II commissioning
- Problem to solve
- Revision of the kick-map theory
- Higher order effects found in NSLS-II IDs
- Integration of the s-dependent Hamiltonian
- Example: soft-edged quadrupole
- Summary

NSLS-II Phase I Commissioning

•First turn was obtained after minor orbit correction.
•There was no need to adjust dipoles
•Beam stored after orbit correction.
•First time tune measurement shows 0.42 (H) and 0.17 (v), compared to the design values of 0.22(h) and 0.26(v).
•Two loose contact springs in the bellows took us about 2 weeks to identify
•Close to 100% injection efficiency was achieved after tune, orbit, and beta beat correction.
•The dynamic aperture is >12mm and the beam lifetime is >3h.
•Phase I commissioning took about 5 weeks.





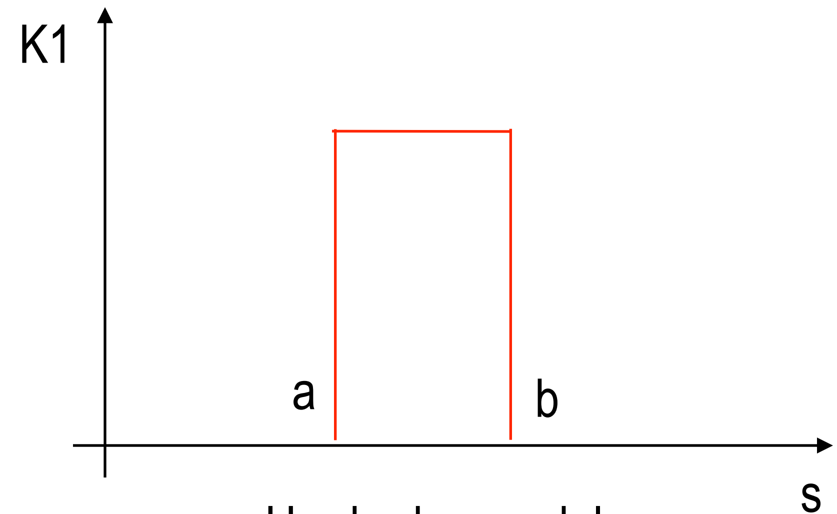
Problem to Solve

- Hard-edge model

Q1: QUAD, K1=1.2, L=0.3

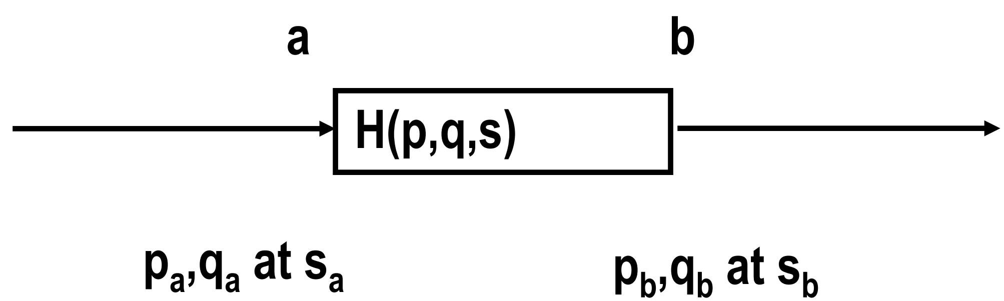
$B_y=b_1\,x$, $B_x = b_1\,y$, $B_z = 0$

$A_x= A_y =0$, $A_s= \frac{1}{2}\,b_1(x^2-y^2)$



Hard-edge model

- 3-d Hamiltonian system



- Applications
  - Modeling of insertion devices—extending the Kick-map theory
  - Modeling of fringe fields

The Concept of Pascal's Kick-map

$$\Psi(x_0, y_0, z) = \left(\int_{z_a}^{z} dz_1 B_y(x_0, y_0, z_1)\right)^2 + \left(\int_{z_a}^{z} dz_1 B_x(x_0, y_0, z_1)\right)^2$$

$$\Delta x'(x_0, y_0) = -\frac{\alpha^2}{2}\frac{\partial}{\partial x_0}\int_{z_a}^{z_b} \Psi(x_0, y_0, z)$$

where $\alpha = \frac{e}{\gamma m v} = \frac{1}{B\rho}$.

$$\Delta y'(x_0, y_0) = -\frac{\alpha^2}{2}\frac{\partial}{\partial y_0}\int_{z_a}^{z_b} \Psi(x_0, y_0, z)$$

- The integration is along a straight line instead of the reference trajectory: contradictory to edge focusing
- The kicks depend only on the initial position→ symplecticity
- Kicks at (x,y) can be obtained from interpolation on the grid

Assumptions:

- The first and second integrals are negligible
- Δx and Δy are not defined→ incomplete phase space transformation
- Only the leading order term is kept

Revision of KM: Equations of Motion

$$\gamma m \frac{d^2x}{dt^2} = e(v_z B_y - v_y B_z)$$

$$\gamma m \frac{d^2z}{dt^2} = e(v_y B_x - v_x B_y)$$

$$\gamma m \frac{d^2y}{dt^2} = e(v_x B_z - v_z B_x)$$

$$x'' = -\alpha\sqrt{1 + x'^2 + y'^2}[y' B_z + x' y' B_x - (1 + x'^2) B_y].$$

$$y'' = \alpha\sqrt{1 + x'^2 + y'^2}[x' B_z + x' y' B_y - (1 + y'^2) B_x].$$

where $x' = \frac{dx}{dz} = \frac{p_x}{p_z}$, $y' = \frac{dy}{dz} = \frac{p_y}{p_z}$ $\alpha = e/\gamma m v = 1/B\rho$

First and Second Order Equations

Now expand x and y as power series of α,

$$\begin{aligned} x &= x_0 + x_0' z + \alpha x_1 + \alpha^2 x_2 + \cdots \\ y &= y_0 + y_0' z + \alpha y_1 + \alpha^2 y_2 + \cdots \end{aligned}$$

Expanding the field around the initial position (x_0, y_0) as

$$\begin{aligned} B_u(x, y, z) &= B_u(x_0, y_0, z) + \frac{\partial}{\partial x_0} B_u(x_0, y_0, z) x_0' z + \frac{\partial}{\partial y_0} B_u(x_0, y_0, z) y_0' z \\ &+ \alpha \frac{\partial}{\partial x_0} B_u(x_0, y_0, z) x_1 + \alpha \frac{\partial}{\partial y_0} B_u(x_0, y_0, z) y_1 + \cdots, \end{aligned}$$

$$\begin{aligned} x_1'' &\approx \sqrt{1 + x_0'^2 + y_0'^2} [B_{y0}(z) - B_{z0}(z) y_0' + B_{y0}^{(x)}(z) x_0' z + B_{y0}^{(y)}(z) y_0' z] \\ y_1'' &\approx \sqrt{1 + x_0'^2 + y_0'^2} [-B_{x0}(z) + B_{z0}(z) x_0' - B_{x0}^{(x)}(z) x_0' z - B_{x0}^{(y)}(z) y_0' z]. \end{aligned}$$

$$\begin{aligned} x_2'' &\approx \sqrt{1 + x_0'^2 + y_0'^2} [\frac{\partial}{\partial x_0} B_y(z) x_1 - B_z(z) y_1' + \frac{\partial}{\partial y_0} B_y(z) y_1] \\ y_2'' &\approx -\sqrt{1 + x_0'^2 + y_0'^2} [\frac{\partial}{\partial x_0} B_x(z) x_1 - B_z(z) x_1' + \frac{\partial}{\partial y_0} B_x(z) y_1] \end{aligned}$$

Phase Space Transformation up to α^2

$$
\begin{aligned}
x(z_B) &= x_0(z_A) + x'_0(z_A)(z_B - z_A) + \alpha F_v \int_{z_A}^{z_B} dz_2 \int_{z_A}^{z_2} dz F_x \\
&- \frac{1}{2}\alpha^2 F_v \frac{\partial}{\partial x_0} \int_{z_A}^{z_B} dz_2 \int_{z_A}^{z_2} dz_1 \Psi + \alpha^2 F_v \int_{z_A}^{z_B} dz x'_K(z) \\
x'(z_B) &= x'_0(z_A) + \alpha F_v \int_{z_A}^{z_B} dz F_x - \frac{1}{2}\alpha^2 F_v \frac{\partial}{\partial x_0} \int_{z_A}^{z_B} dz_1 \Psi + \alpha^2 F_v x'_K(z_B) \\
y(z_B) &= y_0(z_A) + y'_0(z_A)(z_B - z_A) + \alpha F_v \int_{z_A}^{z_B} dz_2 \int_{z_A}^{z_2} dz F_y \\
&- \frac{1}{2}\alpha^2 F_v \frac{\partial}{\partial y_0} \int_{z_A}^{z_B} dz_2 \int_{z_A}^{z_2} dz_1 \Psi + \alpha^2 F_v \int_{z_A}^{z_B} dz y'_K(z) \\
y'(z_B) &= y'_0(z_A) + \alpha F_v \int_{z_A}^{z_B} dz F_y - \frac{1}{2}\alpha^2 F_v \frac{\partial}{\partial y_0} \int_{z_A}^{z_B} dz_1 \Psi + \alpha^2 F_v y'_K(z_B),
\end{aligned}
$$

where

$$
\Psi = \left(\int_{z_a}^{z_1} dz B_{y0}(z)\right)^2 + \left(\int_{z_a}^{z_1} dz B_{x0}(z)\right)^2
$$

$$
F_v = \sqrt{1 + {x'_0}^2 + {y'_0}^2}.
$$

$$
\begin{aligned}
F_x &= B_{y0}(z) - B_{z0}(z) y'_0 + B_{y0}^{(x)}(z) x'_0 z + B_{y0}^{(y)}(z) y'_0 z \\
F_y &= -B_{x0}(z) + B_{z0}(z) x'_0 - B_{x0}^{(x)}(z) x'_0 z - B_{x0}^{(y)}(z) y'_0 z.
\end{aligned}
$$

The Supplemental Terms

- The leading extra terms are the 1st and 2nd integrals
- The kicks depend on both the initial position and angle
- Kicks can be also obtained from interpolation of many maps

$$\int dzB_x,\ \int dzB_y,\ \int dzB_z,\ \frac{\partial}{\partial x}\int dz(B_y z),\ \frac{\partial}{\partial y}\int dz(B_y z),\ \frac{\partial}{\partial x}\int dz(B_x z),\ \frac{\partial}{\partial y}\int dz(B_x z$$

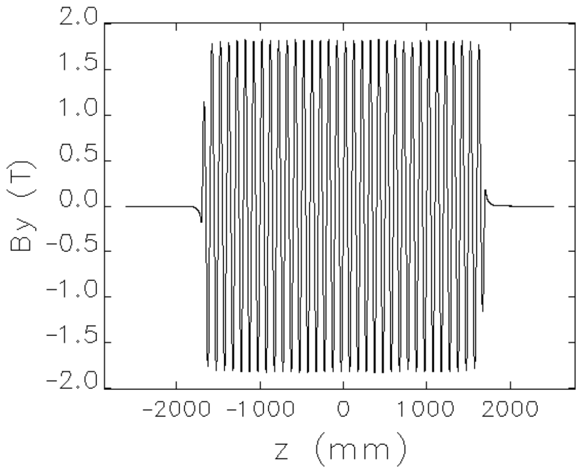
and double integrals.

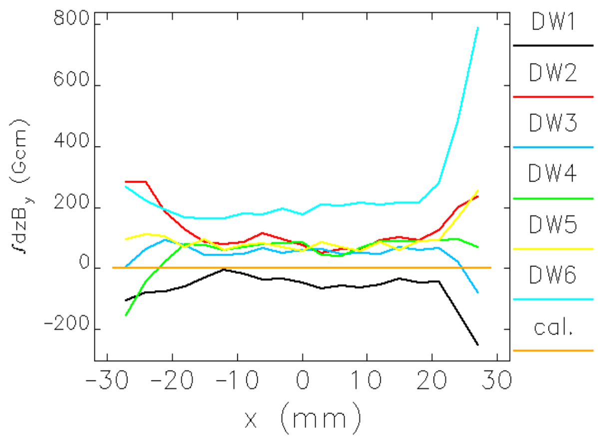
Questions:

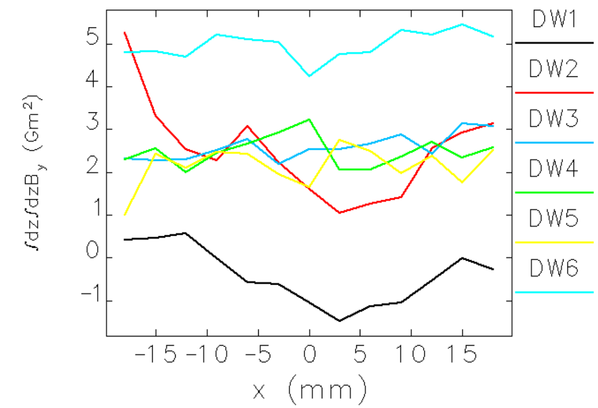
- How to make the transformation symplectic
- How to calculate the path length
- What is the magnitude of these terms in IDs
- How to extend to even higher order

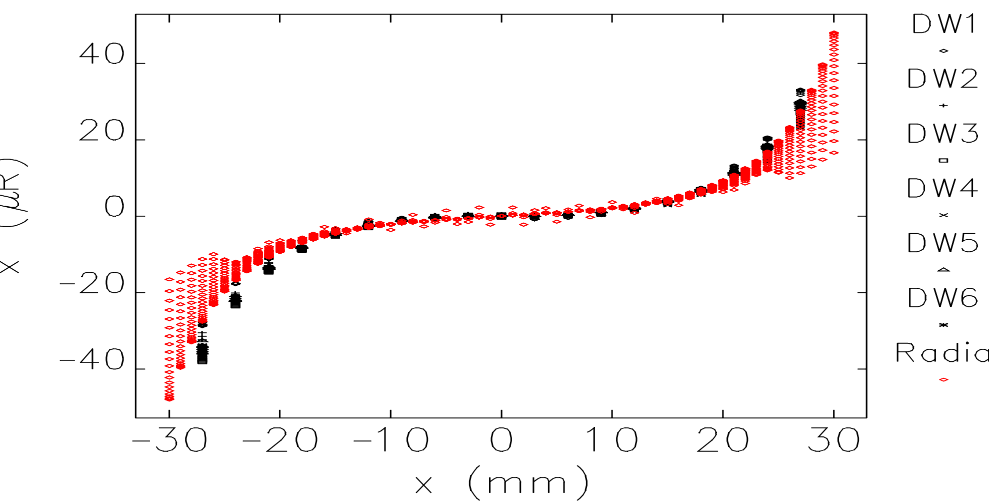
Results from NSLS-II Damping Wigglers

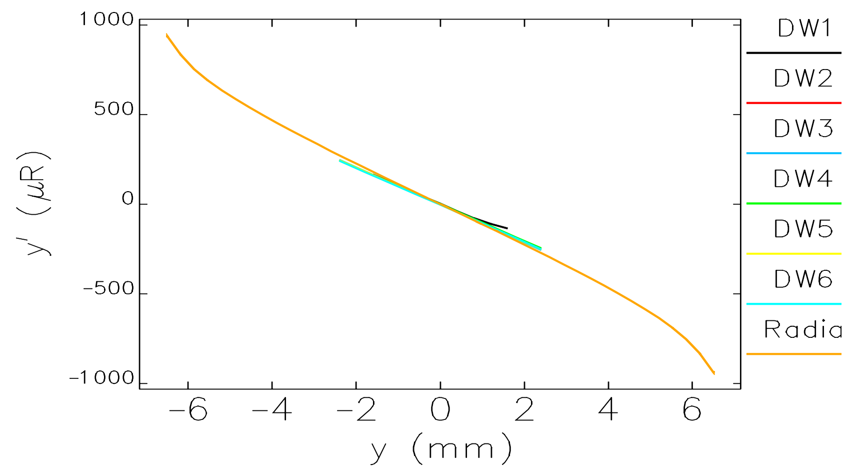
NSLS-II damping wiggler: 33 periods+2 end halves, λ=0.1m, B=2T









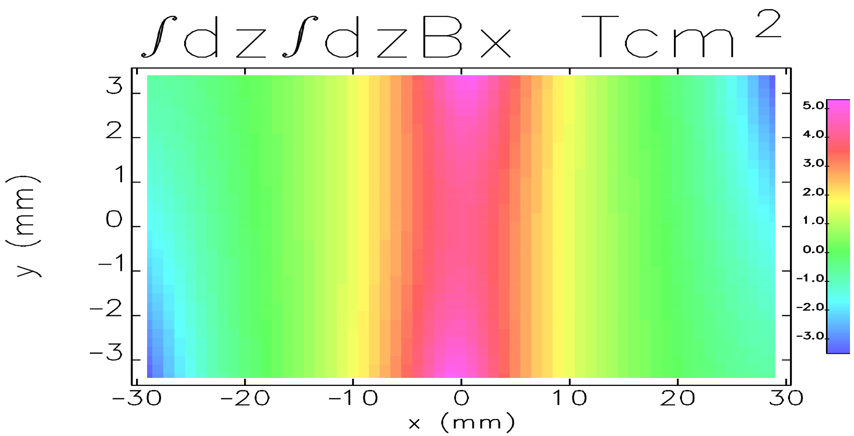


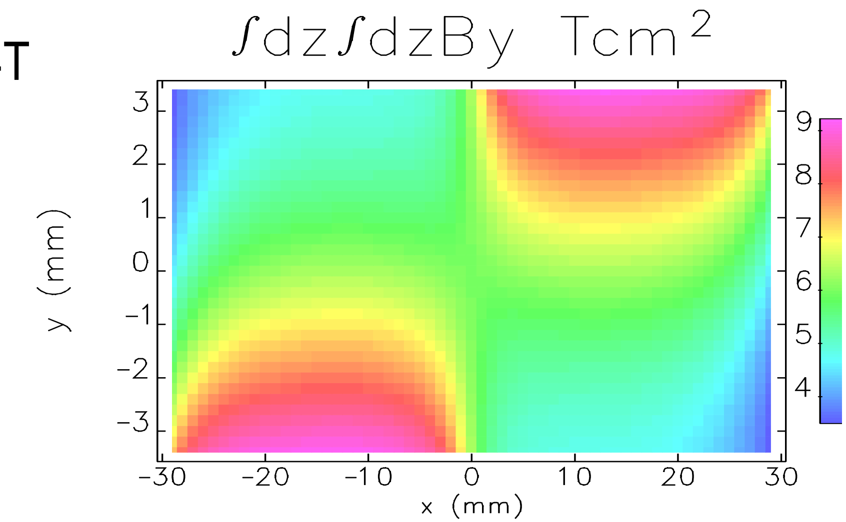
U.S. DEPARTMENT OF ENERGY

BROOKHAVEN NATIONAL LABORATORY
BROOKHAVEN SCIENCE ASSOCIATES

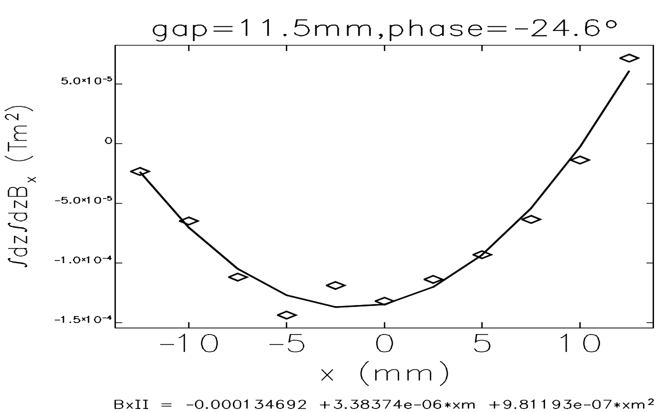
Results from NSLS-II EPU49

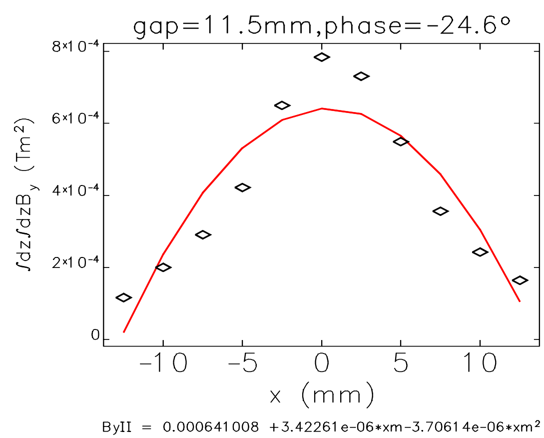
NSLS-II EPU49: 40 periods, λ=0.049m, B=0.4T

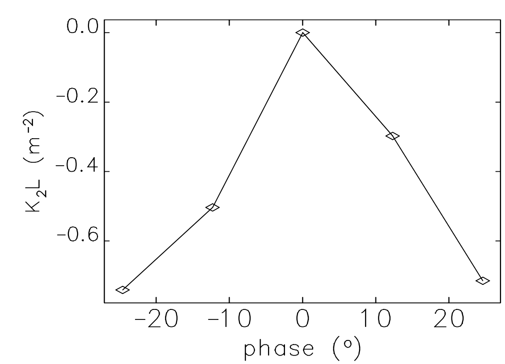




Upper: design data, gap=11.5mm, phase=14.3 degree, helical mode
Lower: measurement, gap=11.5mm, phase=-24.6 degree, helical mode







U.S. DEPARTMENT OF ENERGY

BROOKHAVEN NATIONAL LABORATORY
BROOKHAVEN SCIENCE ASSOCIATES

Hamiltonian Integration: Taylor Map

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t} = - : H : f + \frac{\partial f}{\partial t}$$

$$= -[H, f] + \frac{\partial f}{\partial t} = \vdots H \vdots f$$

$$f(p,q)|_{t=t_a} = e^{-:H:t} f(p,q)|_{t=t_a}$$

$$f(p,q,t)|_{t=t_a} = e^{-:H:t+\frac{\partial}{\partial t}} f(p,q,t)|_{t=t_a} = e^{\vdots H \vdots} f(p,q,t)|_{t=t_a}$$

$$\begin{aligned} f(t_a + \Delta t) &= f(t_a) + \frac{d}{dt} f(t_a) \Delta t + \frac{1}{2} \frac{d^2}{dt^2} f(t_a) \Delta t^2 + \cdots \\ &= f(t_a) + \Delta t \vdots H \vdots f(t_a) + \frac{1}{2} (\Delta t \vdots H \vdots)^2 z_i(t_a) + \cdots, \\ &= \exp(\Delta t \vdots H \vdots) f(t_a) \end{aligned}$$

Integration I

$$
\begin{aligned}
x(s_1+\Delta s) &= x(s_1) + \frac{\partial H}{\partial p}|_{s=s_1}\Delta s + \frac{1}{2!}(\frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p} + \frac{\partial}{\partial s})\frac{\partial H}{\partial p}|_{s=s_1}\Delta s^2 \\
&+ \frac{1}{3!}(\frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p} + \frac{\partial}{\partial s})(\frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p} + \frac{\partial}{\partial s})\frac{\partial H}{\partial p}|_{s=s_1}\Delta s^3 + \cdots
\end{aligned}
$$

Collecting all the $\frac{\partial^n}{\partial s^n}\frac{\partial H}{\partial p}$ terms, one finds

$$
\begin{aligned}
\dot{\tilde{T}}_1 x &= \frac{\partial H}{\partial p}|_{s=s_1}\Delta s + \frac{1}{2!}\frac{\partial}{\partial s}\frac{\partial H}{\partial p}|_{s=s_1}\Delta s^2 + \frac{1}{3!}\frac{\partial^2}{\partial s^2}\frac{\partial H}{\partial p}|_{s=s_1}\Delta s^3 + \cdots \\
&= \frac{\partial}{\partial s}\int ds\frac{\partial H}{\partial p}|_{s=s_1}\Delta s + \frac{1}{2!}\frac{\partial^2}{\partial s^2}\int ds\frac{\partial H}{\partial p}|_{s=s_1}\Delta s^2 + \frac{1}{3!}\frac{\partial^3}{\partial s^3}\int ds\frac{\partial H}{\partial p}|_{s=s_1}\Delta s^3 + \cdots
\end{aligned}
$$

$$
\dot{\tilde{T}}_1 x = t_{x,1}(s_1+\Delta_s) - t_{x,1}(s_1) = \int_{s_1}^{s_1+\Delta_s} ds\frac{\partial H}{\partial p}.
$$

Integration II

$$
\begin{aligned}
x(s_1+\Delta s) &= x(s_1) + \frac{\partial H}{\partial p}|_{s=s_1}\Delta s + \frac{1}{2!}(-:H:+\frac{\partial}{\partial s})\frac{\partial H}{\partial p}|_{s=s_1}\Delta s^2 \\
&+ \frac{1}{3!}(-:H:+\frac{\partial}{\partial s})(-:H:+\frac{\partial}{\partial s})\frac{\partial H}{\partial p}|_{s=s_1}\Delta s^3 \\
&+ \frac{1}{4!}(-:H:+\frac{\partial}{\partial s})(-:H:+\frac{\partial}{\partial s})(-:H:+\frac{\partial}{\partial s})\frac{\partial H}{\partial p}|_{s=s_1}\Delta s^4 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\dot{\tilde{T}}_2 x &= \frac{1}{2!}(-:H:\frac{\partial H}{\partial p})|_{s=s_1}\Delta s^2 + \frac{1}{3!}[\frac{\partial}{\partial s}(-:H:)\frac{\partial H}{\partial p}]|_{s=s_1}\Delta s^3 + \frac{1}{4!}[\frac{\partial^2}{\partial s^2}(-:H:)\frac{\partial H}{\partial p}]|_{s=s_1}\Delta s^4 + \cdots \\
&= \frac{1}{2!}[\frac{\partial^2}{\partial s^2}\int_{s_a}^{s} ds_1 \int_{s_a}^{s_1} ds_2(-:H:)\frac{\partial H}{\partial p}]|_{s=s_1}\Delta s^2 + \frac{1}{3!}[\frac{\partial^3}{\partial s^3}\int_{s_a}^{s} ds_1 \int_{s_a}^{s_1} ds_2(-:H:)\frac{\partial H}{\partial p}]|_{s=s_1}\Delta s^3 \\
&+ \frac{1}{4!}[\frac{\partial^4}{\partial s^4}\int_{s_a}^{s} ds_1 \int_{s_a}^{s_1} ds_2(-:H:)\frac{\partial H}{\partial p}]|_{s=s_1}\Delta s^4 + \cdots \\
&= \int_{s_a}^{s_b} ds_1 \int_{s_a}^{s_1} ds_2(-:H:)\frac{\partial H}{\partial p}
\end{aligned}
$$

Integration-All Terms

General solution:

$$
\begin{aligned}
x(s_b) &= x(s_a) + \int_{s_a}^{s_b} ds\,{:}H{:}x \\
&+ \int_{s_a}^{s_b} ds_1 \int_{s_a}^{s_1} ds_2 (- : H : {:}H{:})x \\
&+ \int_{s_a}^{s_b} ds_1 \int_{s_a}^{s_1} ds_2 \int_{s_a}^{s_2} ds_3 (- : H : {:}H{:}^{2})x \\
&\vdots \\
&+ \int_{s_a}^{s_b} ds_1 \cdots \int_{s_a}^{s_{n-1}} ds_n (- : H : {:}H{:}^{n-1})x + \cdots
\end{aligned}
$$

Equivalent formula:

$$
\begin{aligned}
e^{(s_b - s_a){:}H{:}} x(s_a) &= 1 + \int_{s_a}^{s_b} ds\,{:}H{:} \\
&+ \sum_{n=2}^{\infty} \int_{s_a}^{s_b} ds_1 \cdots \int_{s_a}^{s_{n-1}} ds_n (- : H : {:}H{:}^{n-1})
\end{aligned}
$$

U.S. DEPARTMENT OF ENERGY

BROOKHAVEN NATIONAL LABORATORY
BROOKHAVEN SCIENCE ASSOCIATES

Remarks

- It is not a time-ordered series
- The two methods are equivalent
- The integration is carried out for elements of full length L, convergence
- The physical interpretation of the multiple integrals: $\int f(z) z^n \, dz$
- The method can be used to solve differential equations.

Modelling of Soft-edged Quadrupoles

$$\dddot{I}_1 x = P_x, \quad \dddot{I}_2 x = x B_1, \quad \dddot{I}_3 x = P_x B_1, \quad \dddot{I}_4 x = x B_1^2 + P_x B_1',$$

where $B_1 = \frac{e}{P_0}\frac{b_1}{1+\delta}$, and $P_x = \frac{p_x}{1+\delta}$.

For hard-edged model a magnetic length is defined as:

$$L = \int_{s_a}^{s_b} ds B_1(s)/B_m,$$

Bm is the peak field

$$\begin{aligned} x_b &= x_a \cos\sqrt{-B_1}L + P_{x,a}\frac{\sin\sqrt{-B_1}L}{\sqrt{-B_1}} \\ &= x_a(1+\frac{1}{2}B_1L^2+\frac{1}{24}B_1^2L^4+\cdots)+P_{x,a}(L+\frac{1}{6}B_1L^3+\cdots). \end{aligned}$$

$$\begin{aligned} x_b &= x_a(1+\frac{(\int_{s_a}^{s_b} dsB_1(s))^2}{2B_m}+\frac{(\int_{s_a}^{s_b} dsB_1(s))^4}{24B_m^2}+\cdots) \\ &+ P_{x,a}(L+\frac{(\int_{s_a}^{s_b} dsB_1(s))^3}{6B_m^2}+\cdots). \end{aligned}$$

$$\begin{aligned} x_b &= x_a[1+\int_{s_a}^{s_b} ds_1 \int_{s_a}^{s_1} ds_2 B_1(s_2) \\ &+ \int_{s_a}^{s_b} ds_1 \int_{s_a}^{s_1} ds_2 \int_{s_a}^{s_2} ds_3 \int_{s_a}^{s_3} ds_4 B_1^2(s_4)+\cdots] \\ &+ P_{x,a}[L+\int_{s_a}^{s_b} ds_1 \int_{s_a}^{s_1} ds_2 \int_{s_a}^{s_2} ds_3 B_1(s_3) \\ &+ \int_{s_a}^{s_b} ds_1 \int_{s_a}^{s_1} ds_2 \int_{s_a}^{s_2} ds_3 \int_{s_a}^{s_3} ds_4 B_1'(s_4)+\cdots]. \end{aligned}$$

Hard-edge model ← Differs in two aspects → Hamiltonian Integration

Summary

- Revised Pascal's Kick-map theory, and kept all the lower order terms
- For NSLS-II EPUs we found the 2^{nd} integrals introduce a sextupole effect
- Showed an equivalent 3-d Hamiltonian integration method
- This method can be used to model 3-dimentional electromagnetic devices as well as solving differential equations.

Second Order Transfer Map

$$H_2 \approx -1 + \frac{1}{2}[(P_x - a_x)^2 + (P_y - a_y)^2].$$

$$P_x(z_b) = P_x(z_a) + \int_{z_a}^{z_b} dz \dddot{I}_1 P_x + \int_{z_a}^{z_b} dz \int_{z_a}^{z} dz_1 \dddot{I}_2 P_x$$

$$\dddot{I}_1 P_x = (P_x - a_x)a_x^{(x)} + (P_y - a_y)a_y^{(x)}$$

$$\begin{aligned} \dddot{I}_2 P_x \;&=\; -(\alpha B_z a_x^{(x)} + a_x a_x^{(xy)} + 2a_y a_y^{(xy)} + a_x a_y^{(xx)})P_y \\ &+\; (\alpha B_z a_y^{(x)} - a_y a_x^{(xy)} - 2a_x a_x^{(xx)} - a_y a_y^{(xx)})P_x. \end{aligned}$$

Here we kept terms to the first order of P_x and P_y, and up to α^2.

Comparison of the Two Approaches

$$A_x(x,y,z) = -\int dz B_y(x,y,z)$$
$$A_y(x,y,z) = \int dz B_x(x,y,z)$$

$$\frac{d}{dz}(p_x + aA_x)$$

$$\begin{aligned}
&= \alpha(x' A_x^{(x)} + y' A_y^{(x)}) \\
&= \alpha(A_x^{(x)}(x_0,y_0,z)x_0' + A_y^{(x)}(x_0,y_0,z)y_0') \\
&+ \alpha^2(A_x^{(x)}(x_0,y_0,z)x_1' + A_y^{(x)}(x_0,y_0,z)y_1') \\
&+ \alpha^2(A_x^{(xx)}(x_0,y_0,z)x_1 + A_x^{(xy)}(x_0,y_0,z)y_1)x_0' \\
&+ \alpha^2(A_y^{(xx)}(x_0,y_0,z)x_1 + A_y^{(xy)}(x_0,y_0,z)y_1)y_0' \\
&+ \alpha^2(A_x^{(xx)}(x_0,y_0,z)x_1' + A_y^{(xx)}(x_0,y_0,z)y_1')x_0' z \\
&+ \alpha^2(A_x^{(xy)}(x_0,y_0,z)x_1' + A_y^{(xy)}(x_0,y_0,z)y_1')y_0' z
\end{aligned}$$

Substituting

$$\begin{aligned}
x_1' &\approx -A_x(x_0,y_0,z) - y_0' \int_{-\infty}^{z} B_z(x_0,y_0,z_1)dz_1 \\
y_1' &\approx -A_y(x_0,y_0,z) + x_0' \int_{-\infty}^{z} B_z(x_0,y_0,z_1)dz_1.
\end{aligned}$$

$$\begin{aligned}
&= \alpha(A_x^{(x)}(x_0,y_0,z)x_0' + A_y^{(x)}(x_0,y_0,z)y_0') \\
&- \frac{1}{2}\alpha^2\frac{\partial}{\partial x_0}(A_x^2(x_0,y_0,z) + A_y^2(x_0,y_0,z)) \\
&+ \alpha^2\int_{-\infty}^{z} dz_1(x_0' A_y^{(x)}(x_0,y_0,z_1)B_z(x_0,y_0,z_1) - y_0' A_x^{(x)}(x_0,y_0,z_1)B_z(x_0,y_0,z_1)) \\
&- \alpha^2\int_{-\infty}^{z} dz_1 x_0'(2A_x^{(xx)}(x_0,y_0,z_1)A_x(x_0,y_0,z_1) + A_x^{(xy)}(x_0,y_0,z_1)A_y(x_0,y_0,z_1) \\
&\quad +A_y^{(xx)}(x_0,y_0,z_1)A_y(x_0,y_0,z_1)) \\
&- \alpha^2\int_{-\infty}^{z} dz_1 y_0'(A_y^{(xx)}(x_0,y_0,z_1)A_x(x_0,y_0,z_1) + 2A_y^{(xy)}(x_0,y_0,z_1)A_y(x_0,y_0,z_1) \\
&\quad +A_x^{(xy)}(x_0,y_0,z_1)A_x(x_0,y_0,z_1))
\end{aligned}$$

Here we kept terms to the first order of x_0' and y_0', and up to α^2.

Remarks

- The multiple integrals come from three sources:

  1) the single or double integration of the field expansion terms, such as $\frac{\partial^n}{\partial x^n} B_y(z)(x_0' z)^n$

  2) The iterative relations such as $x_2'' = x_1$

  3) The crossing terms of the above two types.

- If the magnetic field is sinusoidal, the multiple integral will converges as $(1/k)^n$, where k=2p/l is the wave number. For insertion devices k_z~100 which result in quick convergence.
- Momentum and longitudinal position can be added to the phase space variables. Radiation energy loss can be included.
- Even though not presented here the path length can be calculated analytically using this method.
- The expansion to >3rd order becomes very messy for the general Hamiltonian; however, it is not that complicated for a particular magnet type. We proposed to keep terms up to 8th order in the paper, and can be realized by a script.
- Truncation at any order will lead to non-symplecticity of the transformation. However the deviation will be of higher order. This is similar to Taylor map.

Application to the Halbach Field

$$
\begin{aligned}
B_x &= \frac{k_x}{k_y} B_0 \sinh k_x x \sinh k_y y \cos k_z z \\
B_z &= -\frac{k_z}{k_y} B_0 \cosh k_x x \sinh k_y y \sin k_z z \\
B_y &= B_0 \cosh k_x x \cosh k_y y \cos k_z z
\end{aligned}
$$

Where $k_x^2 + k_y^2 = k_z^2 = (\frac{2\pi}{\lambda_w})^2$

λ_w is the period length.

B_0=1.8T, k_z=62.83 1/m, k_x=0.875 1/m, α=0.1 1/(Tm), λw=0.1m

ΔX=x_F+x_y

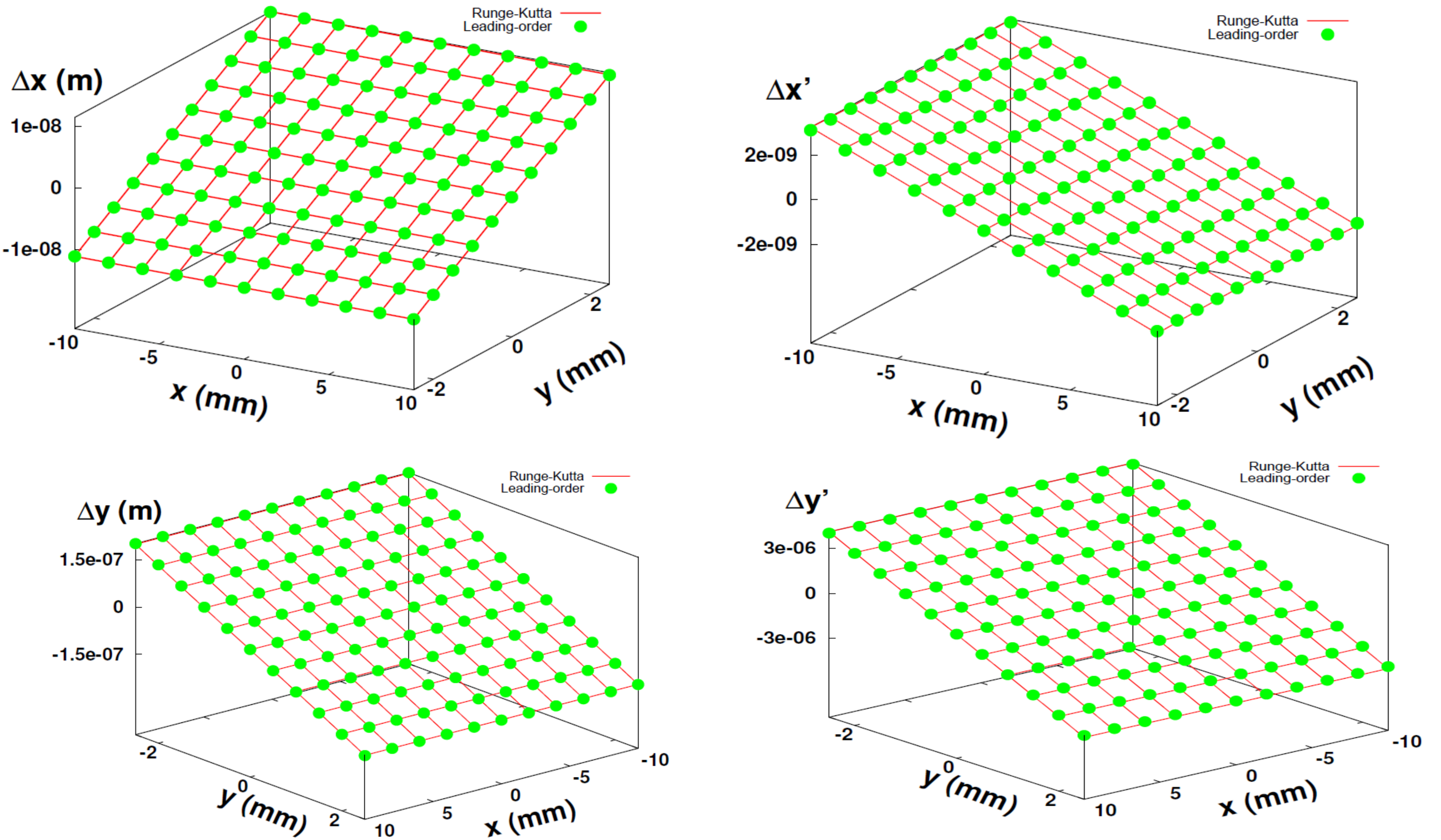
ΔX'=x_F'+x_y'

ΔY=y_F+Y_y

ΔY'=y_F'+y_y'

$$
\begin{aligned}
x_F &= 2\alpha B_0 F_v \pi \frac{1}{k_z^3}(k_y^2 y_0' y_0 + k_x^2 x_0' x_0) \approx 0.018 y_0' y_0 + 3.5 \times 10^{-6} x_0' x_0 \\
x_\Psi &= -\pi^2 \alpha^2 B_0^2 F_v \frac{k_x^2}{k_z^4} x_0 \approx -1.6 \times 10^{-8} x_0 \\
x_F' &= 0 \\
x_\Psi' &= -\pi \alpha^2 B_0^2 F_v \frac{k_x^2}{k_z^3} x_0 \approx -3.1 \times 10^{-7} x_0 \\
y_F &= -2\alpha B_0 F_v \pi \frac{k_x^2}{k_z^3}(x_0' y_0 + y_0' x_0) \approx -3.5 \times 10^{-6}(x_0' y_0 + y_0' x_0) \\
y_\Psi &= -\pi^2 \alpha^2 B_0^2 F_v \frac{k_y^2}{k_z^4} y_0 \approx -8.1 \times 10^{-5} y_0 \\
y_F' &= 0 \\
y_\Psi' &= -\pi \alpha^2 B_0^2 F_v \frac{k_y^2}{k_z^3} y_0 \approx -1.6 \times 10^{-3} y_0.
\end{aligned}
$$

Comparison with Runge-Kutta Tracking



Modelling of a 3-d Quadrupole

$$B_x = b_1 y - \frac{1}{6} b_1'' y^3 + \frac{1}{5!} b_1^{(4)} y^5 - \frac{1}{7!} b_1^{(6)} y^7 + \frac{1}{9!} b_1^{(8)} y^9 + \cdots$$

$$B_s = b_1' xy - \frac{1}{6} b_1^{(3)} xy^3 + \frac{1}{5!} b_1^{(5)} xy^5 - \frac{1}{7!} b_1^{(7)} xy^7 + \frac{1}{9!} b_1^{(9)} xy^9 + \cdots$$

$$B_y = b_1 x - \frac{1}{2} b_1'' xy^2 + \frac{1}{4!} b_1^{(4)} xy^4 - \frac{1}{6!} b_1^{(6)} xy^6 + \frac{1}{8!} b_1^{(8)} xy^8 + \cdots.$$

Vector potential

$$A_x = \frac{1}{2} b_1' xy^2 - \frac{1}{4!} b_1^{(3)} xy^4 + \frac{1}{6!} b_1^{(5)} xy^6 - \frac{1}{8!} b_1^{(7)} xy^8 + \frac{1}{10!} b_1^{(9)} xy^{10} + \cdots$$

$$A_s = \frac{1}{2} b_1 x^2 - \frac{1}{2} b_1 y^2 + \frac{1}{4!} b_1'' y^4 - \frac{1}{6!} b_1^{(4)} y^6 + \frac{1}{8!} b_1^{(6)} y^8 - \frac{1}{10!} b_1^{(8)} y^{10} + \cdots$$

$$A_y = 0.$$

Sextupole and Octupole harmonics

$$A_{s,h} = \epsilon \frac{b_2}{3}(x^3 - 3xy^2) + \epsilon \frac{a_2}{3}(y^3 - 3x^2 y) + \frac{b_3}{4}(x^4 - 6x^2y^2 + y^4) + a_3(xy^3 - x^3 y).$$